Name: ________________________________

Signature: ________________________________

Math 5651. Lecture 001 (V. Reiner) Midterm Exam I
Tuesday, September 28, 2010

This is a 115 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
5.	
6.	
7.	
Total:	

Problem 1. *(10 points total)* When rolling a fair 6-sided dice having $1, 2, 3, 4, 5, 6$ on its sides, consider the two events A, B where A is rolling an even number, and B is rolling a number divisible by three. Are A and B dependent or independent? You must support your answer with calculations to receive any credit.

Problem 2. *(15 points total; 5 points each)* Compute the probabilities of the following events when rolling 8 times repeatedly a fair 6-sided dice as in Problem 1. Your answer in each case can be left as a fraction– you do not need to convert it to a decimal.

a. (5 points) The sum of the numbers rolled is exactly 8.

b. (5 points) The sum of the numbers rolled is exactly 9.

c. (5 points) Each of the six values appears this many times:

value	1	2	3	4	5	6
number of occurrences	0	2	4	1	0	1

Problem 3. *(15 points total)* Prove that if the events A, B, C are independent, then the events A and $B^c \cup C$ are also independent.

Problem 4. *(15 points total)* A fair coin with alternatives heads or tails (H or T) is flipped 10 times repeatedly. Let A be the event that exactly 8 heads occur among the 10 flips, and let B be the event that the first three flips are (H, T, H).

a. (7 points) Compute the probability $\Pr(A)$.

b. (8 points) Compute the conditional probability $\Pr(A|B)$.

Problem 5. *(15 points)* For four events A_1, A_2, A_3, A_4, express the conditional probability $\Pr(A_1 \cup A_2 \cup A_3 | A_4)$ in terms of any or all of these probabilities: $\Pr(A_i), \Pr(A_i A_j), \Pr(A_i A_j A_k)$ and $\Pr(A_1 A_2 A_3 A_4)$.

Problem 6. *(15 points total; 5 points each)* A radioactive material is emitting particles, and each particle independently has a $\frac{1}{7}$ chance of pentrating through a shield. What are the probabilities of the following events?

a. (5 points) After 100 emissions, exactly 4 particles have penetrated.

b. (5 points) After 100 emissions, at least two particles have penetrated.

c. (5 points) The first particle that penetrates is the k^{th} one emitted. (Your answer should be a function of k.)

Problem 7. *(15 points)* At your favorite restaurant, there are three possible chefs, Emeril Lagasse, Bobby Flay, and Rick Bayless. They are randomly scheduled each week for which of the 7 evenings they will work, although each works a different number of days. Also, each has their own well-established percentage chance of overcooking the asparagus. This information is summarized here:

chef	Lagasse	Flay	Bayless
number of evenings worked per week	1	4	2
% chance of overcooking asparagus	100	20	50

If the asparagus that you ordered was overcooked, what's the probability that Emeril Lagasse was the chef that evening?

Brief solutions:

1. Calculate

$$\begin{aligned} S &= \{1,2,3,4,5,6\} \\ A &= \{2,4,6\} \text{ so } \Pr(A) = \frac{3}{6} = \frac{1}{2} \\ B &= \{3,6\} \text{ so } \Pr(B) = \frac{2}{6} = \frac{1}{3} \\ AB &= \{6\} \text{ so } \Pr(AB) = \frac{1}{6} = \frac{1}{2} \cdot \frac{1}{3} = \Pr(A)\Pr(B) \end{aligned}$$

so, yes, they're independent.

2. $|S| = 6^8$.
(a)The event A of rolling a sum of 8 can only occur as $(1,1,1,1,1,1,1,1)$, so $\Pr(A) = \frac{1}{6^8}$.
(b)The event B of rolling a sum of 9 can only occur in these 8 different ways $(2,1,1,1,1,1,1,1),(1,2,1,1,1,1,1,1),\ldots,(1,1,1,1,1,1,1,2)$ so $\Pr(B) = \frac{8}{6^8}$.
(c)Choosing die rolls for this event C is the same as choosing a word with letters $1,2,3,4,5,6$ having each letter occur as many times as in the table, so $\Pr(C) = \frac{\binom{8}{0,2,4,1,0,1}}{6^8} = \frac{8!}{6^8 \cdot 2!4!1!1!}$.

3. We need to show that $\Pr(A(B^c \cup C)) = \Pr(A)\Pr(B^c \cup C)$, so for example, we could just try to compute both sides in terms of intersection probabilities:

$$\begin{aligned} \Pr(B^c \cup C) &= \Pr(B^c) + \Pr(C) - \Pr(B^c C) \\ &= 1 - \Pr(B) + \Pr(BC) \text{ since } C = BC \sqcup B^c C \\ &= 1 - \Pr(B) + \Pr(B)\Pr(C) \text{ due to independence of } B, C. \end{aligned}$$

meanwhile

$$\begin{aligned} \Pr(A(B^c \cup C)) &= \Pr(AB^c \cup AC) \\ &= \Pr(AB^c) + \Pr(AC) - \Pr(AB^c AC) \\ &= \Pr(A) - \Pr(AB) + \Pr(AC) - \Pr(AB^c C) \text{ since } A = AB \sqcup AB^c \\ &= \Pr(A) - \Pr(AB) + \Pr(ABC) \text{ since } AC = ABC \sqcup AB^c C \\ &= \Pr(A) - \Pr(A)\Pr(B) + \Pr(A)\Pr(B)\Pr(C) \text{ due to independence of } A, B, \\ &= \Pr(A)(1 - \Pr(B) + \Pr(B)\Pr(C)) \\ &= \Pr(A)\Pr(B^c \cup C) \end{aligned}$$

as desired.

4. Event A is exactly 8 heads occur among the 10 flips, and event B is that the first three flips are (H, T, H).
(a) $\Pr(A) = \binom{10}{8}\left(\frac{1}{2}\right)^8\left(\frac{1}{2}\right)^2 = \frac{\binom{10}{8}}{2^{10}}$.
(b) $\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\frac{1}{2^3}\cdot\binom{7}{6}\frac{1}{2^7}}{\frac{1}{2^3}} = \frac{\binom{7}{6}}{2^7} = \frac{7}{2^7}$. Thinking about it (slightly) differently, let C be the event that the last seven flips contain exactly 6 heads. Then one has $A = BC$ with B, C independent, so $\Pr(A|B) = \Pr(BC|B) = \Pr(C|B) = \Pr(C) = \frac{\binom{7}{6}}{2^7} = \frac{7}{2^7}$.

5.

$$\begin{aligned}
\Pr(A_1 \cup A_2 \cup A_3 | A_4) &= \frac{\Pr((A_1 \cup A_2 \cup A_3) \cap A_4)}{\Pr(A_4)} \\
&= \frac{1}{\Pr(A_4)}\left[\Pr((A_1A_4) \cup (A_2A_4) \cup (A_3A_4))\right] \\
&= \frac{1}{\Pr(A_4)}\left[\Pr(A_1A_4) + \Pr(A_2A_4) + \Pr(A_3A_4)\right. \\
&\qquad - \Pr(A_1A_2A_4) - \Pr(A_1A_3A_4) - \Pr(A_2A_3A_4) \\
&\qquad \left. + \Pr(A_1A_2A_3A_4)\right]
\end{aligned}$$

6. (a)$\Pr(\text{exactly 4 penetrated}) = \binom{100}{4}\left(\frac{1}{7}\right)^4\left(\frac{6}{7}\right)^{96}$
(b)

$$\begin{aligned}
\Pr(\text{at least two penetrated}) &= 1 - \Pr(\text{zero or one penetrated}) \\
&= 1 - \binom{100}{0}\left(\frac{1}{7}\right)^0\left(\frac{6}{7}\right)^{100} - \binom{100}{1}\left(\frac{1}{7}\right)^1\left(\frac{6}{7}\right)^{99}
\end{aligned}$$

(c)

$$\begin{aligned}
&\Pr(\text{first penetrating particle is the } k^{th} \text{ emitted}) \\
&= \Pr(\text{first } k-1 \text{ emitted don't penetrate, but } k^{th} \text{ does}) \\
&= \left(\frac{6}{7}\right)^{k-1}\frac{1}{7}
\end{aligned}$$

6. Let L, F, B be the events that Lagasse, Flay, Bayless was chef, and O the event that the asparagus was overcooked. Then Bayes' Theorem

says

$$\begin{aligned}\Pr(L|A) &= \frac{\Pr(A|L)\Pr(L)}{\Pr(A|L)\Pr(L)+\Pr(A|F)\Pr(F)+\Pr(A|B)\Pr(B)}\\ &= \frac{(1.00)(\frac{1}{7})}{(1.00)(\frac{1}{7})+(.20)(\frac{4}{7})+(.50)(\frac{2}{7})} = \frac{5}{14}.\end{aligned}$$