Math 5711 Combinatorial optimization
Spring 2006, Vic Reiner
Midterm exam 1- Due Wednesday April 5, in class

Instructions: This is an open book, open library, open notes, open web, take-home exam, but you are *not* allowed to collaborate. The instructor is the only human source you are allowed to consult.

1. (20 points) Schrijver's Problem 2.25 on p. 37. He asserts the equality (formula (64)) of a max and a min, but forgot to ask you to prove it. Prove it.

2. (20 points) Schrijver Problem 1.3 on p. 18. Draw the associated directed graph, along with the last Bellman-Ford function on its vertices, and a tree rooted at vertex A consisting of shortest (A, v)-paths for all other vertices v.

3. (20 points) In Schrijver's knapsack problem example (Application 1.3), assume that object 2 is no longer available, so one has

article	volume	value
1	5	4
3	2	3
4	2	5
5	1	4

Solve this new knapsack problem by drawing an appropriate directed graph and using the Bellman-Ford algorithm.

4. (20 points) Give an example of a directed graph $D = (V, A)$ and a length function on the arcs $\ell : A \to \mathbb{Z}$ with these properties:

- $|V| = 3$, i.e. there are exactly 3 vertices, labelled $V = \{s, r, t\}$,
- there are no directed cycles *at all* (and therefore none of negative length), and
- Dijkstra's algorithm fails to find an $s - t$ directed path of minimum length, but the Bellman-Ford algorithm works. (Write down the output from both algorithms)

5. Let $G = (V, E)$ be the graph which is a four-vertex cycle on vertices $V = a, b, c, d$ with edges and lengths as indicated below:

$$\begin{array}{llcccc} E & = \{ & ab, & bc, & cd, & ad \quad \} \\ \text{length} & & 1 & 3 & 3 & 2 \end{array}$$

(a) (5 points) List a sequence of forests found during the Dijkstra-Prim method for computing a minimum spanning tree in G.
(b) (5 points) List a sequence of forests found during Kruskal's method for computing a minimum spanning tree in G.
(c) (3 points) Write down *all* of the minimum spanning trees in G.
(d) (7 points) Prove that for any connected graph G, if all of the edge lengths $\ell(e)$ are *distinct* real numbers then the minimum length spanning tree for G is *unique*.

(One possible hint for (d): Suppose

$$T = \{e_1, e_2, \ldots, e_{n-1}\}$$
$$T' = \{e'_1, e'_2, \ldots, e'_{n-1}\}$$

were two minimum length spanning trees in G, with their edges indexed in increasing order of length, i.e.

$$\ell(e_1) < \ell(e_2) < \ldots < \ell(e_{n-1})$$
$$\ell(e'_1) < \ell(e'_2) < \ldots < \ell(e'_{n-1}),$$

and suppose that $e_1 = e'_1, e_2 = e'_2, \ldots, e_k = e'_k$, but $\ell(e_{k+1}) < \ell(e'_{k+1})$. Can you reach a contradiction from this?)