Name: ______________________________

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Math 5651 Lecture 003 (V. Reiner) Midterm Exam II
Thursday, March 29, 2018

This is a 115 minute exam. No books, notes, calculators, cell phones, watches or other electronic devices are allowed. You can leave answers as fractions, with binomial or multinomial coefficients unevaluated.

There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
5.	
Total:	

Reminders:

$$\mathbf{Pr}(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbf{Pr}(A_{i_1} \cap \cdots \cap A_{i_k})$$

$$S = \sqcup_{i=1}^{n} B_i \Rightarrow \mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A \cap B_i) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i) \text{ and } \mathbf{Pr}(B_i|A) = \mathbf{Pr}(A|B_i)\mathbf{Pr}(B_i)/\mathbf{Pr}(A)$$

$$\text{cdf } F(x) := \mathbf{Pr}(X \le x), \text{while pdf } f(x) = \frac{\partial}{\partial x} F(x)$$

$$g_1(x|y) = f(x,y)/f_2(y), \quad g_2(y|x) = f(x,y)/f_1(x)$$

$$f_1(x) = \int_{y=-\infty}^{y=+\infty} f(x,y)dy, \quad f_2(y) = \int_{x=-\infty}^{x=+\infty} f(x,y)dx$$

When $\underline{Y} = \underline{r}(\underline{X}) \Leftrightarrow \underline{X} = \underline{s}(\underline{Y})$, then $f(\underline{x}), g(\underline{y})$ satisfy $g(\underline{y}) = f(\underline{s}(y)) \cdot |J|$ where $J := \det\left(\frac{\partial s_i}{\partial y_j}\right)$

$$\mathbf{E}X = \begin{cases} \sum_k k \cdot f(k) & X \text{ discrete,} \\ \int_{-\infty}^{+\infty} x f(x)dx & X \text{ continuous.} \end{cases}$$

X	p.f. $f(k)$	$\mathbf{E}X$
$\mathrm{Bin}(n,p)$	$\binom{n}{k}p^k(1-p)^{n-k}$ for $k \in \{0,1,\ldots,n\}$	pn
$\mathrm{Hypergeom}(A,B,n)$	$\binom{A}{k}\binom{B}{n-k}/\binom{A+B}{n}$ for $k \in \{0,1,\ldots,\min\{A,n\}\}$	$\frac{A}{A+B}n$
$\mathrm{Poi}(\lambda)$	$e^{-\lambda}\frac{\lambda^k}{k!}$ for $k \in \{0,1,2,\ldots\}$	λ

Problem 1. *(20 points total)* Let X_1, X_2 be a pair of random variables whose joint pdf has the form

$$f(x_1, x_2) = \begin{cases} cx_1^2 x_2 & \text{for } (x_1, x_2) \in [0,1] \times [0,1] \\ 0 & \text{otherwise,} \end{cases}$$

for some constant c.

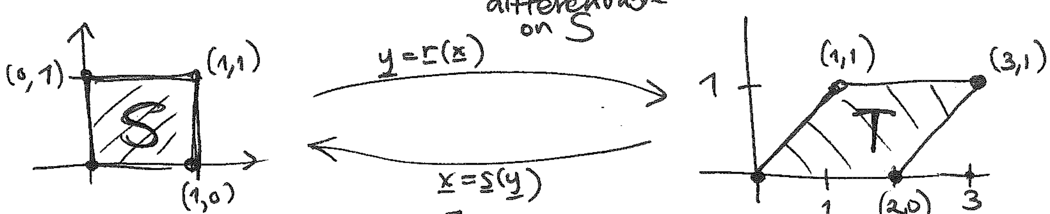
a. (5 points) Determine the constant c.

$$1 = \iint_{(x_1,x_2)\in\mathbb{R}^2} f(x_1,x_2)\,dx_1\,dx_2 = \int_{x_1=0}^{x_1=1} \int_{x_2=0}^{x_2=1} c x_1^2 x_2 \, dx_2\, dx_1 = c\int_{x_1=0}^{x_1=1} x_1^2 \left[\frac{x_2^2}{2}\right]_0^1 dx_1 = \frac{c}{2}\int_{x_1=0}^{x_1=1} x_1^2\,dx_1 = \frac{c}{2}\left[\frac{x_1^3}{3}\right]_0^1 = \frac{c}{6}$$

$$\Longrightarrow \quad c = 6$$

b. (15 points) Compute a pdf $g(y)$ for $Y = 2X_1 + X_2$.

Let $Y_1 = Y = 2X_1 + X_2 \; (= r_1(X_1, X_2))$ and pick $Y_2 = X_2 \; (= r_2(X_1, X_2))$ $\Longleftrightarrow$ invertible, differentiable on S $\quad X_1 = \frac{Y_1 - Y_2}{2} \; (= s_1(Y_1, Y_2))$, $\quad X_2 = Y_2 \; (= s_2(Y_1, Y_2))$



Jacobian $J = \det \begin{bmatrix} \frac{\partial s_1}{\partial y_1} & \frac{\partial s_1}{\partial y_2} \\ \frac{\partial s_2}{\partial y_1} & \frac{\partial s_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}$

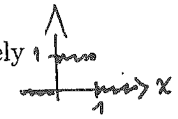
and hence (Y_1, Y_2) have joint pdf $g(y_1,y_2) = \begin{cases} f(s_1(\underline{y}), s_2(\underline{y}))\,|J| = 6\left(\frac{y_1-y_2}{2}\right)^2 y_2 \cdot \left|\frac{1}{2}\right| = \frac{3}{4}(y_1^2 y_2 - 2y_1 y_2^2 + y_2^3) & \text{if } (y_1,y_2)\in T \\ 0 \text{ otherwise} \end{cases}$

and $Y = Y_1$ has pdf

$$g(y_1) = \int_{y_2\in\mathbb{R}} g(y_1,y_2)\,dy_2 = \begin{cases} \frac{3}{4}\int_0^{y_1} (y_1^2 y_2 - 2y_1 y_2^2 + y_2^3)\,dy_2 = \frac{3}{4}\left[\frac{y_1^2 y_2^2}{2} - \frac{2y_1 y_2^3}{3} + \frac{y_2^4}{4}\right]_{y_2=0}^{y_2=y_1} = \frac{3}{4}\left[\frac{y_1^4}{2} - \frac{2y_1^4}{3} + \frac{y_1^4}{4}\right] \\ \frac{3}{4}\int_0^1 (y_1^2 y_2 - 2y_1 y_2^2 + y_2^3)\,dy_2 = \frac{3}{4}\left[\frac{y_1^2 y_2^2}{2} - \frac{2y_1 y_2^3}{3} + \frac{y_2^4}{4}\right]_{y_2=0}^{y_2=1} = \frac{3}{4}\left[\frac{y_1^2}{2} - \frac{2y_1}{3} + \frac{1}{4}\right] \\ \frac{3}{4}\int_{y_1-2}^1 (y_1^2 y_2 - 2y_1 y_2^2 + y_2^3)\,dy_2 = \frac{3}{4}\left[\frac{y_1^2 y_2^2}{2} - \frac{2y_1 y_2^3}{3} + \frac{y_2^4}{4}\right]_{y_2=y_1-2}^{y_2=1} = \frac{3}{4}\left[\frac{y_1^2}{2} - \frac{2y_1}{3} + \frac{1}{4} - \frac{y_1^2(y_1-2)^2}{2} - \frac{2y_1(y_1-2)^3}{3} + \frac{(y_1-2)^4}{4}\right] \\ 0 \text{ otherwise} \end{cases}$$

Problem 2. *(20 points total)*
True or False? Some explanation required for each answer.

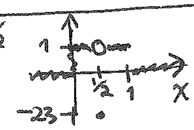
a. (3 points) For a *continuous* random variable X, its *pdf* $f(x)$ is not uniquely determined.



$y=f_1(x)$

TRUE, e.g. $X = \text{Unif}(0,1)$ has a pdf $f_1(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$

but also a pdf $f_2(x) = \begin{cases} 1 & \text{if } x \in (0,1), x \neq \frac{1}{2} \\ -23 & \text{if } x = \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$



$y=f_2(x)$

b. (3 points) For a *continuous* random variable X, its *cdf* $F(x)$ is not uniquely determined.

FALSE, since the cdf value $F(x) = \Pr(X \leq x)$

c. (3 points) For a *discrete* random variable X, its *pf* $f(x)$ is not uniquely determined.

FALSE, since the pf value $f(x) = \Pr(X = x) \left(= \Pr(X \in \{x\}) \right)$

d. (3 points) For a *discrete* random variable X, its *cdf* $F(x)$ is not uniquely determined.

FALSE, since the cdf value $f(x) = \Pr(X \leq x)$

e. (4 points) There exists a continuous random variable X having a pdf

$$f(x) = \begin{cases} \frac{1}{12}(x-1) & \text{for } x \in [0,6], \\ 0 & \text{otherwise.} \end{cases}$$

FALSE, since $f(x) < 0$ for $x \in (0,1)$

f. (4 points) If (X, Y) are random variables with a joint pdf given by

$$f(x,y) = \begin{cases} 4y^3 & \text{for } (x,y) \in [0,1] \times [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

then X and Y are *dependent.*

FALSE, since (X,Y) are independent, due to the fact that $f(x,y) = f_1(x) f_2(y)$ $\left(\forall (x,y) \in \mathbb{R}^2\right)$ where $f_1(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

$f_2(y) = \begin{cases} 4y^3 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

Problem 3. *(20 points total)* Let X be a random variable with a pdf
$$f(x) = \begin{cases} \frac{3}{32}x(4-x) & \text{for } x \in [0,4], \\ 0 & \text{otherwise.} \end{cases}$$

a. (10 points) Calculate its expected value $\mathbf{E}X$.

$$\mathbb{E}X = \int_{x\in\mathbb{R}} x f(x)\,dx = \int_0^4 x \cdot \frac{3}{32}x(4-x)\,dx = \frac{3}{32}\int_0^4 (4x^2 - x^3)\,dx = \frac{3}{32}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^4 = \frac{3}{32}\left[\frac{256}{3} - 64\right] = \frac{3}{32}\cdot\frac{256}{12} = 2$$

b. (10 points) Find a pdf $g(y)$ for the new random variable $Y = X^5$.
Indicate clearly when $g(y)$ is zero.

Since $y = x^5$ is differentiable $\left(\frac{dy}{dx} = 5x^4\right)$ and monotone increasing $\left(\frac{dy}{dx} = 5x^4 > 0\right)$ for $x \in (0,4)$

we can use the direct method:

$y = x^5 = r(x)$ has inverse function $x = y^{1/5} = s(y)$

$$\frac{ds}{dy} = \frac{1}{5}y^{-4/5}$$

$\Rightarrow$ Y has pdf

$$g(y) = \begin{cases} f(s(y))\left|\frac{ds}{dy}\right| & \text{if } y \in r([0,4]) = [0,4^5] = [0,1024] \\ 0 & \text{otherwise} \end{cases}$$

note $\frac{1}{5}y^{-4/5} > 0$ for $y \in [0,1024]$

$$= \begin{cases} \frac{3}{32}y^{\frac{1}{5}}(4-y^{\frac{1}{5}})\left|\frac{1}{5}y^{-\frac{4}{5}}\right| = \frac{3}{160}y^{-\frac{3}{5}}(4-y^{\frac{1}{5}}) & \text{if } y \in [0,1024] \\ 0 & \text{otherwise} \end{cases}$$

Problem 4. *(20 points total)* A group of n restaurant patrons named Person 1, Person 2, ..., Person n each give their hat to the hat-check attendant. Later, the attendant gives them each back a hat, uniformly at random, that is, all distributions are equally likely.

a. (5 points) What is the probability that Person 1 and Person 2 end up with *swapped* hats, that is, Person 1 receives the hat of Person 2 and Person 2 receives the hat of Person 1?

$S = \{\text{all distributions}\}$ has $|S| = n!$

$A = \{\text{those where 1 \& 2 swap hats}\}$ has $|A| = (n-2)!$ Pick the distribution for $3, 4, \dots, n$

$$\Rightarrow \Pr(A) = \frac{|A|}{|S|} = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

b. (15 points) Let X denote the random variable which is the number of pairs (i, j) with $1 \le i < j \le n$ for which Person i and Person j end up with swapped hats. Compute the expected value $\mathbb{E}X$.

Note that $X = \sum_{\substack{\text{pairs } (i,j) \\ \text{with } 1 \le i < j \le n}} X_{ij}$, where $X_{ij} = \begin{cases} 1 & \text{if } i \,\&\, j \text{ swapped hats} \\ 0 & \text{otherwise} \end{cases}$

$$\overset{\text{linearity of}}{\underset{\text{expectation}}{\Longrightarrow}} \quad \mathbb{E}X = \sum_{(i,j)} \mathbb{E}X_{ij}, \quad \text{where } \mathbb{E}X_{ij} = \Pr(\{i \,\&\, j \text{ swapped hats}\})$$

$$= \Pr(\{1 \,\&\, 2 \text{ swapped hats}\})$$

$$= \frac{1}{n(n-1)} \text{ by part (a)}$$

$$= \sum_{(i,j)} \frac{1}{n(n-1)}$$

$$= \#\{\text{pairs } (i,j) \text{ with } 1 \le i < j \le n\} \frac{1}{n(n-1)}$$

$$= \binom{n}{2} \frac{1}{n(n-1)} = \frac{n(n-1)}{2} \cdot \frac{1}{n(n-1)} = \frac{1}{2}$$

Problem 5. *(20 points total)* Define a pair of random variables (X, Y) by first picking X uniformly from the interval $[0,1]$, and then, knowing the value $X = x$, let $Y = \text{Bin}(3, x)$ be a binomial random variable with parameters $n = 3$ and $p = x$.

a. (5 points) Write down a joint pdf $f(x, y)$ for (X, Y), indicating clearly when $f(x, y) = 0$.

$X = \text{Unif}(0,1)$ has pdf $f_1(x) = \begin{cases} 1 & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$

$Y = \text{Bin}(3,x)$ has pf $g_2(y|x) = \begin{cases} \binom{3}{y} x^y (1-x)^{3-y} & \text{if } y \in \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow (X,Y)$ has joint ~~pf~~/pdf

$$f(x,y) = g_2(y|x) f_1(x) = \begin{cases} \binom{3}{y} x^y (1-x)^{3-y} & \text{if } x \in (0,1) \text{ and } y \in \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

b. (10 points) Write down a marginal ~~pdf~~ pf $f_2(y)$ for Y, again indicating clearly when $f_2(y) = 0$.

$$f_2(y) = \int_{x \in \mathbb{R}} f(x,y)\,dx = \begin{cases} \int_{x=0}^{x=1} \binom{3}{y} x^y (1-x)^{3-y}\,dx & \text{if } y \in \{0,1,2,3\} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \int_0^1 (1-x)^3\,dx = \int_0^1 (1-3x+3x^2-x^3)\,dx = \left[x - \frac{3x^2}{2} + x^3 - \frac{x^4}{4}\right]_0^1 = \frac{1}{4} \\ 3\int_0^1 x(1-x)^2\,dx = 3\int_0^1 (x-2x^2+x^3)\,dx = 3\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{1}{4} \\ 3\int_0^1 x^2(1-x)\,dx = 3\int_0^1 (x^2-x^3)\,dx = 3\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{1}{4} \\ \int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} \\ 0 \quad \text{otherwise} \end{cases}$$

c. (5 points) Write down a conditional pdf $g_1(x|2)$ for X given that $Y = 2$, again indicating clearly when $g_1(x|2) = 0$.

$$g_1(x|2) = \frac{f(x,2)}{f_2(2)} = \begin{cases} \dfrac{\binom{3}{2} x^2 (1-x)^{3-2}}{\frac{1}{4}} = 12x^2(1-x) & \text{if } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$