Math 5711 Comb. optimization, Spring 2004 Vic Reiner
Non-Schrijver Homework 5 problems
Due Wednesday April 21, in class

1. Recall Hall's Theorem says a bipartite graph $G = (U \sqcup W, E)$ has a matching that matches all of U if and only if every subset U' of U has $|N(U')| \geq |U|$. Call the latter hypothesis *Hall's condition*.

Recall Tutte's 1-factor Theorem says a graph $G = (V, E)$ has a perfect matching if and only if every subset $V' \subset V$ has the property that the number of odd-sized components remaining after deleting V' from G is at most $|V'|$. Call the latter hypothesis *Tutte's condition*.

The goal of this exercise is to deduce Hall's Theorem from Tutte's. Given a bipartite $G = (U \sqcup W, E)$, create a new graph H by first adding one more vertex to W if $|U| + |W|$ is odd, and then adding in all edges between vertices w, w' in W, so that W now looks like a complete subgraph of G.

(a) Show that G has a matching of size $|U|$ if and only if H has a perfect matching.

(b) Show that G satisfies Hall's condition if and only if H satisfies Tutte's condition.

(c) Explain how to deduce Hall's Theorem from Tutte's Theorem.

2. Write down the stable matching produced by the Gale-Shapley algorithm for the following list of medical students and residency programs.

Students		*Residencies*	
$A:$	y, x, z, w	$w:$	A, B, C, D
$B:$	x, w, y, z	$x:$	C, A, D, B
$C:$	x, z, w, y	$y:$	C, B, D, A
$D:$	y, w, z, x	$z:$	B, A, C, D

3. Use the Gale-Shapley algorithm (and explain how you did it) to decide whether the following list of medical students and residency programs with incomplete lists of preferences have a stable matching.

Students		*Residencies*	
$A:$	x	$x:$	C, A, B
$B:$	z, x, y	$y:$	B, A, C
$C:$	z, x	$z:$	A, B, C