Math 8202 Graduate abstract algebra- Spring 2020, Vic Reiner
Final exam- Due Wednesday May 13, sent in PDF via email

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (40 points total; 5 points each)
True or false; prove or disprove.

(a) Let $\mathbb{K}/\mathbb{Q}$ be a degree 4 Galois extension with $\mathrm{Aut}(\mathbb{K}/\mathbb{Q}) = V_4$, the Klein-four group inside S_4. Then there is at least one *irreducible* degree 4 polynomial $f(x)$ in $\mathbb{Q}[x]$ for which $\mathbb{K}$ is the splitting field over $\mathbb{Q}$.

(b) Let $\mathbb{K}/\mathbb{Q}$ be a degree 4 Galois extension with $\mathrm{Aut}(\mathbb{K}/\mathbb{Q}) = V_4$, the Klein-four group inside S_4. Then there is at least one *reducible* degree 4 polynomial $g(x)$ in $\mathbb{Q}[x]$ for which $\mathbb{K}$ is the splitting field over $\mathbb{Q}$.

(c) Let R be a PID, and $A = (a_{ij})$ in $R^{m\times n}$ with Smith normal form matrix S having nonzero diagonal entries $d_1, d_2, \ldots, d_r$, with d_i dividing d_{i+1} for $i = 1, 2, \ldots, r-1$. Then

$$d_1 = \gcd\left(\{a_{ij}\}_{\substack{i=1,2,\ldots,m \\ j=1,2,\ldots,n}} \right).$$

(d) The splitting field $\mathbb{K}$ for $x^3 - 13x + 13$ over $\mathbb{Q}$ has $\mathrm{Aut}(\mathbb{K}/\mathbb{Q})$ isomorphic to the symmetric group S_3.

(e) There are no solutions to $a^2 + b^2 + c^2 + d^2 = -1$ with $(a, b, c, d) \in \mathbb{K}^4$ if $\mathbb{K} := \mathbb{Q}(\omega\sqrt[3]{2})$ with $\omega := e^{\frac{2\pi i}{3}}$

(f) One has an abelian group isomorphism

$$\mathbb{Z}/14\mathbb{Z} \oplus \mathbb{Z}/33\mathbb{Z} \oplus \mathbb{Z}/65\mathbb{Z} \cong \mathbb{Z}/30\mathbb{Z} \oplus \mathbb{Z}/1001\mathbb{Z}$$

(g) Matrices A in $\mathbb{F}_5^{n\times n}$ satisfying $A^5 = A$ are diagonalizable over $\mathbb{F}_5$.

(h) Matrices A in $\mathbb{F}_2^{n\times n}$ satisfying $A^5 = A$ are diagonalizable over $\mathbb{F}_2$.

2. (20 points total; 10 points each part).
(a) Regard $V = \mathbb{R}^2$ as an $\mathbb{R}[x]$-module in which x acts on V via the matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

How many $\mathbb{R}[x]$-submodules $W \subseteq V$ are there, counting both $W = \{0\}$ and $W = V$ among them?

(b) Regard $V = \mathbb{C}^n$ as a $\mathbb{C}[x]$-module in which x acts on V via a matrix A in $\mathbb{C}^{n \times n}$.
Assuming that $\det(xI - A) = x^n - 1$, how many $\mathbb{C}[x]$-submodules $W \subseteq V$ are there?
(Your answer should be a function of n.)

3. (20 points) How many different similarity classes of matrices in $\mathbb{Q}^{14 \times 14}$ have (simultaneously)

- their *minimal polynomial* equal to $x^5(x+1)^3(x+2)^2$, and
- their *characteristic polynomial* equal to $x^8(x+1)^4(x+2)^2$?

4. (20 points total)

Regard this matrix A in $\mathbb{Z}^{3 \times 3}$,

$$A = \begin{bmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{bmatrix},$$

as defining a $\mathbb{Z}$-module map $\mathbb{Z}^3 \xrightarrow{A} \mathbb{Z}^3$ via $x \mapsto Ax$.

(a) (8 points) Express the $\mathbb{Z}$-module $\operatorname{coker}(A)$ in invariant factor form.

(b) (6 points) Express the $\mathbb{Z}$-module $\ker(A)$ in invariant factor form.

(c) (6 points) Express the $\mathbb{Z}$-module $\operatorname{im}(A)$ in invariant factor form.