Math 8202 Graduate abstract algebra- Spring 2014, Vic Reiner
Midterm exam 1- Due Wednesday February 26, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (15 points total; 5 points each)
(a) For each $n = 1, 2, 3, \ldots$, show how to express $2^n = x^2 + y^2$ with integers x, y, that is, find explicit formulas for x, y as functions of n that work.

(b) Prove or disprove: For each $n \geq 1$ one can express $10^n = x^2 + y^2$ with integers x, y.

(c) Prove that if an integer n can be expressed as $n = x^2 + y^2$ with x, y in $\mathbb{Q}$, then it can also be expressed as $n = x^2 + y^2$ with x, y in $\mathbb{Z}$.

2. (25 points total; 5 points each)
Prove or disprove the following statements about a commutative ring R with 1, always interpreting "subring" to mean "subring with 1":

(a) If R is a UFD, then any subring of R is a UFD.

(b) If R is a Euclidean domain, then any subring of R is a Euclidean domain.

(c) If I is an ideal in R and R/I is a domain, then R is also a domain.

(d) If R is a UFD and I is a prime ideal of R, then R/I is a UFD.

(e) If R is a Noetherian ring, then any subring of R is a Noetherian ring.

3. (15 points) Let φ be the ring homomorphism $\mathbb{Z}[x, y] \xrightarrow{\varphi} \mathbb{Z}[t]$ that sends $f(x, y) \longmapsto f(t^4, t^6)$. Find, with proof, a generating set of minimum cardinality for the ideal $I = \ker \varphi$ in $\mathbb{Z}[x, y]$.

4. (10 points; 5 points each part) Let $R = \mathbb{Z}[x]/(x^3 + x + 1, 5)$.

(a) What is the cardinality of R?

(b) Is R a domain? Is R a field?

5. (15 points total; 5 points each)
(a) Prove $x^2 + y^2 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

(b) Prove $x^n - p$ is irreducible in $\mathbb{Z}[i][x]$ for all positive integers n and all odd primes p in $\mathbb{Z}$.

(c) Prove $x^8 + y^4 + z^6$ is reducible in $\mathbb{F}_p[x, y, z]$ when $p = 2$, but irreducible in $\mathbb{F}_p[x, y, z]$ for all odd primes p.

6. (20 points total; 5 points each)
Let $\mathbb{F}$ be a field. Then a polynomial $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ is *homogeneous of degree* d if every monomial $x_1^{a_1} \cdots x_n^{a_n}$ occurring in f with nonzero coefficient has the same degree $d = \sum_i a_i$. By segregating monomials according to their degree, one can express any polynomial f uniquely as $f = f_0 + f_1 + \cdots + f_d$ with f_i homogeneous of degree i. The f_i are called the *homogeneous components of* f.

(a) Prove that for $f(\mathbf{x})$ homogeneous of degree d, and λ in $\mathbb{F}$, one has $f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x})$.

(b) Prove that for $f(\mathbf{x})$ homogeneous of degree d, one has $\displaystyle\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f$.

(c) An ideal I in $\mathbb{F}[x_1, \ldots, x_n]$ is said to be *homogeneous* if every homogeneous component f_i of any f in I also lies in I. Show that I is a homogeneous ideal if and only if it can be generated by a collection of homogeneous polynomials.

(d) If $n = 2$ and $\mathbb{F}$ is algebraically closed (e.g. $\mathbb{F} = \mathbb{C}$), show that every homogeneous polynomial f in $\mathbb{F}[x, y]$ in two variables can be factored as a product of linear (degree 1) polynomials, that is, $f(x, y) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y)$ for some α_i, β_i in $\mathbb{F}$.