Math 8201 Graduate abstract algebra- Fall 2013, Vic Reiner
Final exam - Due Wednesday December 11, in class

Instructions: This is an open book, library, web, notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1 (15 points) Let G be a finite group, and let n be an integer with $\gcd(|G|, n) = 1$. Show that for every g in G there exists a unique n^{th} *root* $x = \sqrt[n]{g}$ in G, that is, a unique x with $x^n = g$.
(Hint: You might try reasoning about the set map $x \mapsto x^n$ on G.)

2. (15 points total) Let (a, b, c) lie in $\{1, 2, \ldots\}$, with b *dividing* ac.

(a)(5 points) Prove that the map

$$\begin{array}{rcl} \mathbb{Z}/a\mathbb{Z} & \stackrel{\varphi}{\longrightarrow} & \mathbb{Z}/b\mathbb{Z} \\ x \bmod a & \longmapsto & cx \bmod b \end{array}$$

is well-defined and a group homomorphism.

(b)(5 points) Under what conditions on (a, b, c) is φ surjective?

(c)(5 points) Under what conditions on (a, b, c) is φ injective?

3. (10 points) Exhibit a group G along with three normal subgroups $N_1, N_2, N_3 \lhd G$ such that $N_1 \cap N_2 \cap N_3 = 1$ and $G = N_1 N_2 N_3$, but

$$G \not\cong N_1 \times N_2 \times N_3.$$

(Hint: It can be done with G finite, abelian.)

4. (15 points) Show that any family $\{\varphi_i\}_{i \in I}$ of linear operators $V \xrightarrow{\varphi_i} V$ on a finite-dimensional $\mathbb{C}$-vector space V that *pairwise commute*

$$\varphi_i \varphi_j = \varphi_j \varphi_i \quad \text{for all } i, j \in I,$$

can be simultaneously triangularized. That is, show that there exists a single basis $(v_1, \ldots, v_n)$ for V in which the matrices that represent the $\{\varphi_i\}_{i \in I}$ are *simultaneously* all upper triangular.

5. (15 points) Let A in $\mathbb{C}^{n\times n}$ be a diagonalizable matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ in $\mathbb{C}$. Prove that the following matrix B in $\mathbb{C}^{3n\times 3n}$

$$B = \begin{bmatrix} 0 & 0 & A \\ A & 0 & 0 \\ 0 & A & 0 \end{bmatrix}$$

is also diagonalizable, and write down its list of $3n$ eigenvalues in $\mathbb{C}$.

6. (15 points) Consider finite-dimensional $\mathbb{F}$-vector spaces

$$V \text{ with ordered basis } (v_1, \ldots, v_n),$$
$$W \text{ with ordered basis } (w_1, \ldots, w_m).$$

Recall that every t in $V \otimes W$ can be written uniquely as

$$t = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j}\ v_i \otimes w_j, \tag{1}$$

and that t is called *decomposable* if $t = v \otimes w$ for some $v \in V, w \in W$. Show that t is decomposable if and only if the matrix of coefficients $A = (a_{ij})$ in $\mathbb{F}^{n\times m}$ that appear in (1) has $\text{rank}(A)$ at most one.

7. (15 points) Given a linear operator $V \xrightarrow{\varphi} V$, where $\dim_{\mathbb{F}} V = n$, let c_k be the coefficient of t^k in its characteristic polynomial:

$$\det(t \cdot 1_V - \varphi) = c_0 + c_1 t^1 + c_2 t^2 + \cdots + c_{n-1} t^{n-1} + c_n t^n.$$

(a)(10 points) Prove that

$$\begin{aligned} c_n &= 1, \\ c_{n-1} &= -\mathrm{Tr}(\varphi), \\ c_0 &= (-1)^n \det(\varphi). \end{aligned}$$

(b)(5 points) Prove more generally that

$$c_{n-k} = (-1)^k \mathrm{Tr}(\wedge^k \varphi)$$

where recall that $\wedge^k \varphi$ is defined by

$$\begin{array}{ccc} \wedge^k V & \xrightarrow{\wedge^k \varphi} & \wedge^k V \\ v_1 \wedge \cdots \wedge v_k & \longmapsto & \varphi(v_1) \wedge \cdots \wedge \varphi(v_k). \end{array}$$

(If you do part (b) correctly, no need to do part (a) separately.)