Name: ______________________________

Signature: ______________________________

Math 5651 Lecture 001 (V. Reiner) Midterm Exam I
Thursday, February 22, 2018

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	________
2.	________
3.	________
4.	________
5.	________
6.	________
Total:	________

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = B_1 \sqcup \cdots \sqcup B_n \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i),$$

and Bayes' Theorem: $\mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$

$$X = \text{Bin}(n,p) \text{ has p.f. } f(k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } k \in \{0,1,2,\ldots,n\}$$

$$X = \text{Hypergeom}(A,B,n) \text{ has p.f. } f(k) = \binom{A}{k}\binom{B}{n-k} / \binom{A+B}{n} \text{ for } k \in \{0,1,2,\ldots,\min\{A,n\}\}$$

$$X = \text{Poi}(\lambda) \text{ has p.f } f(k) = e^{-\lambda}\frac{\lambda^k}{k!} \text{ for } k \in \{0,1,2,\ldots\}$$

Problem 1. *(10 points)*

a. (3 points) You flip a fair coin 4 times. What is the probability that you never see two of the same flip consecutively, that is, no "heads, heads" and no "tails, tails"?

b. (7 points) Same question, except that this time you flip the coin n times, where $n \geq 2$. (Your answer for the probability should be a function of n.)

Problem 2. *(15 points)* You know that the number of lightning strikes in a certain area during a month is a Poisson random variable with parameter λ.

a. (5 points) What is the probability that next month there are exactly 3 lightning strikes? (The parameter λ will appear in your answer.)

b. (10 points) If you know that last month there was *at least one* lightning strike, then what is the probability that there were between 2-4 lightning strikes?

Problem 3. *(15 points)* Your favorite radio station chooses its morning disc jockey randomly among four possibilities: Charlotte, Emily, Anne, who each work 2 mornings per week on average, and Bramwell who works 1 morning per week on average. Charlotte, Emily and Anne all like to play classical music $\frac{1}{3}$ of the time, and non-classical music the other $\frac{2}{3}$, while Bramwell plays *only* classical music.

a. (5 points) What is the *a priori* chance that, when you wake up tomorrow morning, Charlotte will be the disc jockey?

b. (10 points) If you wake up and hear classical music playing, what is the chance that Bramwell is the disc jockey that morning?

Problem 4. *(20 points total)*
True or False? Each of your answers **must be justified by calculation**.

a. (5 points) Given events A, B with $\mathbf{Pr}(A), \mathbf{Pr}(B) > 0$, if $A \cap B = \varnothing$, then A, B are independent.

b. (5 points) Given an event A with $0 < \mathbf{Pr}(A) < 1$, and if we let $B = A$, then A, B are dependent.

c. (5 points) When sampling **without replacement** $r > 0$ red and $w > 0$ white balls from a box, let R_i, W_i, respectively, be the events that the i^{th} ball sampled is red, white, respectively.

Then $\mathbf{Pr}(R_2|W_1) > \mathbf{Pr}(R_2) > \mathbf{Pr}(R_2|R_1)$.

d. (5 points) With notation as in part (c), events W_1 and R_2 are independent.

Problem 5. *(20 points)* Let A, B, C be events with $\mathbf{Pr}(C) > 0$. Prove that $\mathbf{Pr}(A \cup B|C) = \mathbf{Pr}(A|C) + \mathbf{Pr}(B|C) - \mathbf{Pr}(A \cap B|C)$.

Problem 6. *(20 points total)* You have two coins in your pocket, one *fair* coin having probability of heads 1/2, and one *unfair* coin in which the probability of heads is 2/3. You pull one of the two out of your pocket, with either coin equally probable, and start flipping it, generating a sequence of heads and tails.

a. (10 points) What is the probability that the *first* tails is on the 7^{th} flip?

b. (10 points) After you pull it out of your pocket, you do one test flip, and get heads. What is the probability that you pulled the fair coin out?