Math 5705 Undergraduate enumerative combinatorics
Spring 2005, Vic Reiner
Midterm exam 2- Due Wednesday March 2, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Explain your reasoning- answers without justification or proof (where appropriate) will receive **no credit**.

1. Recall the Ramsey number $R(m, n)$ is the smallest positive integer N having the following property: in any room of N people, there exists either m (pairwise) mutual acquaintances or n (pairwise) mutual strangers.

(a) (5 points) Give a simple explicit formula for $R(m, 2)$ and for $R(2, n)$.

(b) (5 points) Prove that $R(m, n) = R(n, m)$.

(c) (8 points) Problem 69 on page 29. (Hint: Problem 103).

(d) (8 points) Prove or disprove: $R(m, n) = \binom{m+n-2}{m-1}$.

(e) (8 points) Assume that there really is a solution to Problem 70 on page 29 (and there is: for example, arrange the vertices of K_8 in a circle, then color the edges between any pair of vertices that are 1 or 2 steps away around the circle red, and color all other edges green).

Now do problem 71, that is, find $R(4, 3)$.

2. Let r_n be the maximum number of regions into which the plane can be divided by n straight lines (assuming that the lines really stretch infinitely far in both directions). For example,

$$r_0 = 1, r_1 = 2, r_2 = 4, r_3 = 7.$$

(a) (10 points) Find an initial condition and recurrence for r_n that define the sequence $r_0, r_1, r_2, \ldots$ uniquely.

(b) (10 points) Solve this recurrence to find an explicit formula for r_n as a function of n (involving no summations, products, etc.)

3. (20 points) Problem 123 on page 59.

4. (a) (5 points) How many trees are there are on vertex set $[100] := \{1, 2, \ldots, 100\}$?

(b) (5 points) How many of the trees in part (a) have vertices $1, 2, \ldots, 50$ each of degree 1 and vertices $51, 52, 53, \ldots, 100$ each of degree 3?

(c) (6 points) How many of the trees in part (a) have vertices $1, 2, \ldots, 50, 51$ each of degree 1 and vertices $52, 53, \ldots, 100$ each of degree 3?

Define a polynomial f_n in the variables $x_1, x_2, \ldots, x_n$ as follows:

$$f_n := \sum_T x_1^{\deg_T(1)} x_2^{\deg_T(2)} \cdots x_n^{\deg_T(n)}$$

where the sum ranges over all trees T on vertex set $[n] := \{1, 2, \ldots, n\}$, and where $\deg_T(i)$ denotes the degree of vertex i in the tree T.

For example,

$$f_3 = x_1^2 x_2^1 x_3^1 + x_1^1 x_2^2 x_3^1 + x_1^1 x_2^1 x_3^2 = x_1 x_2 x_3 (x_1 + x_2 + x_3).$$

(d) (10 points) Prove the following formula for f_n:

$$f_n = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$