Math 5707 Graph theory
Spring 2023, Vic Reiner
Midterm exam 2- Due Wednesday, April 12 by midnight at the class Canvas site

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1.(20 points total) **True** or **False**. To receive full credit, assertions which are true must be proven, and those that are false must have a counterexample exhibited.

(a) (5 points) Given a graph G with connected components $G_1, \ldots, G_r$, its chromatic number $\chi(G)$ satisfies

$$\chi(G) = \max\{\chi(G_1), \ldots, \chi(G_r)\}.$$

(b) (5 points) Recall (from lecture or Bondy & Murty's Section 3.2) that for every graph G, one can define 2-vertex-connected subgraphs $B_1, \ldots, B_s$ called its *blocks* or *2-connected components*, whose union is G. Then the chromatic number $\chi(G)$ satisfies

$$\chi(G) = \max\{\chi(B_1), \ldots, \chi(B_s)\}.$$

(c) (5 points) Given a network digraph $D = (V, A)$ with distinguished vertices s, t in V and edge capacities $c : A \to \mathbb{R}_{\geq 0}$, there is always a **unique** $s-t$ flow $f : A \to \mathbb{R}_{\geq 0}$ obeying c that achieves the maximum value $\mathrm{val}(f)$. That is, if both f_1, f_2 obey c and achieve the maximum flow value $\mathrm{val}(f_1) = \mathrm{val}(f_2)$, then $f_1(a) = f_2(a)$ for all a in A.

(d) (5 points) Given the same notation as in part (c), there is always a **unique** $s-t$ cut $(S, \bar{S})$ (where $\bar{S} = V - S$) that achieves the minimum capacity $c(S, \bar{S}) := \sum_{a \text{ from } S \text{ to } \bar{S}} c(a)$. That is, if both S_1, S_2 contain s but not t and achieve the minimum capacity $c(S_1, \bar{S}_1) = c(S_2, \bar{S}_2)$, then $S_1 = S_2$.

2.(20 points) Show that for any simple graph $G = (V, E)$, one has

$$|E| \geq \binom{\chi(G)}{2} := \frac{\chi(G)(\chi(G) - 1)}{2}.$$

(Hint: you might try to exhibit an algorithm which, if $|E| < \binom{k}{2}$, will produce a proper vertex-coloring of G with $k - 1$ colors.)

3. (20 points) Given a simple graph $G = (V, E)$, a subset $V' \subseteq V$ is called an *independent* or *stable* set of vertices if there are no edges $\{x, y\}$ in E with $x, y \in V'$. Let

$$\alpha(G) := \max\{|V'| : V' \subseteq V \text{ is an independent subset}\}.$$

Prove the following inequalities. When proving any of the later parts, you may assume the inequalities from any of the previous parts, whether or not you were able to prove them.

(a) (5 points) Show the complement graph $\bar{G}$ to G has $\chi(\bar{G}) \geq \alpha(G)$.

(b) (5 points) Show that $\chi(G) \cdot \alpha(G) \geq |V|$
(Hint: Use a proper vertex-coloring of G to decompose its vertex set $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_{\chi(G)}$ into its color-classes. How does this help?)

(c) (5 points) Show that $\chi(G) \cdot \chi(\bar{G}) \geq |V|$

(d) (5 points) Show that $\chi(G) + \chi(\bar{G}) \geq 2\sqrt{|V|}$
(Hint: arithmetic versus geometric means are relevant here.))

4. (20 points total) Let $G = (X \sqcup Y, E)$ be a bipartite graph, and assume that there are integers $d_X, d_Y \geq 1$ such that every x in X has the same degree $d_G(x) = d_X$ and every y in Y has the same degree $d_G(y) = d_Y$.

(a) (10 points) Prove that $d_X/d_Y = |Y|/|X|$.

(b) (10 points) Prove that if $d_X \geq d_Y$ then there exists a matching $M \subseteq E$ that matches every x in X.

5. (20 points) Let $G = (X \sqcup Y, E)$ be a bipartite graph which is d-regular (all vertices have degree d) with $d \geq 2$. If G is connected, prove that G has no cut-edge, that is, $G - \{e\}$ is also connected for every e in E.