Name: ______________________________

Signature: ______________________________

**Math 5651. Lecture 001 (V. Reiner) Midterm Exam II**
**Thursday, October 21, 2010**

This is a 115 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

| Problem | Score |
|---|---|
| 1. | ________ |
| 2. | ________ |
| 3. | ________ |
| 4. | ________ |
| 5. | ________ |
| Total: | ________ |

**Problem 1.** *(25 points total; 5 points each)* Suppose the two random variables $(X, Y)$ are continuous, with joint probability density function taking the form $f(x, y) = y + cxy$ for $0 \leq x, y \leq 1$ and $f(x, y) = 0$ for all other $x, y$. Here $c$ is some real number constant.

a. (5 points) Determine the constant $c$.

b. (5 points) Compute the probability that $Y > X$.

c. (5 points) Compute both of the marginal probability density functions, $f_1(x)$ for $X$ and $f_2(y)$ for $Y$.

d. (5 points) Are $(X, Y)$ independent? You must justify your answer.

e. (5 points) Compute all of the conditional probability density functions $g_1(x|y)$ and $g_2(y|x)$.

**Problem 2.** *(20 points total)* Let $X$ be a random variable uniformly distributed on the interval $[1, 10]$.

a. (5 points) Write down the explicit formula for the probability density function $f(x)$ of $X$, for all values of $x$ in $(-\infty, \infty)$.

b. (5 points) Write down the explicit formula for the distribution function $F(x) = \Pr(X \leq x)$, for all values of $x$ in $(-\infty, \infty)$.

c. (10 points) Define a new random variable $Y = X^4$, and write down the explicit formula for the probability density function $g(y)$ of $Y$, for all values of $y$ in $(-\infty, \infty)$.

**Problem 3.** *(20 points total)* Recall that a Poisson random variable $X$ with mean $\mu$ has probability function $f(k) = \Pr(X = k) = \frac{e^{-\mu}\mu^k}{k!}$ for $k = 0, 1, 2, \ldots$.

Professors Sneezy and Grumpy are identical twins that share a teaching position half-and-half: each shows up to teach half the time, choosing randomly which one will teach that day. The only detectable difference between them is in their level of allergies: the number of times that Prof. Sneezy sneezes in an hour is a Poisson random variable with mean 20, while for Prof. Grumpy it is a Poisson random variable with mean 1.

a. (10 points) On a given day, with no other information given, what is the probability that the professor who shows up will *not sneeze at all* in an hour?

b. (10 points) If on one particular day you noticed that the professor who showed up sneezed 5 times in an hour, what is the probability that this was Prof. Sneezy who showed up?

**Problem 4.** *(15 points total)* Let $X$ be a continuous random variable with probability density function $f(x)$, and let $Y = X^{\frac{1}{3}}$. Express the probability density function $g(y)$ for $Y$ in terms of the function $f(x)$.

**Problem 5.** *(20 points total; 5 points each)* Let $(X_1, X_2)$ be a pair of random variables uniformly distributed on the rectangular region $[0, 2] \times [0, 3]$, that is, the region $0 \le X_1 \le 2$ and $0 \le X_2 \le 3$. Let $f(x_1, x_2)$ be the joint probability density function for $(X_1, X_2)$.

Create a new pair of random variables $(Y_1, Y_2) = (X_1 X_2\,,\, X_2)$, meaning that $Y_1 = X_1 X_2$ and $Y_2 = X_2$. Let $g(y_1, y_2)$ denote the joint probability density function for $(Y_1, Y_2)$.

a. (5 points) Write down the explicit formula for the joint p.d.f. $f(x_1, x_2)$ for the *original* variables $(X_1, X_2)$ for every $(x_1, x_2)$ in $\mathbb{R}^2$.

b. (5 points) Describe explicitly the inequalities that define the subset of the $(Y_1, Y_2)$-plane where $g(y_1, y_2) > 0$, that is, where the new variables $(Y_1, Y_2)$ have positive density. Sketch this region reasonably accurately on a pair of labelled $(y_1, y_2)$ axes.

c. (5 points) Write down the explicit formula for the joint p.d.f. $g(y_1, y_2)$ of $(Y_1, Y_2)$ for every $(y_1, y_2)$ in $\mathbb{R}^2$.

d. (5 points) Find the probability density function $g_1(y_1)$ for the random variable $Y_1 = X_1 X_2$.

**Brief solutions:**

1.(a) We must have

$$1 = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (y + cxy)dydx = \int_{x=0}^{x=1} \frac{1}{2} 1 + cxydx = \frac{1}{2}\left(1 + \frac{c}{2}\right)$$

which forces $c = 2$.

(b)

$$\Pr(Y > X) = \int_{y=0}^{y=1} \int_{x=0}^{x=y} (y+2xy)dxdy = \int_{y=0}^{y=1} (y^2+y^3)dy = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

(c)

$$f_1(x) = \int_{y=0}^{y=1} (y + 2xy)dy = \begin{cases} \frac{1}{2}(1 + 2x) & \text{for } x \in [0, 1] \\ 0 & \text{for other } x. \end{cases}$$

$$f_2(y) = \int_{x=0}^{x=1} (y + 2xy)dx = \begin{cases} 2y & \text{for } y \in [0, 1] \\ 0 & \text{for other } y. \end{cases}$$

(d) Yes, $(X, Y)$ are independent, since their joint p.d.f.

$$f(x, y) = \begin{cases} y + 2xy & \text{for } x, y \in [0, 1] \times [0, 1] \\ 0 & \text{else.} \end{cases}$$

is the same, for every point $(x, y)$ in $\mathbb{R}^2$, as the product of the marginals:

$$f_1(x)f_2(y) = \begin{cases} \frac{1}{2}(1 + 2x) \cdot 2y = y + 2xy & \text{for } x, y \in [0, 1] \times [0, 1] \\ 0 & \text{else.} \end{cases}$$

(e) Due to independence, the conditional probability density functions $g_1(x|y)$ and $g_2(y|x)$ are the same as the marginal distributions, i.e. $f_1(x) = g_1(x|y)$ and $f_2(y) = g_2(y|x)$. (One could, of course, compute these via integrating out the $x$ or $y$ variable from the joint p.d.f. $f(x, y)$.

2.(a) The p.d.f. for a random variable $X$ uniformly distributed on $[1, 10]$ is

$$f(x) = \begin{cases} \frac{1}{10-1} = \frac{1}{9} & \text{for } x \in [1, 10] \\ 0 & \text{otherwise.} \end{cases}$$

(b) The d.f. is

$$F(x) = \int_{t=-\infty}^{t=x} f(t)dt = \begin{cases} 0 & \text{for } x \leq 1 \\ \frac{x-1}{9} & \text{for } x \in [1, 10] \\ 1 & \text{for } x \geq 10. \end{cases}$$

(c) If $Y = X^4$, then as $X$ ranges over $[1, 10]$, one has $Y$ ranging over $[1^4, 10^4] = [1, 10000]$. Therefore $g(y) = 0$ for $y$ not in $[1, 10000]$. Since $y = r(x) = x^4$ is increasing for $x$ in $[1, 10]$, with inverse $x = s(y) = y^{\frac{1}{4}}$, having

$$\frac{ds}{dy} = \frac{1}{4} y^{\frac{-3}{4}},$$

one can compute

$$g(y) = f(s(y)) \left| \frac{ds}{dy} \right| = \frac{1}{9} \cdot \left| \frac{1}{4} y^{\frac{-3}{4}} \right| = \frac{1}{36} y^{\frac{-3}{4}}.$$

3. Let $X$ be the number of sneezes observed in an hour, for whoever shows up to teach, and let $Y$ be the Poisson parameter (or mean number of sneezes expected per hour) for the teacher who shows up. We are given the marginal p.f. for $Y$ as

$$f_2(y = 1) = \frac{1}{2} = f_2(y = 20)$$

and the conditional p.d.f.'s for $X$ given $Y$ as

$$g_1(x|y) = e^{-y} \frac{y^k}{k!}.$$

(a) The problem is asking for the marginal value $f_1(x = 0)$, which is

$$\begin{aligned} f_1(x = 0) &= g_1(x = 0|y = 1) f_2(y = 1) + g_1(x = 0|y = 20) f_2(y = 20) \\ &= \left( e^{-1} \frac{1^0}{0!} \right) \frac{1}{2} + \left( e^{-20} \frac{20^0}{0!} \right) \frac{1}{2} \\ &= \frac{1}{2} (e^{-1} + e^{-20}). \end{aligned}$$

(b) The problem is asking for the conditional p.d.f. value

$$\begin{aligned} g_2(y = 20|x = 5) &= \frac{f(x = 5, y = 20)}{f_1(x = 5)} \\ &= \frac{g_1(x = 5|y = 20) f_2(y = 20)}{g_1(x = 5|y = 20) f_2(y = 20) + g_1(x = 5|y = 1) f_2(y = 1)} \\ &= \frac{e^{-20} \frac{20^5}{5!} \cdot \frac{1}{2}}{e^{-20} \frac{20^5}{5!} \cdot \frac{1}{2} + e^{-1} \frac{1^5}{5!} \cdot \frac{1}{2}} \\ &= \frac{e^{-20} 20^5}{e^{-20} 20^5 + e^{-1}} \end{aligned}$$

4. Since $y = r(x) = x^{\frac{1}{3}}$ is an increasing function for all $x$, with inverse $x = s(y) = y^3$ having derivative $\frac{ds}{dy} = 3y^2$, one concludes that for all $y$, the p.d.f. for $Y = X^{\frac{1}{3}}$ is

$$g(y) = f(s(y)) \left| \frac{ds}{dy} \right| = f(y^3)|3y^2| = 3y^2 f(y^3).$$

5. (a) Since the region $[0,2] \times [0,3]$ has area $2 \cdot 3 = 6$, the joint p.d.f. for the uniform distribution on this region is

$$f(x,y) = \begin{cases} \frac{1}{6} & \text{for } (x,y) \in [0,2] \times [0,3] \\ 0 & \text{otherwise.} \end{cases}$$

(b) The transformation $(Y_1, Y_2) = (X_1 X_2, X_2)$ has inverse

$$(X_1, X_2) = \left( \frac{Y_1}{Y_2}, Y_2 \right).$$

The inequalities $0 \leq X_1 \leq 2$ and $0 \leq X_2 \leq 3$ become $0 \leq \frac{Y_1}{Y_2} \leq 2$, or equivalently, $0 \leq Y_1 \leq 2Y_2$, and $0 \leq Y_2 \leq 3$. This bounds a triangular region of the $(Y_1, Y_2)$-plane having vertices $(0,0), (0,3), (6,3)$.
(c) The Jacobian determinant for the inverse transformation is

$$J = \det \begin{bmatrix} \frac{\partial s_1}{\partial y_1} & \frac{\partial s_1}{\partial y_2} \\ \frac{\partial s_2}{\partial y_1} & \frac{\partial s_2}{\partial y_2} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{y_2} & \frac{-y_1}{y_2} \\ 0 & 1 \end{bmatrix} = \frac{1}{y_2}$$

and therefore the joint p.d.f. for $(Y_1, Y_2)$ is

$$g(y_1, y_2) = f(s_1(y_1, y_2), s_2(y_1, y_2))|J| = \frac{1}{6} \cdot \left| \frac{1}{y_1} \right| = \frac{1}{6y_1}$$

for $(y_1, y_2)$ in the triangular region defined in (b), and $f(y_1, y_2) = 0$ for all other $(y_1, y_2)$.
(d) The marginal p.d.f. for $Y_1$ is

$$\begin{aligned} g_1(y_1) &= \int_{y_2=-\infty}^{y_2=\infty} g(y_1, y_2) dy_2 \\ &= \int_{y_2=\frac{y_1}{2}}^{y_2=3} \frac{1}{6y_2} dy_2 \\ &= \frac{1}{6} \left[ \log(3) - \log \frac{y_1}{2} \right] \\ &= \frac{1}{6} \log \frac{6}{y_1} \end{aligned}$$

for $0 \leq y_1 \leq 6$, and 0 for all other $y_1$.