Math 8202 Graduate abstract algebra
Spring 2011, Vic Reiner
Final exam- Due Friday May 6, in class

Instructions: This is an open book, open library, open notes, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult.

1. (15 points) Dummit and Foote §9.5, Problem 7, page 315.

2. (15 points total; 5 points each)
(a) Consider the factorization of $f(x) = x^{17^2} - x$ into irreducible polynomials in $\mathbb{F}_{17}[x]$. For each possible degree $d = 1, 2, 3, \ldots$, how many irreducible factors will there be of degree d?
(b) Same question for $g(x) = x^{17^4} - x$.
(c) Same question for $h(x) = x^{\frac{17^2-1}{2}} + 1$ (which divides $f(x)$).

3. (10 points) Let $\mathbb{K} = \mathbb{F}(a, b, c, d)$ be the field of rational functions in indeterminates a, b, c, d with coefficients in some field $\mathbb{F}$. Show that the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ is irreducible in $\mathbb{K}[x]$.

4. (10 points) Dummit and Foote §14.7, Problem 3, page 636. Assume that the necessary and sufficient condition for $\mathbb{F}(\sqrt{\alpha}) = \mathbb{F}(\sqrt{\beta})$ they want you to prove is this: the product $\alpha\beta$ should be a perfect square in $\mathbb{F}$. Don't forget to answer their second question.

5.(15 points total; 5 points each)
On that same page 636 of Dummit and Foote §14.7, do
(a) Problem 4.
(b) Problem 5.
(c) Problem 6.

6. (10 points) How many intermediate fields lie strictly between $\mathbb{Q}$ and and $\mathbb{Q}(\zeta_{19})$ where ζ_{19} is a primitive 19^{th} root of unity? You need not describe them all explicitly, but you must explain your answer.

7. (25 points total; 5 points each)
Let V be an n-dimensional $\mathbb{F}$-vector space where $\mathbb{F}$ is an algebraically closed field, and let $T : V \longrightarrow V$ be an $\mathbb{F}$-linear operator. For each λ in $\mathbb{F}$, define the *generalized λ-eigenspace of T* by

$$V_{T,\lambda} := \{v \in V : \text{ there exists } m > 0 \text{ with } (T - \lambda \cdot 1_V)^m v = 0\}.$$

(a) Show the inclusions

$$\ker(T - \lambda \cdot 1_V) \subseteq V_{T,\lambda} \subseteq \ker(T - \lambda \cdot 1_V)^n.$$

(b) Show that $V_{T,\lambda}$ is T-stable, meaning that $T(V_{T,\lambda}) \subseteq V_{T,\lambda}$.

(c) Show that one has a direct sum of $\mathbb{F}$-vector spaces $V = \bigoplus_{\lambda \in \mathbb{F}} V_{T,\lambda}$. (Hint: how does this relate to Jordan canonical form?)

(d) Let V_1, V_2 be $\mathbb{F}$-vector spaces of dimensions n_1, n_2, respectively, and let

$$V_1 \xrightarrow{T_1} V_2$$
$$V_2 \xrightarrow{T_2} V_1$$

be $\mathbb{F}$-linear maps between them, so that one can form their composite maps in either order:

$$V_1 \xrightarrow{T_2 \circ T_1} V_1$$
$$V_2 \xrightarrow{T_1 \circ T_2} V_2$$

Show that for each *nonzero* λ in $\mathbb{F}^\times$, the maps T_1 and T_2 restrict to give *isomorphisms*

$$(V_1)_{T_2 \circ T_1, \lambda} \xrightarrow{T_1} (V_2)_{T_1 \circ T_2, \lambda}$$
$$(V_2)_{T_1 \circ T_2, \lambda} \xrightarrow{T_2} (V_1)_{T_2 \circ T_1, \lambda}.$$

(e) Deduce that the characteristic polynomials of the two composite maps $T_2 \circ T_1$ and $T_1 \circ T_2$ differ only by powers of t; specifically, show

$$t^{n_1} \det(t \cdot 1_{V_2} - T_1 \circ T_2) = t^{n_2} \det(t \cdot 1_{V_1} - T_2 \circ T_1).$$