Parabolic subgroups and longest element

We've already encountered...

DEF'N: For $J \subseteq S$ in a Coxeter system (W,S), call the subgroup they generate $\langle J \rangle =: W_J$ a (standard) parabolic subgroup of W

PROP: (W_J, J) is in itself a Coxeter system. with expected Coxeter matrix/diagram $(m_{ij})_{i,j\in J}$.

proof: Build the Coxeter system (W', J) that has that expected matrix $(m_{ij})_{i,j\in J}$ and its own geom. rep'n. $W' \xrightarrow{\sigma'} GL(V')$ where $V' \cong \mathbb{R}^{\#J}$.
But then note that V' is isometric to the $\#J$-dim'l subspace $V_J := \mathrm{span}_{\mathbb{R}}\{\alpha_j\}_{j\in J} \subset V$ on which W_J acts faithfully, and one obtains an isomorphism

$$\begin{array}{ccc} W' & \overset{\sigma'}{\hookrightarrow} & GL(V') \\ \vdots\wr & & \| \wr \\ W_J & \overset{\sigma}{\hookrightarrow} & GL(V_J) \end{array}$$

∎

Let's also take care of our earlier worry about $\ell_J(w) \geq \ell(w)$:

PROPOSITION: When $w \in W_J$, $\ell_J(w) = \ell(w)$ $(= \ell_S(w))$.

proof: Starting with any word $w = s_{j_1} s_{j_2} \cdots s_{j_\ell}$ that only uses $s_j \in J$, the Deletion Condition lets us repeatedly omit letters to get to a reduced word (of length $\ell(w)$) still only using $s_j \in J$, ∎

COROLLARY: For $s_i \in S$, $s_i \in W_J \iff s_i \in J$

so the map

$$\begin{aligned} 2^S &\longrightarrow \{\text{standard parabolic subgroups}\} \\ J &\longmapsto W_J \end{aligned}$$

is a bijection.

(In particular, S minimally generates W.)

proof: If $s_i \in W_J$, then

$$1 = \ell(s_i) = \ell_J(s_i) \Rightarrow s_i \in J \quad ∎$$

A ubiquitous tool that we've already encountered:

THEOREM (Parabolic factorization)

Given a Coxeter system (W,S) and $J \subseteq S$, every $w \in W$ has a unique factorization
$$w = v \cdot u$$
where (a) $u \in W_J$

and (b) $v \in W^J \overset{\text{DEFN}}{:=} \{v \in W : \ell(vs) > \ell(v) \ \forall s \in J\}$.

Furthermore it satisfies...

(c) $\ell(w) = \ell(v) + \ell(u)$ (length-additive)

(d) $\ell(v) = \min\{\ell(w') : w' \in wW_J\}$

(e) v is the unique element of the coset wW_J achieving this minimum length.

REMARK: As usual, $w \leftrightarrow w^{-1}$ gives this version.

THM: Every $w \in W$ has a unique factorization
$$w = u \cdot v$$
with (a) $\ell(w) = \ell(u) + \ell(v)$

(b) $u \in W_J$

(c) $v \in {}^JW := \{v \in W : \ell(sv) > \ell(v) \ \forall s \in J\}$

= min length coset reps for $W_J w$

Before proving it, let's understand it for...

EXAMPLE $\mathfrak{S}_n = W\left(\underset{s_1}{\circ} - \underset{s_2}{\circ} - \cdots - \underset{s_{n-2}}{\circ} - \underset{s_{n-1}}{\circ} \right)$

$$\{ \underset{s_1}{(12)}, \underset{s_2}{(23)}, \ldots, \underset{s_{n-1}}{(n-1,n)} \} = S$$

For $J \subseteq S$, $W_J = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_m} =: \mathfrak{S}_\alpha$ a Young subgroup

for a composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of n

e.g. $\mathfrak{S}_n = W\left(\underset{(12)}{\overset{s_1}{\circ}} - \underset{(23)}{\overset{s_2}{\circ}} - \cancel{\underset{(34)}{\overset{s_3}{\circ}}} - \underset{(45)}{\overset{s_4}{\circ}} - \underset{(56)}{\overset{s_5}{\circ}} - \underset{(67)}{\overset{s_6}{\circ}} - \cancel{\underset{(78)}{\overset{s_7}{\circ}}} - \underset{(89)}{\overset{s_8}{\circ}} \right) = W$

$J = \{s_1, s_2, s_4, s_5, s_6, s_8\}$

$\mathfrak{S}_{(3,4,2)} = W\left(\underset{(12)}{\overset{s_1}{\circ}} - \underset{(23)}{\overset{s_2}{\circ}} \quad \underset{(45)}{\overset{s_4}{\circ}} - \underset{(56)}{\overset{s_5}{\circ}} - \underset{(67)}{\overset{s_6}{\circ}} \quad \underset{(89)}{\overset{s_8}{\circ}} \right) = W_J$

$= \mathfrak{S}_3 \times \mathfrak{S}_4 \times \mathfrak{S}_2$

$= \mathfrak{S}_{\{1,2,3\}} \times \mathfrak{S}_{\{4,5,6,7\}} \times \mathfrak{S}_{\{8,9\}}$

Parabolic factorizations of $w = \begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 7&1&9&2&8&5&4&6&3 \end{pmatrix}$:

$$w = \left(\begin{array}{ccc|cccc|cc} 1&2&3&4&5&6&7&8&9 \\ 7&1&9&2&8&5&4&6&3 \end{array}\right) = \left(\begin{array}{ccc|cccc|cc} 1&2&3&4&5&6&7&8&9 \\ 1&7&9&2&4&5&8&3&6 \end{array}\right) \left(\begin{array}{ccc|cccc|cc} 1&2&3&4&5&6&7&8&9 \\ 2&1&3&4&7&6&5&9&8 \end{array}\right)$$

sorted between bars

$= v \cdot u$ in $W^J \cdot W_J$

$$w = \begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 7&1&9&2&8&5&4&6&3 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 1&2&3&7&5&4&6&9&8 \end{pmatrix} \begin{pmatrix} 1&2&3&4&5&6&7&8&9 \\ 4&1&8&2&9&5&6&7&3 \end{pmatrix}$$

$= u \cdot v$ in $W_J \cdot {}^J W$

a shuffle of increasing alphabets $123 \uplus 4567 \uplus 89$

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Furthermore it satisfies...

(c) $\ell(w) = \ell(v) + \ell(u)$ (length-additive)

(d) $\ell(v) = \min\{\ell(w') : w' \in wW_J\}$

(e) v is the unique element of the coset wW_J achieving this minimum length

proof: Existence: Pick any $v \in wW_J$ of minimum length,
so $\ell(vs) > \ell(v)\ \forall s \in J$ and hence $v \in W^J$.
Then $w = v \cdot u$ for some $u \in W_J$.

To show (c), note $\ell(w) \leq \ell(v) + \ell(u)$. But we claim
this must be an equality: otherwise,
write $\begin{cases} v = s_{i_1} s_{i_2} \cdots s_{i_{\ell(v)}} & \text{reduced} \\ u = s_{j_1} s_{j_2} \cdots s_{j_{\ell(u)}} & \text{reduced and } s_j \in J, \end{cases}$
and Deletion Condition says we can omit two
generators from
$$w = s_{i_1} s_{i_2} \cdots s_{i_{\ell(v)}} \cdot s_{j_1} s_{j_2} \cdots s_{j_{\ell(u)}}$$
to shorten it.
Neither one can come from v, by our choice of v,
so both would come from u,
contradicting $u = s_{j_1} s_{j_2} \cdots s_{j_{\ell(u)}}$ reduced.

Uniqueness: If we had another such factorization
$w = v' \cdot u'$ with $u' \in W_J$
$v' \in W^J$

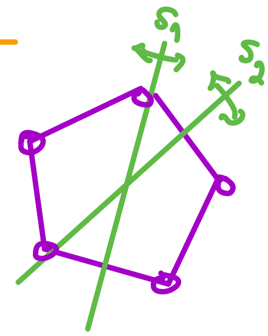
then $v' \in wW_J$ shows $v' = v \cdot u''$ with $u'' \in W_J$
and v achieving min length in wW_J from before,
and $\ell(v') = \ell(v) + \ell(u'')$, from the proof of (c).
But then $\ell(v's) > \ell(v') \ \forall s \in J$
forces $u'' = 1$: else pick $s \in J$ with $\ell(u''s) < \ell(u'')$
and get the contradiction
$$\ell(v's) = \ell(vu''s) \le \ell(v) + \ell(u''s)$$
$$< \ell(v) + \ell(u'') = \ell(v').$$

CONCLUSION: $v' = v$,
and every $v \in W^J$ is the unique
element of vW_J achieving the
minimum length in that coset. ∎

The uniqueness of length additive parabolic
factorization $W = W^J \cdot W_J$
$w = v \cdot u$
$$\ell(w) = \ell(v) + \ell(u)$$
has consequences for length-generating functions...

DEF'N: Given a Cox. system (W,S) and $A \subseteq W$, define $A(q) := \sum_{w \in A} q^{\ell(w)} \in \mathbb{Z}[[q]]$

EXAMPLES: $W(q) = \sum_{w \in W} q^{\ell(w)}$

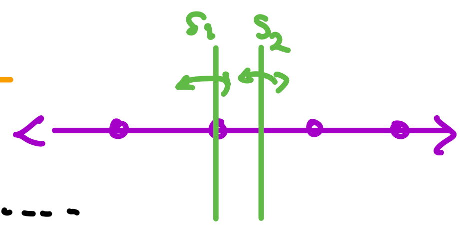


(1) For $W = W\left(\overset{m}{\underset{s_1}{\circ}\!\!-\!\!\underset{s_2}{\circ}}\right) = I_2(m)$,

If $m < \infty$, one has

$$\begin{array}{r|c|c|c|c|c|c|c} W = \{ & 1, & s_1, & s_1s_2, & s_1s_2s_1, & \cdots, & s_1s_2s_1\cdots & s_1s_2s_1\cdots \quad \} \\ & & s_2, & s_2s_1, & s_2s_1s_2, & \cdots & s_2s_1s_2\cdots & (=s_2s_1s_1\cdots) \\ \ell(w) = & 0 & 1 & 2 & 3 & \cdots & m-1 & m \end{array}$$

$$\Rightarrow W(q) = q^0 + 2q^1 + 2q^2 + 2q^3 + \dots + 2q^{m-1} + q^m$$
$$= (1+q)(1+q+q^2+\dots+q^{m-1})$$
$$= [2]_q [m]_q$$



If $m = \infty$, one has

$$W(q) = 1 + 2q + 2q^2 + 2q^3 + \dots$$
$$= 1 + 2\frac{q}{1-q} = \frac{1-q+2q}{1-q} = \frac{1+q}{1-q}$$

(2) For $W = W\left(\underset{s_1}{\circ} \overset{3}{—} \underset{s_2}{\circ} \overset{3}{—} \cdots \overset{3}{—} \underset{s_{n-1}}{\circ}\right) = \mathfrak{S}_n$,

$$W(q) = \sum_{w \in W} q^{\ell(w)} = \sum_{w \in \mathfrak{S}_n} q^{\mathrm{inv}(w)}$$

CLAIM
$$= (1+q+q^2+\dots+q^{n-1}) \sum_{\hat{w} \in \mathfrak{S}_{n-1}} q^{\mathrm{inv}(\hat{w})} = [n]_q \cdot \hat{W}(q)$$

where $\hat{W} = \mathfrak{S}_{n-1}$

since for each $\hat{w} \in \mathfrak{S}_{n-1}$, can add new letter n before $n-1$ letters, creating $n-1$ inversions

or $n-2$	$n-2$
$\vdots$	$\vdots$
2	2
1	1
0	0

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e.g. $n=4$

In $\mathfrak{S}_3$, $\hat{w} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$\mathrm{inv}(\hat{w}) = 2$

w in $\mathfrak{S}_4$	$\mathrm{inv}(w)$
4 3 1 2	2+3=5
3 4 1 2	2+2=4
3 1 4 2	2+1=3
3 1 2 4	2+0=2

COROLLARY: $W(q) = [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q =: [n]!_q$ for $W = \mathfrak{S}_n$

(by induction on n)

PROPOSITION: For (W,S) and $J \subseteq S$

one has $$W(q) = W^J(q) \cdot W_J(q)$$

or equivalently $$W^J(q) = \frac{W(q)}{W_J(q)}$$

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proof: Comes from unique length-additive factorization

$$w = v \cdot u$$
$$\ell(w) = \ell(v) + \ell(u) \quad \text{with } v \in W^J,\ u \in W_J$$

$$\Rightarrow W(q) = \sum_{w \in W} q^{\ell(w)} = \sum_{\substack{(v,u) \\ \in W^J \times W_J}} q^{\ell(v)+\ell(u)} = \sum_{v \in W^J} q^{\ell(v)} \sum_{u \in W_J} q^{\ell(u)}$$

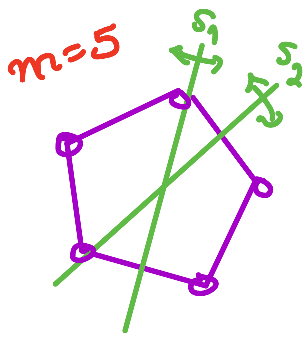
$$= W^J(q) \cdot W_J(q). \blacksquare$$

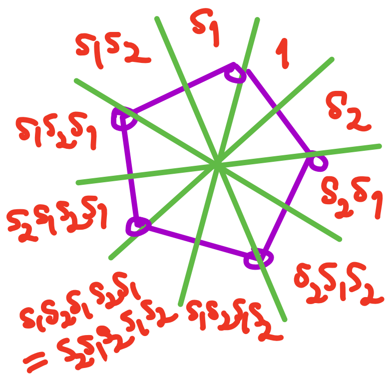
EXAMPLES (1) For $W\left(\underset{s_1}{\circ} \overset{m}{—} \underset{s_2}{\circ}\right) = I_2(m)$, take $J = \{s_1\}$

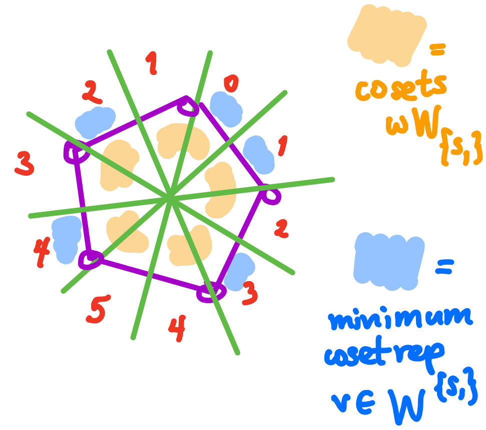
$$W(q) = [2]_q [m]_q \quad \text{if } m < \infty$$

$$= (1+q)(1+q+q^2+ \cdots + q^{m-1})$$

$$= W_{\{s_1\}}(q) \cdot W^{\{s_1\}}(q)$$

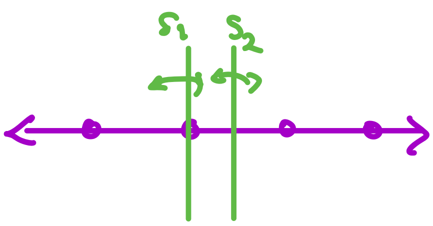


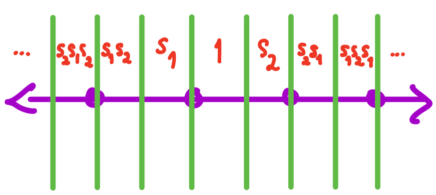


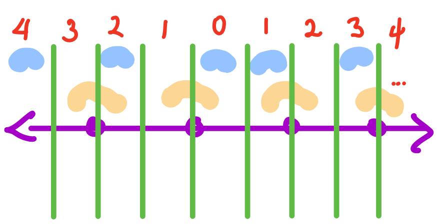


$W_{\{s_1\}} = \{1, s_1\}$ $\qquad$ $W^{\{s_1\}} = \{1, s_2, s_1s_2, s_2s_1s_2, \dots\}$

(and same when $m=\infty$, $W(q) = \frac{1+q}{1-q} = (1+q)(1+q+q^2+\dots)$)







(2) When $W = W\left(\underset{s_1}{\circ}\overset{3}{-}\underset{s_2}{\circ}\overset{3}{-}\cdots\overset{3}{-}\underset{s_{n-1}}{\circ}\right) = \mathfrak{S}_n$, we saw any $J \subseteq S$ corresponds to $W_J = \mathfrak{S}_\alpha = \mathfrak{S}_{\alpha_1} \times \mathfrak{S}_{\alpha_2} \times \cdots \times \mathfrak{S}_{\alpha_m}$ for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ some composition of n.

Then $W(q) = \mathfrak{S}_n(q) = [n]!_q$

$$W_J(q) = \mathfrak{S}_{\alpha_1}(q) \cdots \mathfrak{S}_{\alpha_m}(q) = [\alpha_1]!_q [\alpha_2]!_q \cdots [\alpha_m]!_q$$

and
$$W^J(q) = \frac{W(q)}{W_J(q)} = \frac{[n]!_q}{[\alpha_1]!_q \cdots [\alpha_m]!_q} = \begin{bmatrix} n \\ \alpha_1 \; \alpha_2 \; \cdots \; \alpha_m \end{bmatrix}_q$$

q-multinomial coefficient

$$W^J(q) = {}^J W(q) = \sum_{\substack{\text{shuffles } w \text{ of} \\ \text{alphabets} \\ 1<2<\cdots<\alpha_1, \\ \alpha_1+1<\alpha_1+2<\cdots<\alpha_1+\alpha_2, \\ \vdots}} q^{\mathrm{inv}(w)} = \sum_{\substack{\text{multiset shuffles } w \text{ of} \\ \alpha_1 \; 1\text{'s} \\ \alpha_2 \; 2\text{'s} \\ \alpha_3 \; 3\text{'s} \\ \vdots}} q^{\mathrm{inv}(w)}$$

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e.g. $n=4$ $\quad \alpha = (2,2)$

$w \in {}^J W$	shuffle of 1122	$\mathrm{inv}(w)$
1234	1122	0
1324	1212	1
1342	1221	2
3124	2112	2
3142	2121	3
3412	2211	4

$$1 + q + 2q^2 + q^3 + q^4$$

$\overset{s_1}{\circ}-\overset{s_2}{\circ}-\overset{s_3}{\circ}$ $\quad W = \mathfrak{S}_4$

$\circ \qquad \circ \quad W_J = \mathfrak{S}_2 \times \mathfrak{S}_2$

$$W^J(q) = {}^J W(q) = \frac{[4]!_q}{[2]!_q [2]!_q} = \begin{bmatrix} 4 \\ 2,2 \end{bmatrix}_q$$

$$= \frac{[4]_q [3]_q}{[2]_q [1]_q} = \frac{(1+q+q^2+q^3)(1+q+q^2)}{(1+q)(1)}$$

$$= (1+q^2)(1+q+q^2)$$

$$= 1 + q + 2q + q^3 + q^4$$

Conjugacy of simple systems & longest element

Recall that $\Pi = \{\alpha_1, -, \alpha_n\} \subset \Phi$

(i) are a basis for V and

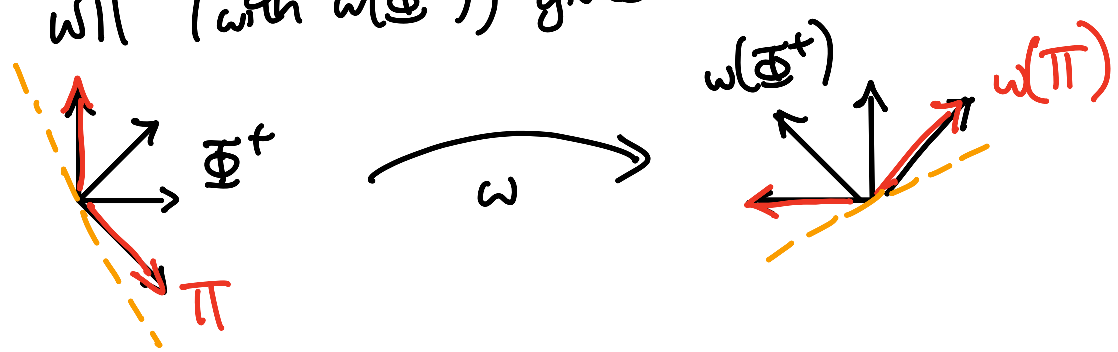
(ii) $\Phi^+ := \mathbb{R}_{\geq 0} \cdot \Pi \cap \Phi$
has $\Phi = \Phi^+ \sqcup -\Phi^+$ disjoint union

DEF'N: Call any subset $\Pi \subset \Phi$ satisfying (i), (ii) a simple system for Φ (and Φ^+ its associated positive system)

NOTE: Π uniquely determines Φ^+ by (ii), but also Φ^+ uniquely determines Π by

$$\Pi = \left\{ \beta \in \Phi^+ : \beta \neq \sum_{i=1}^{m} c_i \beta_i \text{ with } \beta_i \in \Phi^+ \setminus \{\beta\} \text{ and at least two } c_i > 0 \right\}$$

Simple systems are not unique in general, since $\forall w \in W$, $w\Pi$ (with $w(\Phi^+)$) gives another one

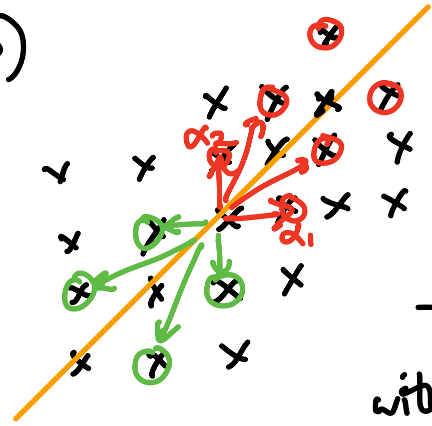


When W is finite, there are no others:

PROPOSITION: A finite W acts simply transitively on the simple systems $\Pi \subset \Phi$, i.e. any two Π, Π' have a unique $w \in W$ with $w(\Pi) = \Pi'$.

REMARK: False if $\# W = \infty$, e.g.

$W = I_2(\infty)$



Π, Φ^+

$\Pi', (\Phi^+)'$
$\| \quad \|$
$-\Pi \quad -\Phi^+$

There is no $w \in W$ with $w(\Pi) = \Pi'$.

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proof: For transitivity, given Π, Π'

induct on $\# \Phi^+ \cap -(\Phi^+)'$ ($< \infty$ since W is finite)

BASE CASE: $\# \Phi^+ \cap -(\Phi^+)' = 0$

$\Rightarrow \Phi^+ = (\Phi^+)'$

$\Rightarrow \Pi = \Pi'$

INDUCTIVE STEP: If $\#\Phi^+ \cap -(\Phi^+)' =: r > 0$,

then $\exists$ some $\alpha_i \in \Pi \cap -(\Phi^+)'$.

But then $s_i(\Phi^+ \setminus \{\alpha_i\}) = \Phi^+ \setminus \{\alpha_i\}$

and $s_i(\alpha_i) = -\alpha_i \in (\Phi^+)'$,

so $s_i(\Phi^+) \cap -(\Phi^+)' = r-1$

and by induction, $\exists\, w \in W$ with $w s_i(\Phi^+) = (\Phi^+)'$.

Simple transitivity then follows because

if $w\Pi = \Pi$ then $w(\alpha_i) \in \Pi \subset \Phi^+ \;\; \forall \alpha_i \in \Pi$

$\Rightarrow \ell(ws_i) > 1 \;\; \forall s_i \in S$

$\Rightarrow w = 1$ ▨

~~DEF'N~~ COROLLARY: In a finite ref'n group W or in any Coxeter system (W,S) with W finite,

$\exists$ a unique element, called the longest element w_0, characterized by any of these properties:

(a) $w\Pi = -\Pi$

(b) $w\Phi^+ = \Phi^-$

(c) $N(w) = \Phi^+$

(d) $\ell(w) = |\Phi^+| = |T|$

(e) $\ell(ws_i) < \ell(w) \;\; \forall s_i \in S$

(f) $w(\alpha_i) \in \Phi^- \;\; \forall \alpha_i \in \Pi$ i.e. $w(\Pi) \subset \Phi^-$

proof: Since $-\Pi$ is another simple system, simple transitivity of W shows $\exists$ a unique $w_0 \in W$ with (a) $w\Pi = -\Pi$.

Then check (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f) $\Rightarrow$ (b)

are all pretty easy. $\blacksquare$

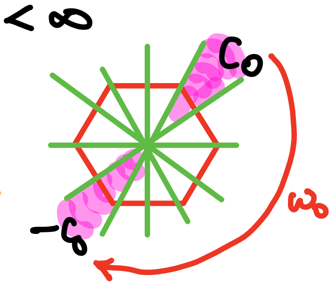
EXAMPLES

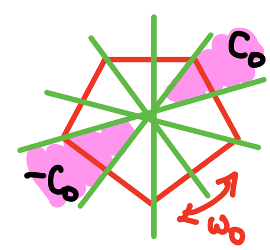
(1) $I_2(m) = W\left(\underset{s_1}{\circ} \overset{m}{\text{———}} \underset{s_2}{\circ} \right)$ has $w_0 = \underbrace{s_1 s_2 s_1 \cdots}_{m} = \underbrace{s_2 s_1 s_2 \cdots}_{m}$

$= \{1, s_1, s_2, s_1s_2, s_2s_1, \dots, w_0\}$ if $m < \infty$

$\ell(w_0) = m = |T| = |\Phi^+|$

$w_0 = \begin{cases} \text{rotation through } 180^\circ & \text{if } m \text{ even} \\ \text{a reflection} & \text{if } m \text{ odd} \end{cases}$





(and if $m = \infty$, $I_2(\infty)$ contains no w_0)

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(2) $\mathfrak{S}_n = W\left(\underset{s_1}{\circ} \overset{3}{—} \underset{s_2}{\circ} \overset{3}{—} \cdots \overset{3}{—} \underset{s_{n-1}}{\circ} \right)$

has $w_0 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ n & n-1 & n-2 & \cdots & 2 & 1 \end{pmatrix}$ with $\ell(w_0) = \mathrm{inv}(w_0) = |T| = |\Phi^+|$

$= \#\{(i,j) : 1 \le i < j \le n\} = \binom{n}{2}$

Having an element like $w_0 \in W$ actually characterizes the finite case:

PROPOSITION: For a Cox. sys. (W,S)

$\exists\, w \in W$ with $w(\Pi) = -\Pi \iff W$ is finite.

proof: We showed $(\Leftarrow)$ already.

For $(\Rightarrow)$, $w(\Pi) = -\Pi \Rightarrow w(\Phi^+) = -\Phi^+$

$\Rightarrow N(w) = \Phi^+$

$\Rightarrow \Phi^+$ is finite

$\Rightarrow \Phi$ is finite

$\Rightarrow \mathfrak{S}_\Phi$ is a finite group

But the permutation rep'n
$$\begin{aligned} W &\longrightarrow \mathfrak{S}_\Phi \quad \text{on roots} \\ w &\longmapsto \left(\beta \overset{w}{\longmapsto} w(\beta)\right) \end{aligned}$$
is faithful (injective): if $w(\beta) = \beta \;\forall \beta \in \Phi$

then $w(\alpha_i) = \alpha_i \;\forall \alpha_i \in \Pi$

so $w = 1$.

Hence $\# W \leq |\mathfrak{S}_\Phi| = |\Phi|!$ (finite) $\blacksquare$