Math 5707 Graph theory
Spring 2013, Vic Reiner
Final exam- Due Wednesday May 8, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult. Clearly indicate on the exam any outside sources consulted, and make sure to understand and process the solution sufficiently to *explain it in your own words.* Solutions which the instructor views as insignificant alterations of an outside source will receive no credit.

1. (15 points) Given G a graph with minimum degree $\delta(G) \geq 2$, show there exists a *connected* graph H with the same degrees: $d_H(x) = d_G(x)$ for all x in $V = V(H) = V(G)$.

2. (15 points total) Given a simple graph $G = (V, E)$, recall that its complement $\bar{G} = (V, \bar{E})$ has the same vertex set V, but the complementary set of edges

$$\bar{E} := \{\{x, y\} \subset V : \{x, y\} \notin E\}.$$

Prove the following inequalities involving the chromatic numbers $\chi(G)$ and $\chi(\bar{G})$.

(a) (2 points) Show $\chi(\bar{G}) \geq \alpha(G)$, where $\alpha(G)$ is the size of the largest independent/stable set of vertices in G.

(b) (3 points) Show $\chi(G) \cdot \chi(\bar{G}) \geq n := |V|$.

(c) (10 points) Show $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{n}$.
(Hint: even if you didn't prove (b), you can assume it in part (c).)

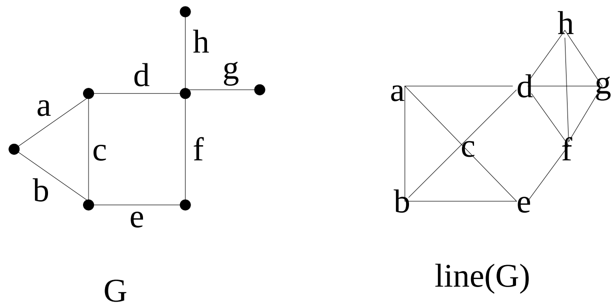
3. (15 points) Prove that for a connected graph G with n vertices, one always $a_i \geq \binom{n-1}{i}$ in the chromatic polynomial expansion

$$\chi(G, k) = \sum_{i=0}^{n-1} (-1)^i a_i k^{n-i}.$$

4. (20 points total) Given a simple graph $G = (V, E)$, recall that its *line graph*

$$\mathrm{line}(G) = (V_{\mathrm{line}(G)}, E_{\mathrm{line}(G)})$$

has vertex set $V_{\mathrm{line}(G)} = E$, the edge set of G, and has an edge $\{e, e'\}$ in $E_{\mathrm{line}(G)}$ whenever e, e' were incident at some vertex x in V of G. An example is illustrated below.



(a) (5 points) Show that an edge $e = \{x, y\}$ in E of G gives rise to a vertex of line(G) having $\deg_{\mathrm{line}(G)}(e) = d_G(x) + d_G(y) - 2$. Explain why this implies both that

(i) a d-regular graph G has line(G) being $2(d-1)$-regular, and
(ii) a bipartite graph $G = (X \sqcup Y, E)$ which is (d_X, d_Y)-*regular*, in the sense that $d_G(x) = d_X$ and $d_G(y) = d_Y$ for all $x \in X, y \in Y$, has line(G) being $(d_X + d_Y - 2)$-regular.

(b) (15 points) Prove that a connected simple graph G has line(G) a k-regular graph for some $k \geq 0$ if and only if either

(i) G is d-regular, with $2d = k + 2$, or
(ii) G is bipartite and (d_X, d_Y)-regular, with d_X, d_y positive integers whose sum is $k + 2$.

5. (20 points) Given a graph $G = (V, E)$ and an edge *strength* function $s : E \to R$, define the *reliability* $r_G(u, v)$ between two vertices u, v in V to be the maximum over all paths P from u to v of the function $\min\{s(e) : e \text{ an edge on } P\}$. In other words,

$$r_G(u, v) = \max_{u,v-\text{paths } P} \min_{e \in P}\{s(e)\}.$$

Show that any spanning tree T in G that *maximizes* $s(T) := \sum_{e \in T} s(e)$ allows one to quickly calculate simultaneously all of these reliability values, by proving that for such a tree T one has

$$r_G(u, v) = r_T(u, v) \text{ for all } u, v \text{ in } V.$$

6. (20 points total) Let G be a simple planar connected graph G with $n \geq 3$ vertices.

(a) (10 points) Show the degree sequence $d(G) = (d_1, \ldots, d_n)$ satisfies

$$\sum_{\substack{i=1,2,\ldots,n \\ \text{with } d_i \leq 6}} (6 - d_i) \geq \sum_{i=1}^{n} (6 - d_i) \geq 12.$$

(b) (10 points) Explain why this implies that ...
(i) if the min degree $\delta(G) \geq 5$ then G has at least 12 degree 5 vertices,
(ii) and if $\delta(G) \geq 4$ then G has at least 6 vertices of degree 4 or 5.