Name: ______________________________

Signature: ______________________________

# Math 5651 Lecture 002 (V. Reiner) Midterm Exam II
## Thursday, March 31, 2016

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

| Problem | Score |
|---|---|
| 1. | ________ |
| 2. | ________ |
| 3. | ________ |
| 4. | ________ |
| 5. | ________ |
| 6. | ________ |
| Total: | ________ |

**Reminders:**

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^n B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^n \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^n \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \le x), \text{while pdf } f(x) = \frac{\partial}{\partial x} F(x)$$

$$g_1(x|y) = f(x,y)/f_2(y), \quad g_2(y|x) = f(x,y)/f_1(x)$$

$$f_1(x) = \int_{y=-\infty}^{y=+\infty} f(x,y)dy, \quad f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

$$\text{When } \underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y}), \text{ then } f(\underline{x}), g(\underline{y}) \text{ satisfy } g(\underline{y}) = f(\underline{s}(y)) \cdot |J| \text{ where } J := \det \left( \frac{\partial s_i}{\partial y_j} \right)$$

$$\mathbf{E}X = \begin{cases} \sum_k k \cdot f(k) & X \text{ discrete,} \\ \int_{-\infty}^{+\infty} x f(x) dx & X \text{ continuous.} \end{cases}$$

| $X$ | p.f. $f(k)$ | $\mathbf{E}X$ |
|---|---|---|
| $\mathrm{Bin}(n,p)$ | $\binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0,1,\ldots,n\}$ | $pn$ |
| $\mathrm{Hypergeom}(A,B,n)$ | $\binom{A}{k}\binom{B}{n-k}/\binom{A+B}{n}$ for $k \in \{0,1,\ldots,\min\{A,n\}\}$ | $\frac{A}{A+B} n$ |
| $\mathrm{Poi}(\lambda)$ | $e^{-\lambda} \frac{\lambda^k}{k!}$ for $k \in \{0,1,2,\ldots\}$ | $\lambda$ |

**Problem 1.** *(20 points)* Let $X_1, X_2$ be random variables with joint pdf

$$f(x_1, x_2) = \begin{cases} 2x_2 & \text{if } 0 < x_1 < 1 \text{ and } 0 < x_2 < 1, \\ 0 & \text{otherwise.} \end{cases}$$

a. (10 points) Are $X_1, X_2$ independent? You must justify your answer.

b. (10 points) Defining the random variable $Y := X_1 - X_2$, compute the pdf $g(y)$ for $Y$ for all $y$ in $\mathbb{R}$.

**Problem 2.** *(15 points)* Assume that a 10 person committee is chosen from among 60 women and 30 men, with all possible choices equally likely. Let $X$ denote the number of women on the committee, and $Y$ the number of men on the committee.

a. (5 points) Calculate $\mathbf{E}X$.

b. (10 points) Calculate $\mathbf{E}(X - Y)$.

**Problem 3.** *(15 points)* Let $X$ be a discrete random variable whose values lie in $\{0, 1, 2, \ldots, n\}$. Prove that

$$\mathbf{E}X = \mathbf{Pr}(X \geq 1) + \mathbf{Pr}(X \geq 2) + \cdots + \mathbf{Pr}(X \geq n-1) + \mathbf{Pr}(X \geq n)$$

**Problem 4.** *(20 points)* Let $X$ be a continuous random variable with pdf

$$f(x) = \begin{cases} c(x^2 - 1) & \text{if } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $c$ is some constant lying in $\mathbb{R}$.

a. (5 points) Find the value of $c$.

b. (5 points) Compute $\mathbf{E}X$.

c. (5 points) Compute the cdf $F(x)$ for $X$.

d. (5 points) If $X_1, X_2$ are independent and identically distributed, both with the same distribution as $X$, then what is $\mathbf{Pr}(X_1 > X_2)$? You only need to write down an explicit integral that calculates it– do not evaluate the integral.

**Problem 5.** *(15 points total)* Let $X$ be a continuous random variable, uniformly distributed on the interval $[0, 4]$.

a. (10 points) Let $Y$ be a continuous random variable chosen uniformly on the interval $[0, x]$ after knowing the value $X = x$. Compute the conditional pdf $g_1(x|y = 1) = g_1(x|1)$ for all values of $x$.

b. (5 points) Compute the pdf $g(z)$ for $Z = X^5$.

**Problem 6.** *(15 points total)* A group of $n$ people walk into a restaurant, hand their hat to the hat-check attendant, and after dinner, the attendant hands back one of the hats uniformly at random to each person.

Let $X$ be the random variable which is the number of people that receive their own hat. Compute $\mathbf{E}X$.

(Hint: Try writing $X$ as a sum of simpler *indicator random variables*, that is, random variables that take on values 0 or 1.)