Tits's geometric representation for a Coxeter system (Humphreys Chap. 5,6)

Rather than just proving properties of real refl'n groups using their roots, let's show they hold for all Coxeter groups, which will also have some notion of roots.

DEF'N: Call a symmetric $n\times n$ matrix $(m_{ij})_{i,j=1,2,\dots,n}$ with $m_{ij}\in\{2,3,\dots\}\cup\{\infty\}$ and $m_{ii}=2\ \forall i$ a Coxeter matrix, with

$$W := \langle \underset{\{s_1,s_2,\dots,s_n\}}{\overset{\|}{S}} \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$$

the associated Coxeter system (W,S), and Coxeter group W,

so $W := \underbrace{F(S)}_{\text{Free group on } S} \Big/ \left(\begin{array}{l}\text{normal subgroup}\\ \text{gen'd by}\\ \{s_i^2, (s_i s_j)^{m_{ij}}\}\end{array} \right)$

e.g. here $s_1 s_2 s_5 s_1 s_1 s_1^{-1} s_1 s_3$
$= s_1 s_2 s_5 s_1 s_1 s_3$

REMARK: Sometimes its handy to use this version of the Coxeter presentation:

$$W := \langle S \mid s_i^2 = 1,\ \underbrace{s_i s_j s_i s_j \cdots}_{m_{ij} \text{ letters}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij} \text{ letters}} \rangle$$

called a braid relation

Why "braid" relation?

Every such Coxeter group W has an associated braid group

$$B_W := \langle \sigma_1, \ldots, \sigma_n \mid \underbrace{\sigma_i \sigma_j \sigma_i \sigma_j \cdots}_{m_{ij}} = \underbrace{\sigma_j \sigma_i \sigma_j \sigma_i \cdots}_{m_{ij}} \rangle$$

$$B_W \twoheadrightarrow W \qquad \sigma_i \mapsto s_i$$

(but $\sigma_i^2 \neq 1$, $\sigma_i \neq \sigma_i^{-1}$)

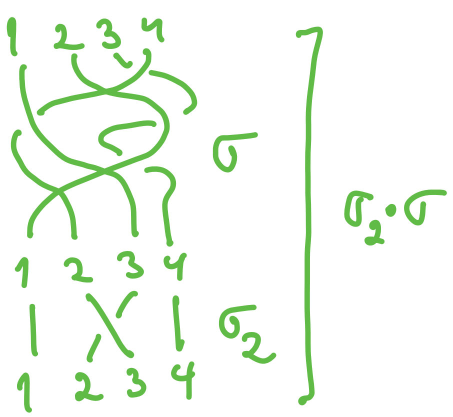
EXAMPLE: $\mathfrak{S}_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ \underset{|i-j|\geq 2}{s_i s_j = s_j s_i},\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$

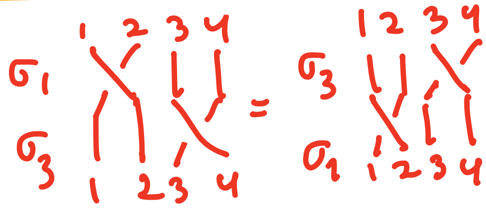
$\uparrow$

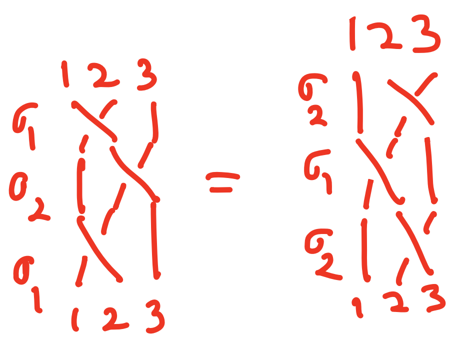
B_n = braid group on n strands

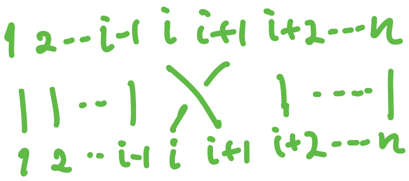
$\cong \langle \sigma_1, \dots, \sigma_{n-1} \mid \underset{|i-j|\geq 2}{\sigma_i \sigma_j = \sigma_j \sigma_i},\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle$

Artin's presentation







$\sigma_i =$ 

$\sigma_1^2 \neq 1$ 

$\sigma_1^{-1} =$ 

$\neq \sigma_1$

Recall group presentations can be tricky!

~~EXAMPLE~~ EXERCISE Check $\langle a,b \mid 1 = a^5 = b^8 = ab \rangle = \{1\}$

Q: Do the relations $(s_i s_j)^{m_{ij}}$ in W force any extra collapsing, e.g. does order of $s_i s_j$ ever strictly divide m_{ij}?

Is $W \neq \{1\}$?! No, at least $W \neq \{1\}$.

DEF'N/PROP: There's a sign homomorphism

$$W \xrightarrow{\epsilon} \{\pm 1\}$$

defined by $s_i \longmapsto -1 \quad \forall i$

$$\left(\text{so} \quad \underset{s_{i_1} s_{i_2} \cdots s_{i_\ell}}{\overset{w}{\parallel}} \overset{\epsilon}{\longmapsto} (-1)^{\ell = \ell(w)} \right)$$

proof: Check the set map $\underset{s_i \mapsto -1}{S \xrightarrow{\epsilon} \{\pm 1\}}$ when extended to $F(S) \to \{\pm 1\}$ has s_i^2 and $(s_i s_j)^{m_{ij}}$ in its kernel. ▨

This has consequences for one of our main tools for inductive proofs.

DEF'N: Given (W,S), define the length function

$$W \xrightarrow{\ell} \{0,1,2,\ldots\}$$
$$w \longmapsto \ell(w) := \min\{\ell : w = s_{i_1} s_{i_2} \cdots s_{i_\ell}\}$$
$$\ell_S(w) :=$$

PROP: (i) $\ell(w) = \ell(w^{-1})$

(ii) $\epsilon(w) = (-1)^{\ell(w)}$

(iii) $\ell(ws) = \ell(w) \pm 1 \;\big(\neq \ell(w)\big)$

(iv) $\ell(uv) \leq \ell(u) + \ell(v)$
$\geq \ell(u) - \ell(v)$

proof: (i) comes from $w^{-1} = s_{i_\ell} s_{i_{\ell-1}} \cdots s_{i_2} s_{i_1}$ if $w = s_{i_1} \cdots s_{i_\ell}$

(ii) comes from previous PROP

(iii) comes from $\ell(ws) \leq \ell(w) + 1$, so also
$\ell(w) = \ell(ws \cdot s) \leq \ell(ws) + 1$

(iv) comes from iterating (iii) ▨

To understand (W,S) better, need some geometry.

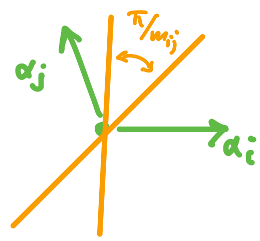
DEF'N: Given a Coxeter matrix $(m_{ij})_{i,j=1,\dots,n}$ introduce a vector space $V \cong \mathbb{R}^n$ with $\mathbb{R}$-basis $\Pi := \{\alpha_1,\dots,\alpha_n\} = \{\alpha_s\}_{s\in S}$, $S = \{s_1,\dots,s_n\}$

and define a symmetric $\mathbb{R}$-bilinear form $B(\cdot,\cdot)$

$$\begin{array}{ccc} V \times V & \xrightarrow{B(\cdot,\cdot)} & \mathbb{R} \\ (x,y) & \longmapsto & B(x,y) \end{array}$$

via its Gram matrix on Π:

$$B(\alpha_i,\alpha_j) := \begin{cases} 1 & \text{if } i=j \\ -\cos\left(\frac{\pi}{m_{ij}}\right) & \text{if } i\neq j \end{cases} \qquad \text{so } B(\alpha_i,\alpha_i)=1$$



Then we (attempt to) define...

DEF'N: Tits's geometric representation for (W,S)

$$W \xrightarrow{\ \sigma\ } GL(V)$$

sending $\quad s_i \longmapsto \sigma_i = \sigma_{s_i}$

where $\sigma_i(x) := x - 2\underbrace{B(x,\alpha_i)}\alpha_i$ ← same as $\frac{B(x,\alpha_i)}{B(\alpha_i,\alpha_i)}$

THEOREM This does extend to a
(i) homomorphism $W \xrightarrow{\sigma} GL(V)$
(ii) with image in the isometry group $O(V, B(\cdot,\cdot))$
(iii) and which is injective.

This **THM** takes work, but let's at least check...

PROP: $\sigma_i^2 = 1 \;\; \forall i$ and σ_i is a B-isometry.

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proof: Could do it directly, but also note

$$V = \mathbb{R}\alpha_i \oplus \alpha_i^\perp \quad \text{where } \alpha_i^\perp := \{x \in V : B(x,\alpha_i) = 0\}$$

since
$$V = \mathbb{R}\alpha_i + \alpha_i^\perp \quad \text{as any } x \in V$$
$$\text{has } x = \underbrace{B(x,\alpha_i)\alpha_i}_{\in \mathbb{R}\alpha_i} + \underbrace{(x - B(x,\alpha_i)\alpha_i)}_{\in \alpha_i^\perp}$$

$$\mathbb{R}\alpha_i \cap \alpha_i^\perp = \{0\} \quad \text{since } B(\alpha_i,\alpha_i) = 1 \neq 0.$$

Then check
$$\sigma_i(\alpha_i) = \alpha_i - 2\alpha_i = -\alpha_i, \text{ so } \sigma_i\big|_{\mathbb{R}\alpha_i} = -1_{\mathbb{R}\alpha_i}$$
$$\sigma_i\big|_{\alpha_i^\perp} = 1_{\alpha_i^\perp}$$

so σ_i acts as an involution & isometry on both summands of the $\perp$ direct sum decomp $V = \mathbb{R}\alpha_i \oplus \alpha_i^\perp$ $\boxtimes$

EXAMPLES

(1) Assume we started with a real ref'n group $W' \subset GL(V')$ for $V' = \mathbb{R}^n$ with inner product $(\cdot,\cdot)$

then produced $\Phi' = (\Phi^+)' \sqcup (\Phi^-)'$

$$\begin{array}{c} (\Phi^+)' \\ \cup \\ \Pi' = \{\alpha_1', \ldots, \alpha_n'\} \end{array}$$

(assuming $V' = \mathrm{span}_{\mathbb{R}}(\Pi')$ WLOG),

Then we could create a Coxeter matrix

(m_{ij}) via $m_{ij} =$ order of $s_{\alpha_i'} s_{\alpha_j'}$

and Coxeter system (W,S) with geom. rep'n

$W \xrightarrow{\sigma} GL(V)$ and $B(\cdot,\cdot)$ on V.

We then have an isometry $(V', (\cdot,\cdot)) \longrightarrow (V, B(\cdot,\cdot))$

$$\alpha_i' \longmapsto \alpha_i$$

inducing an isomorphism $W' \longrightarrow \sigma(W) \underset{\uparrow}{\cong} W$

THM Part (iii)

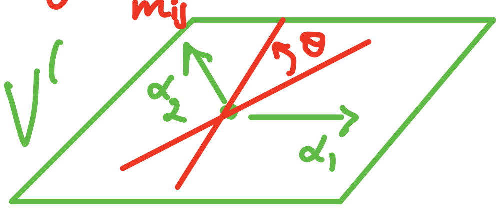
CONCLUSION: Anything we prove about general Coxeter systems (W,S) will apply to real ref'n groups W'.

(2) In particular, when $m_{ij} < \infty$ (and re-index $\substack{i=1 \\ j=2}$)

EXAMPLE (1) above applies to $V' = \mathrm{span}_{\mathbb{R}}\{\alpha_1, \alpha_2\}$ where $B(\cdot,\cdot)$ restricts to a positive definite inner product, and

$$W' := \langle \sigma_1, \sigma_2 \rangle \text{ acts as a real refl'n group}$$
$$\cong I_2(m_{ij}) = \text{dihedral group of order } 2m_{ij}$$

$\theta = \frac{\pi}{m_{ij}}$



Gram matrix of $B(\cdot,\cdot)$ on V' $=$
$$\begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & 1 & -\cos\theta \\ \alpha_2 & -\cos\theta & 1 \end{array}$$

We also have $V = V' \oplus (V')^{\perp}$ since

$$\left\{ \begin{array}{l} V = V' + (V')^{\perp} \text{ as one can pick a } B\text{-orthonormal basis } \{e_1, e_2\} \text{ for } V' \text{ and write any } x \in V \text{ as} \\ \quad x = \underbrace{\big(B(x,e_1)e_1 + B(x,e_2)e_2\big)}_{\in V'} + \underbrace{\big(x - B(x,e_1)e_1 - B(x,e_2)e_2\big)}_{\in (V')^{\perp}} \\ V' \cap (V')^{\perp} = \{0\} \text{ since } B(\cdot,\cdot) \text{ is pos. def. on } V' \end{array} \right.$$

$\Rightarrow$ $\sigma_1\sigma_2$ acts with order m on V', as $1_{(V')^{\perp}}$ on $(V')^{\perp}$,

$\Rightarrow$ $\sigma_1\sigma_2$ acts with order m on V.

This already proves (i), (ii) here (but not (iii) yet):

THEOREM $s_i \xmapsto{\sigma} \sigma_i$ does extend to a

✓ (i) homomorphism $W \xrightarrow{\sigma} GL(V)$

✓ (ii) with image in the isometry group $O(V, B(\cdot,\cdot))$

(iii) and which is injective.

EXAMPLE

(3) $W\left(\underset{s_1}{\circ}\overset{\infty}{\text{———}}\underset{s_2}{\circ}\right)$ i.e. $m_{12} = m_{21} = \infty$

so $B(\cdot,\cdot)$ has Gram matrix on $V = \mathbb{R}^2$

$$\begin{array}{c|cc} & \alpha_1 & \alpha_2 \\ \hline \alpha_1 & 1 & -1 \\ \alpha_2 & -1 & 1 \end{array}$$

degenerate, i.e. singular

with $\lambda := \alpha_1 + \alpha_2 \in V^\perp$ since $B(\lambda, \alpha_i) = 1 + (-1) = 0$ for $i = 1,2$

Check: $\sigma_1(\alpha_2) = \alpha_2 - 2B(\alpha_2, \alpha_1)\alpha_1 = \alpha_2 - 2(-1)\alpha_1 = 2\alpha_1 + \alpha_2 = \lambda + \alpha_1$

$\sigma_2(\alpha_1) = \lambda + \alpha_2$ by symmetry

$\sigma_2\sigma_1(\alpha_2) = \sigma_2(\lambda + \alpha_1) = \lambda + \alpha_1 - B(\lambda+\alpha_1, \alpha_2)\alpha_2$

$= \lambda + \alpha_1 + 2\alpha_2 = 2\lambda + \alpha_2$

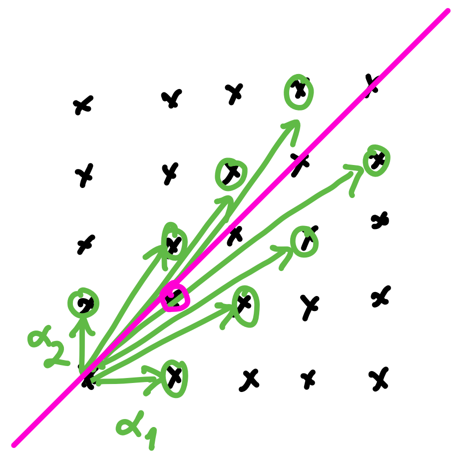
$\sigma_1\sigma_2(\alpha_1) = 2\lambda + \alpha_1$ by symmetry

$(\sigma_1\sigma_2)^k(\alpha_1) = 2k\lambda + \alpha_1$, $(\sigma_2\sigma_1)^k(\alpha_2) = 2k\lambda + \alpha_2$

is easy to check by induction on k.

PICTURE for $W(\overset{\infty}{\underset{s_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{s_2}{\circ}})$

$V = \mathbb{R}^2$



$V^\perp = \mathbb{R}\lambda$, $\lambda = \alpha_1 + \alpha_2$

$(\sigma_2\sigma_1)^k(\alpha_2) = 2k\lambda + \alpha_2$

$(\sigma_1\sigma_2)^k(\alpha_1) = 2k\lambda + \alpha_1$

CONCLUSION: The Coxeter presentation

$$W = \langle S \mid s_i^2 = 1 = (s_i s_j)^{m_{ij}} \rangle$$

always leads to order of $s_i s_j$ being exactly m_{ij}, with no further collapsing.

But why does $W \xrightarrow{\sigma} O(V, B(\cdot,\cdot))$, $s_i \longmapsto \sigma_i$

end up injective ?

Need roots Φ ...

DEF'N: Starting with the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ used to define $V = \mathbb{R}^n$ and $B(\cdot,\cdot)$, and define the (real) root system $\Phi := \{w(\alpha_i) : w \in W, \alpha_i \in \Pi\}$

really means $\sigma(w)(\alpha_i) \in V$

Since $s_i(\alpha_i) = -\alpha_i \in \Phi$, one has $ws_i(\alpha_i) = w(-\alpha_i) = -w(\alpha_i)$,

so $\Phi = -\Phi$

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positive roots $\Phi^+ := \{\alpha \in \Phi :$ the unique expansion $\alpha = \sum_{\alpha_i \in \Pi} c_i \alpha_i$ has $c_i \geq 0 \ \forall i\}$

negative roots $\Phi^- := \{\alpha \in \Phi : \alpha = \sum_{\alpha_i \in \Pi} c_i \alpha_i$ with $c_i \leq 0 \ \forall i\}$

$= -\Phi^+$

$$\Phi \supseteq \Phi^+ \sqcup \Phi^-$$

not clear yet that this is an equality, but it is...

THEOREM: $\Phi := \{w(\alpha_i) : \alpha_i \in \Pi, w \in W\}$
$= \Phi^+ \sqcup \Phi^-$

because (a) $\ell(ws_i) > \ell(w) \Rightarrow w(\alpha_i) \in \Phi^+$
(b) $\ell(ws_i) < \ell(w) \Rightarrow w(\alpha_i) \in \Phi^-$
(and hence both of these $\Rightarrow$ are $\Leftrightarrow$)

This has many consequences, including--

COROLLARY: The geometric rep'n
$W \xrightarrow{\sigma} O(V, B(\cdot,\cdot))$ is injective.

proof of COR from THM: If $w \in \ker(\sigma)$ and $w \neq 1$,
then pick any $s_i \in S$ with $\ell(ws_i) < \ell(w)$
(say $s_i = s_{i_\ell}$ in a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$).
$\ell = \ell(w)$, i.e. minimum length expression
One has $w(\alpha_i) \in \Phi^-$,
so $w(\alpha_i) \neq \alpha_i \ (\in \Phi^+)$ $\lightning$ ∎

Let's prove ...

THEOREM: $\Phi := \{w(\alpha_i) : \alpha_i \in \Pi, w \in W\}$
$$= \Phi^+ \sqcup \Phi^-$$

because (a) $\ell(ws_i) > \ell(w) \Rightarrow w(\alpha_i) \in \Phi^+$

(b) $\ell(ws_i) < \ell(w) \Rightarrow w(\alpha_i) \in \Phi^-$

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proof: Note (b) follows from (a):

if $\ell(ws_i) < \ell(w)$ then $\ell((ws_i)s_i) = \ell(w) > \ell(ws_i)$

so (a) $\Rightarrow \Phi^+ \ni ws_i(\alpha_i) = w(-\alpha_i) = -w(\alpha_i)$

$\Rightarrow \Phi^- \ni w(\alpha_i)$.

To prove (a), induct on $\ell(w)$

with base case $\ell(w)=0$ having $w=1$,

where $\ell(ws_i) = 1 > 0 = \ell(w)$ and $w(\alpha_i) = \alpha_i \in \Phi^+$ ✓

In the inductive step, find some s_j ($\neq s_i$) with $\ell(ws_j) = \ell(w) - 1$, i.e., picking s_j to be rightmost in some reduced expression $w = s_{j_1} s_{j_2} \cdots s_{j_{\ell(w)}}$.

Let's re-index $s_1 = s_i$, $s_2 = s_j$ so $\ell(ws_1) = \ell(w)+1$
$\ell(ws_2) = \ell(w) - 1$

NEW IDEA: parabolic factorization

Factor $w = v \cdot u$

with (a) $u \in W_{\{s_1,s_2\}} := \langle \{s_1, s_2\} \rangle$ called a parabolic subgroup

(b) $\ell(w) = \ell(v) + \ell_{\{s_1,s_2\}}(u)$ length using only s_1, s_2

(c) v shortest ($\ell(v)$ smallest) with (a), (b)

We hope to apply induction on length to v.

Note $\ell(v) \leq \ell(w) - 1$ since $w = ws_2 \cdot s_2$ is one such factorization as in (a), (b).

We claim $\ell(vs_1) > \ell(v)$, else if $\ell(vs_1) < \ell(v)$ then

$$
\begin{aligned}
\ell(w) = \ell(vu) &= \ell(vs_1 \cdot s_1 u) \\
&\leq \ell(vs_1) + \ell(s_1 u) \\
&\leq \ell(vs_1) + \ell_{\{s_1,s_2\}}(s_1 u) \\
&\leq \ell(v) - 1 + \ell_{\{s_1,s_2\}}(u) + 1 \\
&= \ell(v) + \ell_{\{s_1,s_2\}}(u) = \ell(w)
\end{aligned}
$$

$\ell_{\{s_1,s_2\}}(x) \leq \ell(x)$ for $x \in W_{\{s_1,s_2\}}$

forcing equality throughout, including here so vs_1 beats v, a contradiction.

Same argument shows $\ell(vs_2) > \ell(v)$.

Hence induction in THM shows $v(\alpha_1), v(\alpha_2) \in \Phi^+$, and hence any $\beta \in \Phi \cap (\mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2)$ also has $v(\beta) \in \Phi^+$.

Since $w = v \cdot u \implies w(\alpha_1) = v \cdot u(\alpha_1)$, it remains to show $u(\alpha_1) \in \mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2$, which we'll argue with some dihedral geometry inside $\mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$!

Note $\ell_{\{s_1,s_2\}}(us_1) > \ell_{\{s_1,s_2\}}(u)$, else

$$\begin{aligned}
\ell(ws_1) = \ell(vus_1) &\leq \ell(v) + \ell(us_1) \\
&\leq \ell(v) + \ell_{\{s_1,s_2\}}(us_1) \\
&< \ell(v) + \ell_{\{s_1,s_2\}}(u) = \ell(w)
\end{aligned}$$

$\lightning$ to $\ell(ws_1) > \ell(w)$

So a shortest $\{s_1,s_2\}$-word for u ends in s_2, and is either

$$u = s_1s_2s_1s_2 \cdots s_1s_2 = (s_1s_2)^k$$

or $u = s_2s_1s_2s_1s_2 \cdots s_1s_2 = s_2(s_1s_2)^k$ for some k,

with $k < \frac{m}{2}$ if $m = m_{12} < \infty$, since $\underbrace{s_1s_2s_1s_2\cdots}_{m \text{ letters}} = \underbrace{s_2s_1s_2s_1\cdots}_{m \text{ letters}}$

CASE 1. $m = m_{12} = \infty$. We already computed that

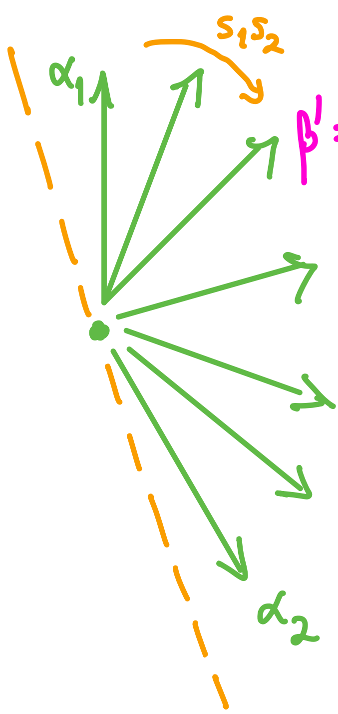
$$(s_1 s_2)^k(\alpha_1) = \alpha_1 + 2k\lambda \quad \text{where } \lambda := \alpha_1 + \alpha_2$$

so
$$s_2(s_1 s_2)^k(\alpha_1) = \alpha_1 + 2k\lambda - 2B(\alpha_1 + 2k\lambda, \alpha_2)\alpha_2$$
$$= \alpha_1 + 2k\lambda + 2\alpha_2 = \alpha_2 + (2k+1)\lambda$$

both lying in $\mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2$, as desired.

CASE 2. $m = m_{12} < \infty$. Letting $\theta := \frac{\pi}{m}$ = dihedral angle of $H_1 = \alpha_1^\perp$, $H_2 = \alpha_2^\perp$,

then $s_1 s_2$ rotates $\frac{2\pi}{m}$ clockwise here:



$\beta' = s_2(s_1 s_2)^k(\alpha_1) = s_2(\beta)$

$\beta = (s_1 s_2)^k(\alpha_1)$ with $k < \frac{m}{2}$

$=$ rotation of α_1 through $k \cdot \frac{2\pi}{m}$ $(< \pi)$

Can check both β, β' lie in $\mathbb{R}_{\geq 0}\alpha_1 + \mathbb{R}_{\geq 0}\alpha_2$,

as desired. ∎