Math 8202 Graduate abstract algebra- Spring 2020, Vic Reiner
Midterm exam 1- Due Wednesday February 26, in class

Instructions: This is an open book, library, notes, web, take-home exam, but you are *not* to collaborate. The instructor is the only human source you are allowed to consult. Indicate outside sources used.

1. (25 points total; 5 points each)
True or false; prove or disprove.

(a) If I is an ideal in a commutative ring R with 1, and R/I is a domain, then R is also a domain.

(b) For n in $\mathbb{Z}$, there exists an expression $n = a^2 + b^2$ with a, b in $\mathbb{Z}$ if and only if there exists such an expression with a, b in $\mathbb{Q}$.

(c) For any field $\mathbb{F}$, the ring of formal power series $\mathbb{F}[[x]]$ has only one prime ideal I with $(0) \subsetneq I \subsetneq (1)$.

(d) For any field $\mathbb{F}$, the ring of formal power series $\mathbb{F}[[x]]$ has only one radical ideal I with $(0) \subsetneq I \subsetneq (1)$.

(e) Given two ideals $I, J \subset \mathbb{F}[x_1, \ldots, x_n]$ for a field $\mathbb{F}$, consider the following ideal K inside the larger ring $\mathbb{F}[x_1, \ldots, x_n, t]$ having an extra variable t:

$$K := (tI) + ((1-t)J) := (t \cdot f(\mathbf{x}) + (1-t)g(\mathbf{x}) : f \in I, g \in J)$$

Then the intersection $K \cap \mathbb{F}[x_1, \ldots, x_n]$ equals $I \cap J$.

2. (15 points) Give a simple characterization of the integer primes p for which the ideal $I = (p, x^3+1, x^2+1)$ in the ring $\mathbb{Z}[x]$ forms a maximal ideal.

3. (20 points total) For integers $a, b, c \geq 1$, consider the quotient ring

$$R := \mathbb{F}_3[x]/\left(\ (x+2)^b(x^2+x+2)^c(x^3+1)^a\ \right)$$

(a) (3 points) Is R a domain?

(b) (7 points) Show that R is finite, and compute (with proof) its cardinality as a function of a, b, c.

(c) (10 points) How many ideals I does R contain, including the ideals $I = (0)$ and $I = (1) = R$? Your answer should again be a function of a, b, c, and must be proven.

4. (25 points total)
(a) (5 points) Prove $x^3 + y^3 - 1$ is irreducible in $\mathbb{Q}[x, y]$.

(b) (10 points) Prove $x^n - p$ is irreducible in $\mathbb{Z}[i][x]$ for all positive integers n and all odd primes p in $\mathbb{Z}$.

(c) (10 points) Prove $x^{18} + y^{30} + z^{45}$ is reducible in $\mathbb{F}_3[x, y, z]$, and irreducible in $\mathbb{F}_p[x, y, z]$ for primes $p \neq 3$.

5. (15 points total; 5 points each)
Let $\mathbb{F}$ be a field. A polynomial $f(\mathbf{x}) \in \mathbb{F}[x_1, \ldots, x_n]$ is called *homogeneous of degree* d if every monomial $x_1^{a_1} \cdots x_n^{a_n}$ occurring in f with nonzero coefficient has the same degree $d = \sum_i a_i$.

(a) Given λ in $\mathbb{F}$, let $f(\lambda\mathbf{x}) := f(\lambda x_1, \ldots, \lambda x_n)$. Prove that for $f(\mathbf{x})$ homogeneous of degree d, one has

$$f(\lambda\mathbf{x}) = \lambda^d f(\mathbf{x}).$$

(b) Prove that for $f(\mathbf{x})$ homogeneous of degree d, one has $\displaystyle\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = d \cdot f(\mathbf{x})$.

(c) Show that every polynomial $f(x, y)$ in $\mathbb{C}[x, y]$ which is homogeneous of degree d can be factored as a product of linear (degree 1) polynomials:

$$f(x, y) = \prod_{i=1}^{d} (\alpha_i x + \beta_i y)$$

for some $\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d$ in $\mathbb{C}$.