Name: ________________________

Signature: ________________________

# Math 5651 Lecture 003 (V. Reiner) Midterm Exam II
## Thursday, March 29, 2018

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

| Problem | Score |
|---|---|
| 1. | |
| 2. | |
| 3. | |
| 4. | |
| 5. | |
| Total: | |

**Reminders:**

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^{n} B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \le x), \text{while pdf } f(x) = \frac{\partial}{\partial x} F(x)$$

$$g_1(x|y) = f(x,y)/f_2(y), \quad g_2(y|x) = f(x,y)/f_1(x)$$

$$f_1(x) = \int_{y=-\infty}^{y=+\infty} f(x,y)dy, \quad f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

When $\underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y})$, then $f(\underline{x}), g(\underline{y})$ satisfy $g(\underline{y}) = f(\underline{s}(y)) \cdot |J|$ where $J := \det \left( \frac{\partial s_i}{\partial y_j} \right)$

$$\mathbf{E}X = \begin{cases} \sum_k k \cdot f(k) & X \text{ discrete,} \\ \int_{-\infty}^{+\infty} x f(x) dx & X \text{ continuous.} \end{cases}$$

| $X$ | p.f. $f(k)$ | $\mathbf{E}X$ |
|---|---|---|
| $\mathrm{Bin}(n,p)$ | $\binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0,1,\ldots,n\}$ | $pn$ |
| $\mathrm{Hypergeom}(A,B,n)$ | $\binom{A}{k}\binom{B}{n-k}/\binom{A+B}{n}$ for $k \in \{0,1,\ldots,\min\{A,n\}\}$ | $\frac{A}{A+B} n$ |
| $\mathrm{Poi}(\lambda)$ | $e^{-\lambda} \frac{\lambda^k}{k!}$ for $k \in \{0,1,2,\ldots\}$ | $\lambda$ |

**Problem 1.** *(20 points total)* Let $X_1, X_2$ be a pair of random variables whose joint pdf has the form

$$f(x_1, x_2) = \begin{cases} cx_1^2 x_2 & \text{for } (x_1, x_2) \in [0,1] \times [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

for some constant $c$.

a. (5 points) Determine the constant $c$.

b. (15 points) Compute a pdf $g(y)$ for $Y = 2X_1 + X_2$.

**Problem 2.** *(20 points total)*
**True or False**? Some explanation required for each answer.

a. (3 points) For a *continuous* random variable $X$, its *pdf* $f(x)$ is not uniquely determined.

b. (3 points) For a *continuous* random variable $X$, its *cdf* $F(x)$ is not uniquely determined.

c. (3 points) For a *discrete* random variable $X$, its *pf* $f(x)$ is not uniquely determined.

d. (3 points) For a *discrete* random variable $X$, its *cdf* $F(x)$ is not uniquely determined.

e. (4 points) There exists a continuous random variable $X$ having a pdf
$$f(x) = \begin{cases} \frac{1}{12}(x-1) & \text{for } x \in [0,6], \\ 0 & \text{otherwise.} \end{cases}$$

f. (4 points) If $(X, Y)$ are random variables with a joint pdf given by
$$f(x,y) = \begin{cases} 4y^3 & \text{for } (x,y) \in [0,1] \times [0,1], \\ 0 & \text{otherwise.} \end{cases}$$
then $X$ and $Y$ are *dependent*.

**Problem 3.** *(20 points total)* Let $X$ be a random variable with a pdf
$$f(x) = \begin{cases} \frac{3}{32}x(4-x) & \text{for } x \in [0,4], \\ 0 & \text{otherwise.} \end{cases}$$

a. (10 points) Calculate its expected value $\mathbf{E}X$.

b. (10 points) Find a pdf $g(y)$ for the new random variable $Y = X^5$. Indicate clearly when $g(y)$ is zero.

**Problem 4.** *(20 points total)* A group of $n$ restaurant patrons named Person 1, Person 2, ..., Person $n$ each give their hat to the hat-check attendant. Later, the attendant gives them each back a hat, uniformly at random, that is, all distributions are equally likely.

a. (5 points) What is the probability that Person 1 and Person 2 end up with *swapped* hats, that is, Person 1 receives the hat of Person 2 and Person 2 receives the hat of Person 1?

b. (15 points) Let $X$ denote the random variable which is the number of pairs $(i, j)$ with $1 \leq i < j \leq n$ for which Person $i$ and Person $j$ end up with swapped hats. Compute the expected value $\mathbf{E}X$.

**Problem 5.** *(20 points total)* Define a pair of random variables $(X, Y)$ by first picking $X$ uniformly from the interval $[0, 1]$, and then, knowing the value $X = x$, let $Y = \text{Bin}(3, x)$ be a binomial random variable with parameters $n = 3$ and $p = x$.

a. (5 points) Write down a joint pdf $f(x, y)$ for $(X, Y)$, indicating clearly when $f(x, y) = 0$.

b. (10 points) Write down a marginal pdf $f_2(y)$ for $Y$, again indicating clearly when $f_2(y) = 0$.

c. (5 points) Write down a conditional pdf $g_1(x|2)$ for $X$ given that $Y = 2$, again indicating clearly when $g_1(x|2) = 0$.