Weak (Bruhat) Order (Björner-Brenti Chap. 3)

- overall a less subtle order than Bruhat

DEFINITION-PROPOSITION
Given (W,S), the following define the same poset $\leq_R$ on W, called the
(right) weak (Bruhat) order,
weaker than Bruhat $\leq$ (meaning $u \leq_R w \Rightarrow u \leq w$)
but with same rank function $\ell(w)$:

(i) $\leq_R$ is the transitive closure of $u <\!\!\cdot_R\, us$ if $s \in S$ and $\ell(u) < \ell(us)$

(ii) $u \leq_R w$ if $w = u s_1 s_2 \cdots s_k$ with $\ell(u s_1 s_2 \cdots s_i) = \ell(u) + i$ for $i = 0,1,\ldots,k$

(iii) $u \leq_R w$ if u, w have reduced words of form
$$\left.\begin{aligned} u &= s_1 s_2 \cdots s_m \\ w &= s_1 s_2 \cdots s_m s_1' s_2' \cdots s_k' \end{aligned}\right] \text{“prefix order”}$$

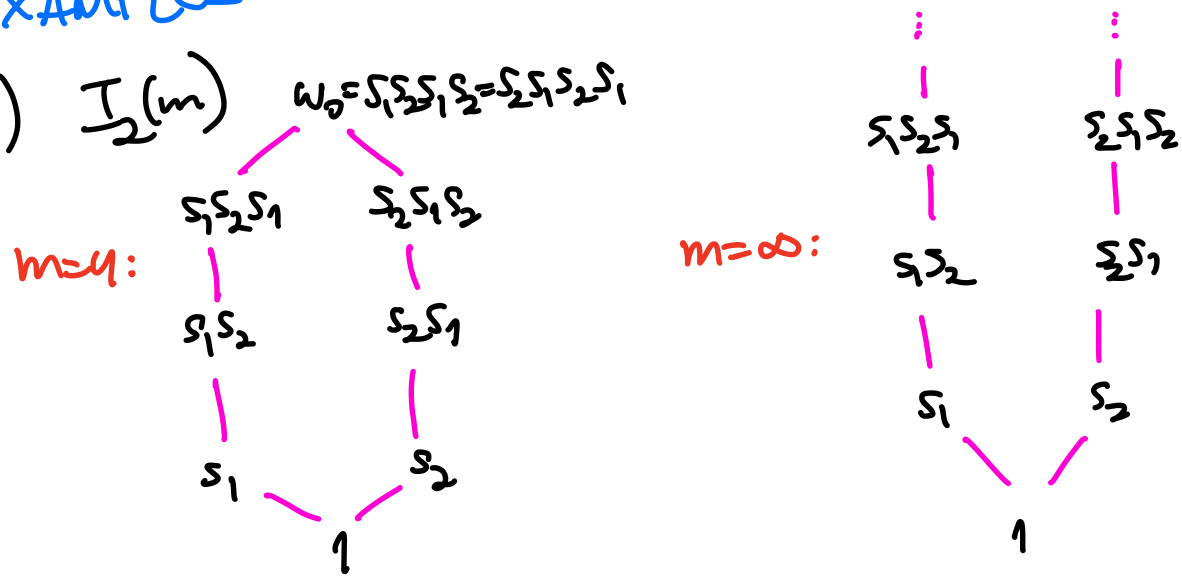
(iv) $u \leq_R w$ if $\ell(u) + \ell(u^{-1}w) = \ell(w)$

proof: All straightforward. $\blacksquare$

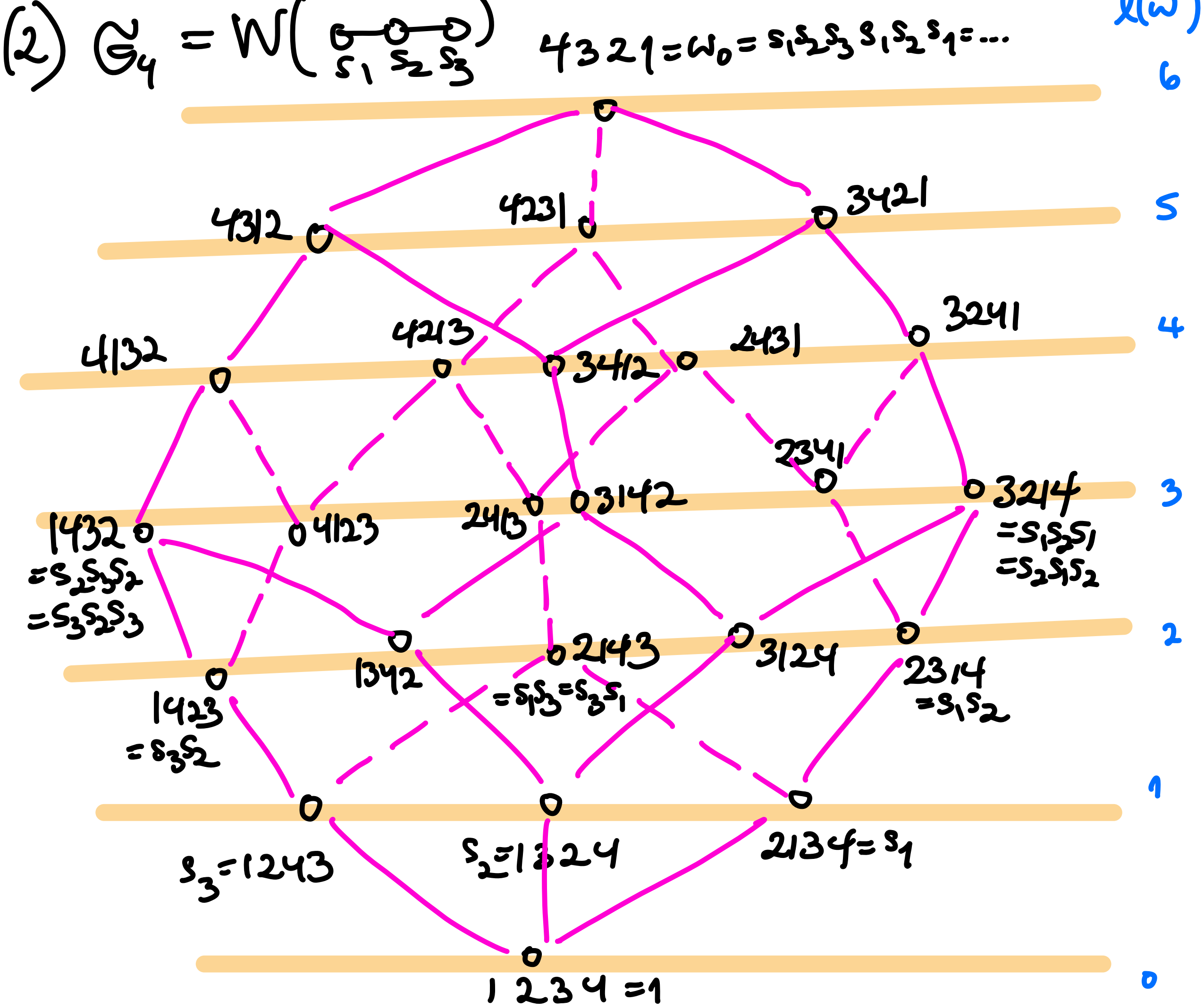
REMARK: Can similarly define left weak order $\leq_L$, isomorphic to $\leq_R$ via $w \mapsto w^{-1}$, but $\leq_R \neq \leq_L$.

EXAMPLES

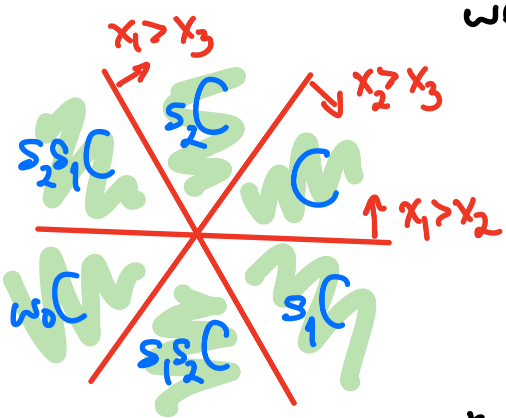
(1) $I_2(m)$

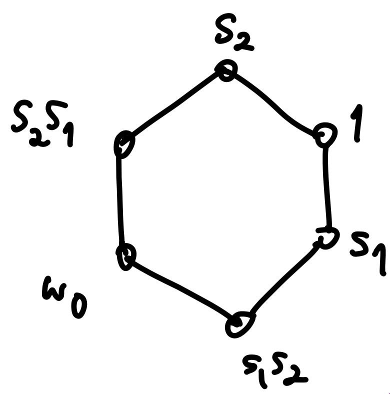


(2) $S_4 = W(\underset{s_1}{\circ}-\underset{s_2}{\circ}-\underset{s_3}{\circ})$



REMARK: By definition, the Hasse diagram of $\leq_R$ as a graph is the dual/ridge graph for the maximal cones/chambers in the Tits cone $\mathcal{U} = \bigcup_{w \in W} w(C) \subset V^*$

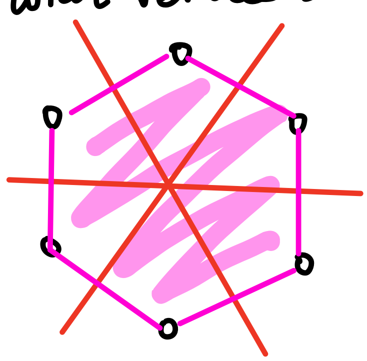




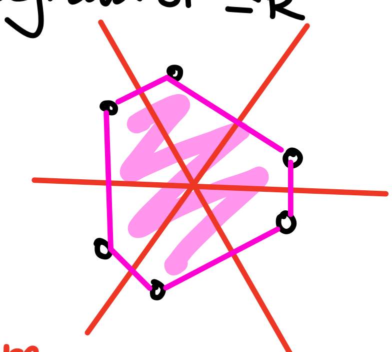
$\leq_R$ increasing this way

$W = \mathfrak{S}_3$, $\mathcal{U} = V^* (\cong V)$

FACT: When W is finite, any vector v_0 in the interior of C has the polytope $P :=$ convex hull of $\{w(v_0)\}_{w \in W}$ with vertices & edges $\cong$ Hasse diagram of $\leq_R$



Called W-permutohedra



Some symmetries of weak order, like Bruhat...

PROPOSITION: When W is finite,

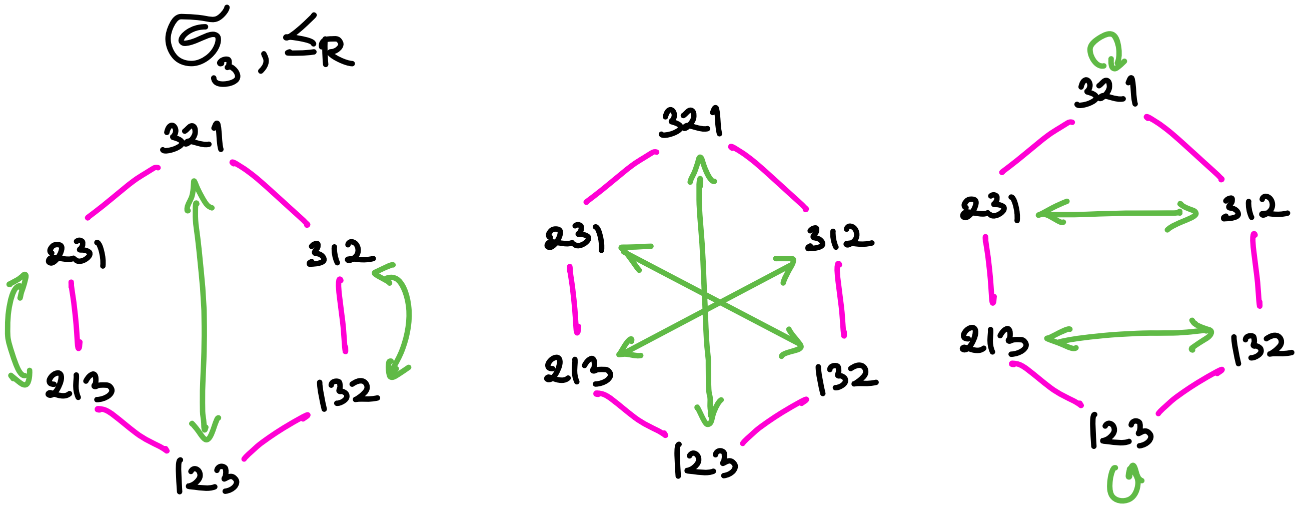
(i) $w_0 \geq_R w \quad \forall w \in W$

(ii) $\left.\begin{array}{l} w \longmapsto w_0 w \\ w \longmapsto w w_0 \end{array}\right\}$ both give poset anti-automorphisms of $\leq_R$

(iii) $w \longmapsto w_0 w w_0$ is a poset automorphism of $\leq_R$

EXAMPLE

$\mathfrak{S}_3, \leq_R$



$w \longmapsto w_0 w$ $\quad$ $w \longmapsto w w_0$ $\quad$ $w \longmapsto w_0 w w_0$

proof: They all follow from

$$u \leq_R w \Leftrightarrow \ell(u) + \ell(u^{-1}w) = \ell(w)$$

$$\ell(w w_0) = \ell(w_0 w) = \ell(w_0) - \ell(w)$$

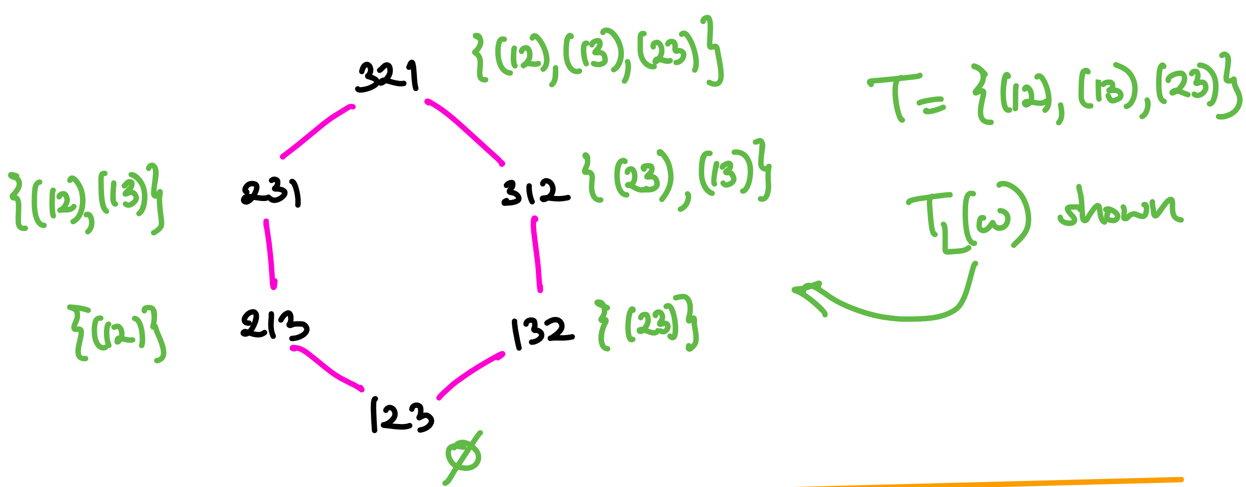
and $w \mapsto w_0 w w_0$ permuting S $\blacksquare$

As with Tableau Criterion for Bruhat, there is a more efficient encoding/rephrasing of $\leq_R$:

PROPOSITION: $u \leq_R w \iff T_L(u) \subseteq T_L(w)$

i.e. $(\mathfrak{S}_n, \leq_R) \hookrightarrow (2^T, \subseteq)$ embeds $\leq_R$ as a subposet
$w \longmapsto T_L(w)$

EXAMPLE: $\mathfrak{S}_3, \leq_R$



COROLLARY: When $u \leq_R w$, then

$$\#[u,w]_{\leq_R} \leq \#[T_L(u), T_L(w)]_{\subseteq} = 2^{\ell(w)-\ell(u)}$$

proof of PROP: The forward implication $(\Rightarrow)$ comes from the prefix characterization:

$u \leq_R w \Rightarrow \left.\begin{array}{l} u = s_1 s_2 \cdots s_k \\ w = s_1 s_2 \cdots s_k s_{k+1} \cdots s_\ell \end{array}\right\}$ reduced

$\Rightarrow T_L(u) \subset T_L(w) = \{\text{palindromes } s_1 s_2 \cdots s_i \cdots s_2 s_1\}_{i=1,2,\ldots,\ell}$

For the backward implication ($\Rightarrow$),

assume $T_L(u) \subseteq T_L(w)$

$$\{s_1, s_1s_2s_1, \ldots, s_1s_2\cdots s_k\cdots s_2s_1\} \text{ for } u = s_1s_2\cdots s_k \text{ reduced.}$$

with $t_1 = s_1$, $t_2 = s_1s_2s_1$, …, $t_k = s_1s_2\cdots s_k\cdots s_2s_1$.

Try to show w has a reduced word of form

$$w \overset{(*)}{=} s_1s_2\cdots s_i\, s_1's_2'\cdots s_{\ell(w)-i}' \quad \text{for each } i = 0,1,2,\ldots,k$$

by induction on i.

BASE CASE: $i=0$ says nothing (w has a reduced word).

INDUCTIVE STEP: Assume true for i.

Note $t_{i+1} = s_1s_2\cdots s_is_{i+1}s_i\cdots s_2s_1 \in T_L(u) \subseteq T_L(w)$,

and $t_{i+1} \neq t_1, t_2, \ldots, t_i$ since $s_1s_2\cdots s_k$ is reduced,

so Strong Exchange applied to $(*)$ shows

$$t_{i+1} = s_1s_2\cdots s_is_1's_2'\cdots s_m'\cdots s_2's_1's_i\cdots s_2s_1 \text{ for some } m \geq 1.$$

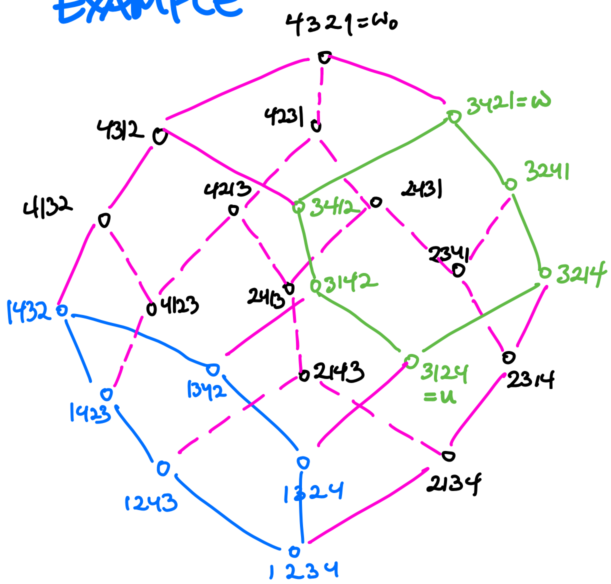
Then
$$w = t_{i+1}^2 \cdot w = (s_1\cdots s_{i+1}\cdots s_1)(s_1s_2\cdots s_is_1'\cdots s_m'\cdots s_1's_i\cdots s_2s_1)\cdot w$$
$$= s_1s_2\cdots s_is_{i+1}s_1's_2'\cdots \hat{s}_m' \cdots s_{\ell(w)-i}' \quad \blacksquare$$

Understanding intervals $[u,w]_{\le_R}$ reduces to $[1,w']_{\le_R}$:

PROPOSITION: One has a poset isomorphism

$$\begin{array}{ccc} [1, u^{-1}w]_{\le_R} & \xrightarrow{\sim} & [u,w]_{\le_R} \\ v & \longmapsto & uv \end{array}$$

EXAMPLE



$$\begin{array}{ccc} [1, u^{-1}w] & \xrightarrow{\sim} & [u,w] \\ = [1234, 1432] & & = [3124, 3421] \\ v & \longmapsto & uv \end{array}$$

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proof: Assume $u \le_R w$. For any $v \in W$, one has

$$\ell(w) = \ell(u) + \ell(u^{-1}w) = \ell(u) + \ell(v \cdot v^{-1}u^{-1}w)$$

$$\underset{(a)}{\le} \ell(u) + \ell(v) + \ell(v^{-1}u^{-1}w) \underset{(b)}{\ge} \ell(uv) + \ell(v^{-1}u^{-1}w) \underset{(c)}{\ge} \ell(w)$$

Now note

$v \in [1, u^{-1}w] \iff$ equality in (a) $\iff$ equality in (b) AND equality in (c) $\iff$ $u \le_R uv$ AND $uv \le_R w$ ∎

The lattice property, and consequences

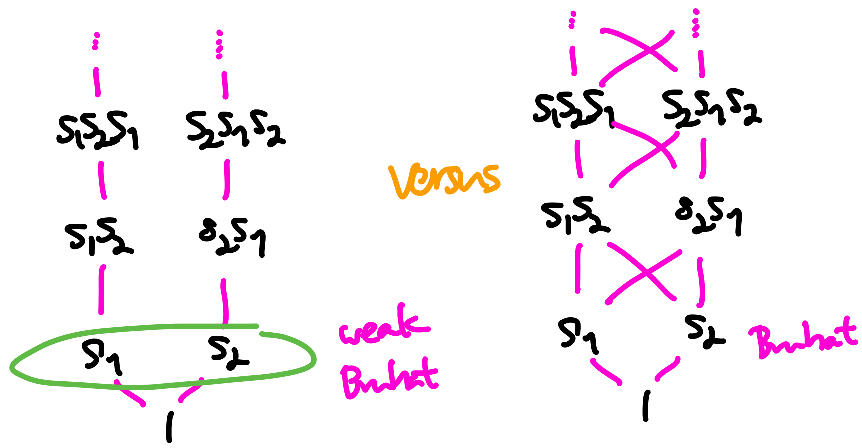
Unlike Bruhat order $\leq$, when W is infinite, weak Bruhat $\leq_R$ is never directed:

EXAMPLE

$W = I_2(\infty)$

$= W(\overset{\infty}{\underset{s_1\quad s_2}{\circ\!-\!\!-\!\circ}})$

no upper bound for s_1, s_2



Why "never"? If $S = \{s_1, s_2, \ldots, s_n\}$ have an upper bound $w \geq_R s_1, s_2, \ldots, s_n$

then $D_L(w) = \{s_1, s_2, \ldots, s_n\} = S \Rightarrow w = w_0$ and W is finite.

an old PROPOSITION

Nevertheless, $\leq_R$ does always have

meets $x \wedge y :=$ greatest lower bound of x, y

and even ∞ meets $\bigwedge A :=$ greatest lower bound of all $a \in A$

PROPOSITION:

(a) For any Cox. sys. (W,S) and any $x,y \in W$, the meet $x \wedge y$ exists in $\leq_R$

Consequently,

(b) $\leq_R$ is a complete meet-semilattice, i.e. all meets $\bigwedge A$ exist $\forall A \subseteq W$

(c) When A has an upper bound, the join $\bigvee A :=$ least upper bound always exists.

(d) In particular, if W is finite, $\leq_R$ is a lattice (meets $\wedge$, joins $\vee$ exist)

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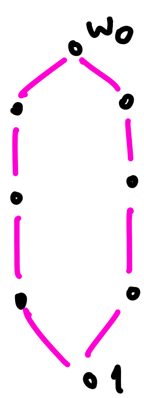
proof: First show (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d).



Given (a), then if $A = \{a_1, a_2, a_3, \ldots\}$

one has $\bigwedge A = \underbrace{(\underbrace{(a_1 \wedge a_2)}_{\text{weakly shorter than } a_2} \wedge a_3)}_{\text{weakly shorter than } a_3} \wedge a_4 \cdots$ ...

and it must terminate since $\ell(w) \in \mathbb{N}$

So (b) follows.

Then (b) $\Rightarrow$ (c) since $\bigvee A = \bigwedge \{ \text{upper bounds of } A \}$

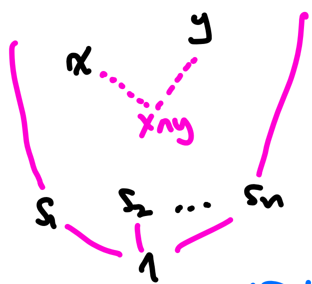
and (c) $\Rightarrow$ (d) since when W is finite, w_0 is an upper bound $\forall A \subseteq W$.

To prove (a), show $x \wedge y$ exists by induction on $\ell(x)$.

BASE CASE: The set

$$L := \{ \text{common lower bounds } z \leq_R x, y \}$$

contains no $s \in S$. (e.g. if $\ell(x) = 0$)



Then $1 = x \wedge y$.

INDUCTIVE STEP: L contains some $s \in S$.

We'll show any $z \in L$ of maximum length has $z = x \wedge y$.

First we claim $s \leq_R x, y \Rightarrow s \leq_R z$:

Otherwise $sz > z$ and if we start with

$$\left.\begin{aligned} z &= s_1 \cdots s_k \text{ reduced} \\ x &= s_1 \cdots s_k s_1' \cdots s_\ell' \text{ reduced} \\ y &= s_1 \cdots s_k s_1'' \cdots s_m'' \text{ reduced} \end{aligned}\right\} \text{since } z \leq_R x, y$$

then $\begin{matrix} sx < x \\ sy < y \end{matrix} \underset{\text{strong exchange}}{\Rightarrow} \begin{matrix} x = s s_1 \cdots s_k s_1' \cdots \hat{s}_i' \cdots s_\ell' & \text{for some } i \\ y = s s_1 \cdots s_k s_1'' \cdots \hat{s}_j'' \cdots s_m'' & \text{for some } j \end{matrix}$

so $sz = s s_1 \cdots s_k \leq_R x, y \Rightarrow sz \in L$, contradiction.

Given $w \in L - \{1\}$, want to show $w \leq_R z$.

Pick any $s \in S$ with $sw < w$

i.e. $s \leq_R w \Rightarrow s \leq_R x, y \underset{\text{by above discussion}}{\Longrightarrow} s \leq_R z$

Now we'll repeatedly use a

"Lifting-like" FACT: If $s \in S$ has $\begin{matrix} su < u \\ sw < w \end{matrix}$ then

$$u \leq_R w \iff su \leq_R sw$$

proof:

$$\begin{matrix} u \leq_R w & \iff & su \leq_R sw \\ \Updownarrow & & \Updownarrow \\ \ell(w) = \ell(u) + \ell(u^{-1}w) & \overset{\checkmark}{\iff} & \underbrace{\ell(sw)}_{\overset{\shortparallel}{\ell(w)-1}} = \underbrace{\ell(su)}_{\overset{\shortparallel}{\ell(u)-1}} + \ell(\underbrace{(su)^{-1}sw}_{u^{-1}w}) \end{matrix}$$

■

Let $z' := sx \wedge sy$, which exists by induction $(\ell(sx) < \ell(x))$

Then

$$w, z \leq_R x, y$$
$$\Downarrow \text{FACT}$$
$$sw, sz \leq_R sx, sy$$
$$\Downarrow$$
$$sw, sz \leq_R sx \wedge sy =: z'$$

Also, $z' \leq_R sx, sy$

$$\Downarrow \text{FACT}$$
$$sz' \leq_R x, y$$
$$\Downarrow$$
$$sz' \in L$$
$$\Downarrow$$
$$\ell(sz') \leq \ell(z)$$

So $sz \leq_R z'$ and $\ell(sz) = \ell(z) - 1 \geq \ell(sz') - 1 = \ell(z')$.

Hence $sz = z'$. But $sw \leq_R z' = sz \overset{\text{FACT}}{\Longrightarrow} w \leq_R z$. ■

The lattice property has some nice consequences.

THEOREM (Word Property B-B Thm 3.3.1)

(i) Every expression $w = s_1 s_2 \cdots s_q$ can be transformed to a reduced expression by a sequence of nil-moves and braid moves

$$\cdots s_i s_i \cdots \rightsquigarrow \cdots\ \cdots \qquad \cdots \overbrace{s_i s_j s_i s_j \cdots s_i}^{m_{ij} \text{ letters}} \cdots \rightsquigarrow \cdots s_j s_i s_j s_i \cdots s_j \cdots$$

(ii) Any 2 reduced expressions for w can be connected by a sequence of braid moves.

proof: Prove (ii) first, by induction on $q = \ell(w)$.

Assume $\left.\begin{aligned} w &= s_1 s_2 \cdots s_k \\ &= s_1' s_2' \cdots s_k' \end{aligned}\right\}$ both reduced.

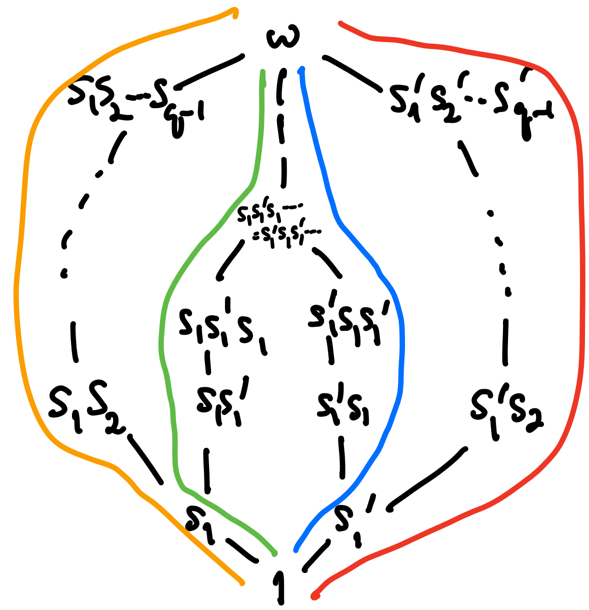
If $s_1 = s_1'$, we're done by induction applied to $s_1 w$.

Otherwise $s_1 \neq s_1'$ and $s_1, s_1' \leq_R w \Rightarrow w_0(W_{\{s_1, s_1'\}}) = s_1 \vee s_1' \leq_R w$.

So can write

$$\begin{aligned} w &= s_1 s_2 \cdots s_k && \text{conn. by induction} \\ &= (s_1 s_1' s_1 \cdots s_1') s_1'' s_2'' \cdots s_t'' && \text{conn by a braid move} \\ &= (s_1' s_1 s_1' \cdots s_1) s_1'' s_2'' \cdots s_t'' && \text{conn. by induction} \\ &= s_1' s_2' \cdots s_k' \end{aligned}$$

Picture:



Now prove (i) by induction on q in $w = s_1 s_2 \cdots s_q$.

If not reduced, so $q > \ell(w)$, find smallest i such that

$$\left.\begin{array}{c} s_q \\ s_{q-1} s_q \\ \vdots \\ s_{i+1} \cdots s_{q-1} s_q \end{array}\right\} \text{all reduced}$$

$$s_i s_{i+1} \cdots s_{q-1} s_q \leftarrow \text{not reduced}$$

By Exchange Property $s_{i+1} \cdots s_{q-1} s_q = s_i s_{i+1} \cdots \hat{s}_j \cdots s_{q-1} s_q$ for some j with $i+1 \leq j \leq q$, and these two reduced words are connected by braid moves using (ii).

Hence
$$w = s_1 \cdots s_i s_{i+1} \cdots\cdots s_q \quad \Big\} \text{ conn. by braid moves}$$
$$s_1 \cdots s_i \, s_i s_{i+1} \cdots \hat{s}_j \cdots s_q \quad \Big\} \text{ nil-move}$$
$$s_1 \cdots s_{i-1} \, s_{i+1} \cdots \hat{s}_j \cdots s_q \leftarrow \text{shorter, so done by induction} \ \blacksquare$$

Here's a topological Möbius function consequence.

COROLLARY: (B-B Thm 3.2.7 or 3.2.8) For any Cox. sys. (W,S) and $u \leq_R w$,

$$\Delta(u,w)_{\leq R} \underset{\text{homotopy equivalent}}{\approx} \begin{cases} \mathbb{S}^{\#J-2} & \text{if } u^{-1}w = w_0(W_J) \text{ for some } J \subseteq S \\ \text{a point (contractible)} & \text{otherwise} \end{cases}$$

and hence $\mu(u,w) = \begin{cases} (-1)^{\#J-2} & \text{if } u^{-1}w = w_0(W_J) \\ 0 & \text{otherwise} \end{cases}$

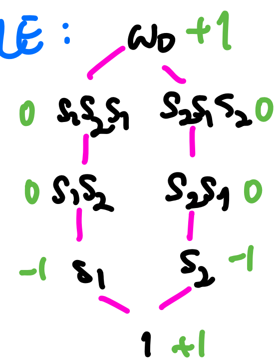
REMARK: We know $[u,w]_{\leq R} \cong [1, u^{-1}w]_{\leq R}$,

so WLOG $u=1$ anyway in thinking about this!

EXAMPLE:

$I_2(4)$

$\leq_R$

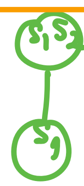


$\mu(1,w)$ labeled

$\Delta(1,s_1) = \{\emptyset\} \cong \mathbb{S}^{1-2} = \mathbb{S}^{-1}$!

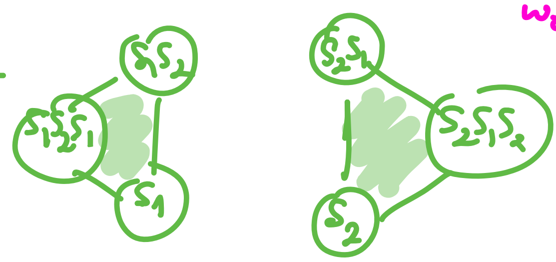
$s_1 = w_0(W_J)$ for $J=\{s_1\}$, $\#J=1$

$\Delta(1,s_1s_2s_1) =$



$\approx$ point, contractible

$\Delta(1,w_0) =$



$\approx \mathbb{S}^0 = \mathbb{S}^{2-2}$

$w_0 = w_0(W_J)$ for $J=S$, $\#J=2$

sketchy proof: As mentioned earlier, WLOG $u=1$.

On the open interval $P := (1, \omega)_{\leq \mathbb{R}}$

the map
$$\begin{aligned} P &\xrightarrow{f} P \\ x &\longmapsto \bigvee_{\substack{s \in S: \\ s \leq_{\mathbb{R}} x}} s \end{aligned}$$

gives a (co-)closure operator on the poset P:

DEF'N: (a) f is order-preserving: $x \leq y \Rightarrow f(x) \leq f(y)$

(b) $f(x) \geq x \quad \forall x \in P$

(c) $f^2 = f$ i.e. $f(f(x)) = f(x)$

(all 3 of (a),(b),(c) are easy to check here)

(Topological poset) LEMMA: For any co-closure map $f: P \to P$

B-B Fact A.2.3.2

on a poset, one has a homotopy equivalence

$$\Delta \underbrace{f(P)}_{= \mathrm{im}(f)} \approx \Delta P$$

(and even a strong deformation retraction

$$\Delta f(P) \hookrightarrow \Delta P \ )$$

Why does this help?

If $w = w_0(W_J)$ then

$f(P)$ = subposet of $\leq_R$ on $\{w_0(W_K)\}_{\emptyset \subsetneq K \subsetneq J}$

$\cong (\emptyset, J)$ inside Boolean algebra 2^J

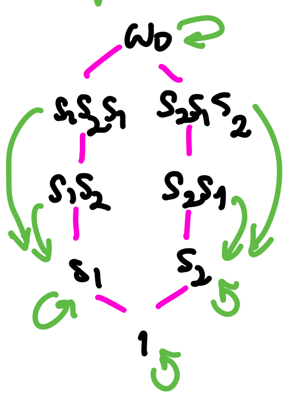
and $\Delta f(P) \cong \Delta(\emptyset, J) \cong$ barycentric subdivision of boundary of simplex with vertex set J $\cong \mathbb{S}^{\#J-2}$

If $w_0 \neq w_0(W_J)$ then

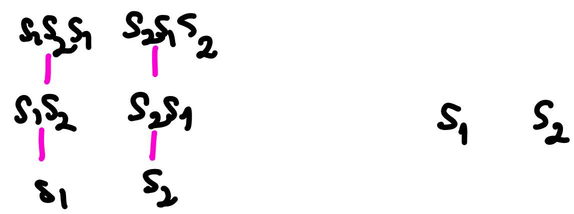
$f(P)$ has $w_0(W_J)$ where $J := \{s \in S : s \leq_R w\}$ as a top element, and $\Delta f(P)$ is a cone, so contractible. $\blacksquare$

EXAMPLES (1) $I_2(4)$

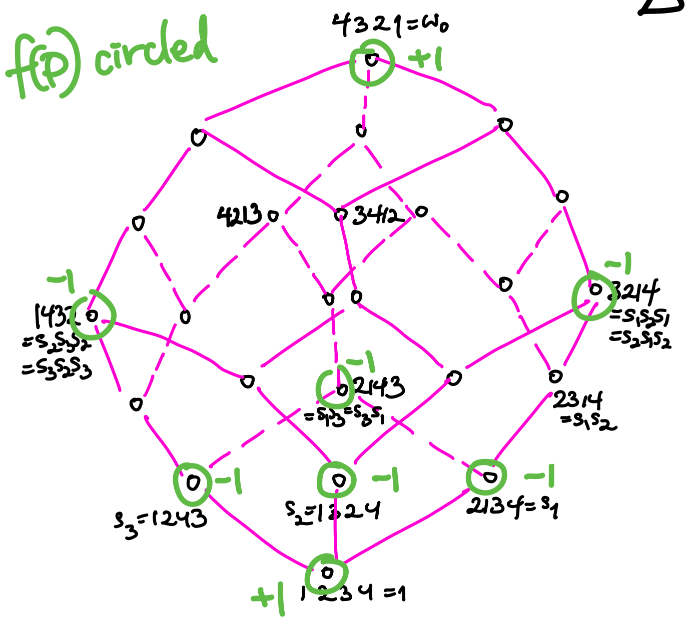
The map f on $[1, w_0]$ is shown



$P = (1, w_0) \xrightarrow{f} f(P)$

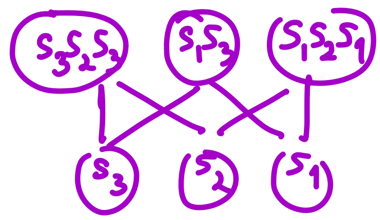


(2) $W = G_4 = W(\underset{s_1}{\circ}\!-\!\underset{s_2}{\circ}\!-\!\underset{s_3}{\circ})$

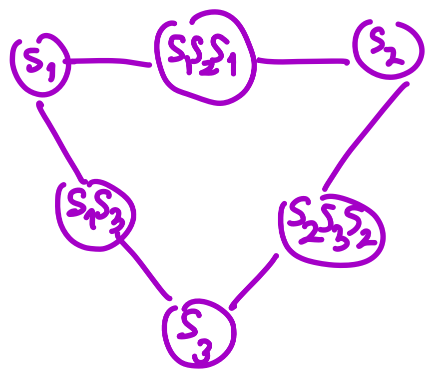


$\Delta(1,w_0) \approx \Delta f(P)$

$=$

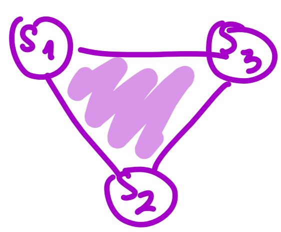


$\cong$

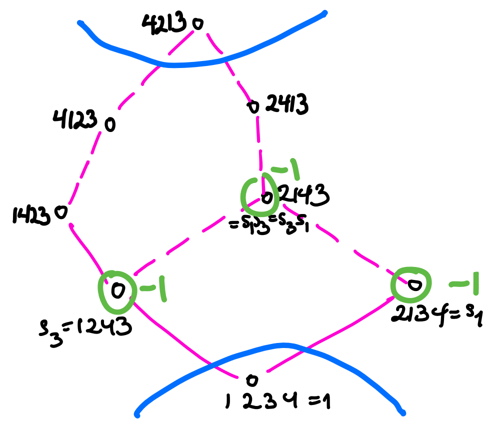


$\cong$

barycentric subdivision of boundary of

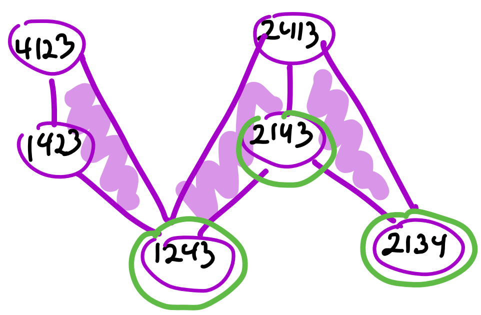


$P = (1, 4213)_{\leq_R} =$

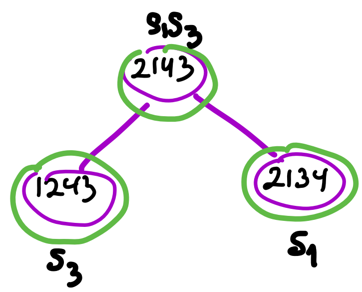


$\Delta(1, 4213)_{\leq_R} =$

$\Delta P =$



$\approx \Delta f(P) =$



$=$

max element of $f(P)$
gives a cone vertex

Coming back to that ...

(Topological poset) LEMMA : For any ω-closure map $f: P \to P$ on a poset, one has a homotopy equivalence

B-B Fact A.2.3.2

$$\underbrace{\Delta f(P)}_{=\,\mathrm{im}(f)} \approx \Delta P$$

(and even a strong deformation retraction

$$\Delta f(P) \hookrightarrow \Delta P \;)$$

sketch proof: The inclusion $f(P) \overset{i}{\hookrightarrow} P$ and the map $P \overset{f}{\twoheadrightarrow} f(P)$ are both order-preserving, so they give simplicial maps (& continuous)

$$\Delta f(P) \overset{i}{\hookrightarrow} \Delta P$$
$$\Delta P \overset{f}{\twoheadrightarrow} \Delta f(P)$$

The composites

$$f(P) \overset{i}{\hookrightarrow} P \overset{f}{\twoheadrightarrow} f(P) \quad \text{have } f\circ i = 1_{f(P)}$$
$$P \overset{f}{\twoheadrightarrow} f(P) \overset{i}{\hookrightarrow} P \quad\quad i\circ f \leq 1_P$$

and poset maps $f,g: P \to Q$ having $f \leq g$ are always homotopic. So $f\circ i = 1_{\Delta f(P)}$, $i\circ f \approx 1_{\Delta P}$.

This makes i a deformation retraction ▨