Math 8669 Introductory Grad Combinatorics
Spring 2010, Vic Reiner
Homework 1- Due Friday February 26, 2010

Hand in at least 6 of the 11 problems.

1. (cf. Stanley, E.C. Vol. I, Chapter 3 Problem 30) A *closure relation* on a poset P is a map

$$\begin{aligned} P &\to P \\ x &\mapsto \bar{x} \end{aligned}$$

satisfying these three properties:

$$\begin{aligned} &x \leq \bar{x} \\ &\bar{\bar{x}} = \bar{x} \\ &x \leq y \quad \Rightarrow \quad \bar{x} \leq \bar{y}. \end{aligned}$$

Let $\bar{P}$ denote the subposet consisting of the *closed* elements, that is, $\bar{P}$ is the image of P or the set of elements with $x = \bar{x}$.

(a) Show that for any $x, y \in \bar{P}$ one has

$$\mu_{\bar{P}}(x,y) = \sum_{y' \in P : x \leq \bar{y'} = y} \mu_P(x, y').$$

(b) Deduce that if P is a poset with bottom, top elements $\hat{0}, \hat{1}$ and a closure relation $x \mapsto \bar{x}$ that restricts to $P - \{\hat{0}, \hat{1}\}$ (i.e. $\hat{0}, \hat{1}$ are the only elements whose closures are $\hat{0}, \hat{1}$ respectively), then

$$\mu_P(\hat{0}, \hat{1}) = \mu_{\bar{P}}(\hat{0}, \hat{1}).$$

2. (a) Let $x < y < z$ in a poset P have the property that every element y' in the open interval (x, z) is comparable to y. Show that $\mu(x, z) = 0$.

(b) Let $x < y < z$ in a finite lattice have the property that every element y' in the open interval (x, z) has $y' \vee y < z$. Show that $\mu(x, z) = 0$.

(Hint for (b): Define a *closure relation* on the open interval by $\bar{y'} := y' \vee y$. Then try to apply Problem 1(b) followed by part (a) of this problem.)

3. In a finite lattice L, show that $\mu(x, y) = 0$ whenever x is not the meet of all the co-atoms in in the interval $[x, y]$. What about if y is not the join of all the atoms in $[x, y]$?
(Hint: consider the closure relation on $[x, y]$ which maps

$$z \mapsto \bigwedge_{\text{coatoms } c \in [x,y] : c \geq z} c.)$$

4. Show that for a lattice, one of the distributive laws

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

implies its dual distributive law

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

(Hint: first show that the inequality

$$x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$$

is valid in *any* lattice.)

5. In a finite lattice, the *atoms* are the elements which cover the bottom element $\hat{0}$, and the *coatoms* are the elements which are covered by the top element $\hat{1}$. Say that L is *atomic* if every element is the join of atoms below it,, and that it is *coatomic* if every element is the meet of the set of coatoms above it. Say that an element $x^\perp$ is a *complement* to the element x if $x \vee x^\perp = \hat{1}$ and $x \wedge x^\perp = \hat{0}$.

For a finite distributive lattice L, show that the following are equivalent:

(i) L is atomic.
(ii) L is coatomic.
(iii) L is complemented (that is, every element has a complement).
(iv) $L = \mathcal{B}_n$ is a Boolean algebra.

6. Extend Birkhoff's Theorem (The fundamental theorem of finite distributive lattices) and its inverse associations

$$\begin{array}{ccc} \{\text{ finite posets }\} & \leftrightarrow & \{\text{ finite distributive lattices }\} \\ P & \rightarrow & J(P) \\ \mathrm{Irr}(L) & \leftarrow & L. \end{array}$$

to a result about maps as follows.

A set-map $P_1 \xrightarrow{f} P_2$ between two posets is *order preserving* if $x \leq y$ implies $f(x) \leq f(y)$. A set-map $L_2 \xrightarrow{\phi} L_1$ is a *lattice-morphism* if

$$\phi(x \vee y) = \phi(x) \vee \phi(y) \text{ and}$$
$$\phi(x \wedge y) = \phi(x) \wedge \phi(y).$$

Say that ϕ is *pointed* if ϕ sends $\hat{0}_{L_2}, \hat{1}_{L_2}$ to $\hat{0}_{L_1}, \hat{1}_{L_1}$ respectively.

(a) Show that an order-preserving map $P_1 \xrightarrow{f} P_2$ induces a pointed lattice-morphism $J(P_2) \xrightarrow{J(f)} J(P_1)$ defined by $I \overset{J(f)}{\mapsto} f^{-1}(I))$.

(b) Show that a pointed lattice-morphism $L_2 \xrightarrow{\phi} L_1$ between two distributive lattices induces an order-preserving map $\mathrm{Irr}(L_1) \xrightarrow{\mathrm{Irr}(\phi)} \mathrm{Irr}(L_2)$ defined by $x \overset{\mathrm{Irr}(\phi)}{\mapsto} \bigwedge_{y \in L_2 : \phi(y) \geq x} y$.

(c) Show that

$$J(f \circ g) = J(g) \circ J(f)$$
$$J(\mathrm{id}) = \mathrm{id}$$
$$\mathrm{Irr}(\phi \circ \psi) = \mathrm{Irr}(\psi) \circ \mathrm{Irr}(\phi)$$
$$\mathrm{Irr}(\mathrm{id}) = \mathrm{id}$$

(d) Show that

$$J(\mathrm{Irr}(\phi)) = \phi$$
$$\mathrm{Irr}(J(f)) = f$$

In other words, J and Irr are *contravariant functors*, and mutually inverse. This whole set-up is sometimes called *Birkhoff-Priestley duality*.

7. In a lattice, an element x is called

$$\begin{array}{rccc} \textit{prime} \text{ if} & x \leq y \vee z & \Rightarrow & x \leq y \text{ or } x \leq z \\ \textit{(join-)irreducible} \text{ if} & x = y \vee z & \Rightarrow & x = y \text{ or } x = z. \end{array}$$

(a) Prove that in any lattice, prime implies irreducible, but not conversely.

(b) Prove that the following conditions are equivalent for a finite lattice L (or if you like, assume only that L satisfies the *descending chain condition*, i.e. every descending chain $x_1 > x_2 > \cdots$ must terminate after finitely many steps):

(i) x is prime if and only x is irreducible,
(ii) every x in L has a decomposition $x = \bigvee_{i=1}^{r} x_i$ into irreducibles x_i, and if the decomposition is *irredundant* (that is, no two x_i are comparable) then the set $\{x_i\}_{i=1}^{r}$ *uniquely determined* by x,
(iii) L is distributive.

8. This problem explores further some of the ways in which the Boolean algebra $\mathcal{B}_n$ should be viewed as the limiting case of vector space lattices $L_n(q)$ when the order q of the field goes to 1.

(a) Given a k-dimensional $\mathbb{F}_q$-subspace V of $\mathbb{F}_q^n$, show that there is a unique $k \times n$ matrix A_V with entries in $\mathbb{F}_q$ whose row-space is V and which is in *row-reduced echelon form*:

(i) each row ends with a (possibly empty) sequence of zeroes and then has its last non-zero entry (called a *pivot*) equal to 1,
(ii) letting c_i denote the column index of the pivot entry in row i, one has $c_1 < \ldots < c_k$,
(iii) the only non-zero entry in each column c_i is the pivot entry 1.

For example, the following matrix is in row-reduced echelon form

$$\begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & 0 & * & 1 & 0 \end{bmatrix}$$

where the $*$'s are some arbitrary elements of the field, and its pivot columns are $\{c_1, c_2, c_3\} = \{3, 6, 8\}$.

(b) Define a map $L_n(q) \xrightarrow{\pi} \mathcal{B}_n$ by sending a subspace V to its set $\{c_1, \ldots, c_k\}$ of pivot columns. Show that π is order-preserving, rank-preserving and surjective.

(c) The usual R-labelling of $\mathcal{B}_n$ induces via this map π a labelling of Hasse diagram edges in $L_n(q)$ as follows: if $V \lessdot V'$ is a covering relation, label this edge by the unique $i \in [n]$ such that $\pi(V') - \pi(V) = \{i\}$. Show that this is an R-labelling, and use it to calculate $\mu(V, V')$ for any pair of subspaces $V \leq V'$.

(d) Do Problem #45 in Chapter 3 of Stanley's E.C. Vol I. Compare with your answer to part (c).

(e) How many maximal chains are there in $\mathcal{B}_n$? How many in $L_n(q)$? Given a sequence of integers $(k_1, \ldots, k_r)$ with $\sum_{i=1}^r k_i = n$, how many chains in $\mathcal{B}_n$ pass through the ranks

$$0, k_1, k_1+k_2, k_1+k_2+k_3, \ldots, k_1+k_2+\cdots+k_r = n$$

and no other ranks? How many in $L_n(q)$ pass through the same set of ranks?

9. The *order dimension* odim$P$ of a poset P is the minimum value d such that P can be embedded into a Cartesian product of d chains. That is, it is the smallest d for which their is an encoding map $P \xrightarrow{\phi} \mathbb{R}^d$ with the property that $p \leq_P p'$ if and only if for each $i = 1, 2, \ldots, d$ one has $\phi(p)_i \leq \phi(p')_i$.

(a) Show that the Boolean algebra $\mathcal{B}_n$ has odim$\mathcal{B}_n = n$.
(Hint: Show that odim$\mathcal{B}_n \geq n$ by showing that the induced subposet P_n on the union of atoms and coatoms in $\mathcal{B}_n$ has odim$P_n \geq n$.)

(b) Let L be a finite distributive lattice, and P its subposet of join-irreducibles. Show that odim$L$ is the size of the largest antichain in P.
(Hint for (b): Dilworth originally proved his theorem as a lemma aimed toward proving this result!)

10. Let P be a finite poset, and $A(P)$ the collection of all antichains in P, partially ordered by saying $A \leq A'$ if for every $a \in A$ there exists some $a' \in A'$ with $a \leq a'$.

(a) Explain why this poset $A(P)$ is isomorphic to the distributive lattice $J(P)$, and prove that the antichains of maximum size form a sublattice of $A(P)$.

(b) Let Aut(P) denote the group of poset automorphisms of P. Prove that there exists a maximum sized antichain in P which is a union of orbits of Aut(P).

(c) Use part (b) to show that if P is ranked and has the property that Aut(P) acts transitively on each rank (that is, given two elements $p, p' \in P$ of the same rank, there is a poset automorphism $\phi : P \to P$ with $\phi(p) = p'$), then P is Sperner.

11. Prove that if P_1, P_2 are two ranked posets, each with a symmetric chain decomposition, then their Cartesian product $P_1 \times P_2$ also has a symmetric chain decomposition.