Math 5705 Undergraduate enumerative combinatorics
Fall 2002, Vic Reiner
Midterm exam 1- Due Friday September 20, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Supplementary problem 9 for Chapter 1 on page 29.

2. A *composition* of a number n into k *parts* is an expression

$$n = n_1 + n_2 + \cdots n_k$$

where the n_i are positive integers, and in which the *order matters*, i.e. $(n_1, \ldots, n_k)$ is considered as an ordered sequence. For example, there are 8 compositions of 4:

$$\begin{aligned} 4 =& 4 \\ \\ =& 3+1 \\ =& 1+3 \\ =& 2+2 \\ \\ =& 2+1+1 \\ =& 1+2+1 \\ =& 1+1+2 \\ \\ =& 1+1+1+1 \end{aligned}$$

(a) (10 points) Supplementary problem 2 for Chapter 1 on page 28.
(b) (10 points) Supplementary problem 1 for Chapter 1 on page 28.

3. (a) (10 points) Supplementary problem 3 for Chapter 1 on page 28.
(b) (10 points) Supplementary problem 4 for Chapter 1 on page 28.
In your solution to part (b), feel free to assume that a Gray code exists for every n, even if you weren't able to prove it in part (a). Also feel free to instead provide a bijection in part (b) that has nothing to do with Gray codes.

4. Let $n \geq m$ be positive integers. Suppose $m+n$ people cast votes sequentially in a two candidate election, where in the finally tally, Candidate 1 received n votes, Candidate 2 received m votes, and throughout the voting process, Candidate 2 was *never* ahead of Candidate 1. We wish to count the number of possible such voting sequences; call this number $C_{n,m}$. For example, if $n = 5, m = 4$, then one such sequence is

$$(1, 2, 1, 1, 2, 2, 1, 2, 1).$$

(a) (5 points) Describe explicitly a bijection between such voting sequences and the lattice paths taking unit steps north or east from $(0,0)$ to (n,m) which lie weakly below the line $y = x$.
(b) (10 points) Using ideas from the solution of Problem 52 on page 21, find a formula for $C_{n,m}$.
(c) (5 points) Express your formula for $C_{m,n}$ using no additions or subtractions, i.e. only with multiplications and divisions (e.g. allowing factorials, binomial coefficients, etc.)

5. Prove by any means:
(a) (10 points)

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(Hints: if you have no other ideas, here are some you might try. In a lattice path from $(0,0)$ to (n,n), where might you cross the line $y = n-x$? In choosing a committee of n people from a pool of n women and n men, how many women might end up on the committee?)
(b) (10 points)

$$\sum_{m=0}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$