Computing $W(q)$

We'll put this together to get a recursion that helps compute $W(q) = \sum_{w\in W} q^{\ell(w)}$.

1st introduce descent sets for (W,S)

DEF'N: Given (W,S) and $w \in W$

$$D_R(w) := \{ s\in S : \underbrace{\ell(ws) < \ell(w)}_{=\ell(w)-1} \} \quad \text{right descent set of } w$$
$$= S \cap T_R(w)$$

(and, of course,

$$D_L(w) := \{ s\in S : \underbrace{\ell(sw) < \ell(w)}_{=\ell(w)-1} \} \quad \text{left descent set of } w$$
$$= S \cap T_L(w)\ )$$

EXAMPLE $\mathfrak{S}_n = W\big(\overset{3}{\underset{s_1}{\circ}} \!-\! \overset{3}{\underset{s_2}{\circ}} \cdots \overset{3}{\underset{s_{n-1}}{\circ}} \big)$

has $D_R(w) = \{ s_i : w_i > w_{i+1} \} \leftrightarrow$ descent set $\mathrm{Des}(w) := \{ i : w_i > w_{i+1} \}$

$w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix}$

$D_L(w) = \{ s_i : w^{-1}_i < w^{-1}_{i+1} \} \leftrightarrow$ inverse descent set $\mathrm{Des}(w^{-1})$

$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$ has $D_R(w) = \{ \overset{s_1}{(12)}, \overset{s_3}{(34)} \}$

$(D_L(w) = \{ (23), (34) \})$

PROPOSITION: For any Cox. sys. (W,S),

$$\sum_{J\subseteq S} (-1)^{\#J} \frac{W(q)}{W_J(q)} = \begin{cases} q^{|T|} = q^{|\Phi^+|} = q^{\ell(w_0)} & \text{if } W \text{ is finite},\\ 0 & \text{if } W \text{ is infinite}.\end{cases}$$

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proof:

$$\sum_{J\subseteq S} (-1)^{\#J} \frac{W(q)}{W_J(q)} = \sum_{J\subseteq S} (-1)^{\#J} W^J(q)$$

Since $W^J := \{w\in W : \ell(ws) > \ell(w)\ \forall s\in J\}$
$= \{w\in W : D_R(w) \subseteq S\setminus J\}$,

can rewrite the RHS above as

$$\sum_{J\subseteq S} (-1)^{\#J} \sum_{\substack{w\in W:\\ D_R(w)\subseteq S\setminus J}} q^{\ell(w)} = \sum_{w\in W} q^{\ell(w)} \Bigg(\sum_{\substack{J\subseteq S:\\ D_R(w)\subseteq S\setminus J}} (-1)^{\#J} \Bigg)$$

$$\Updownarrow$$

$$S\setminus D_R(w) \supseteq J \supseteq \emptyset$$

EXERCISE: $\displaystyle\sum_{\emptyset\subseteq J\subseteq A} (-1)^{\#J} = \begin{cases} 1 & \text{if } A=\emptyset \\ 0 & \text{if } A\neq\emptyset \end{cases}$

So it equals

$$\sum_{\substack{w\in W:\\ S\setminus D_R(w)=\emptyset \\ \text{i.e. } D_R(w_0)=S}} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & \text{if } W \text{ finite}.\\ 0 & \text{if } W \text{ is infinite} \end{cases}$$

COROLLARY: $W(q)$ is a rational function, i.e. in $\mathbb{Q}(q)$

proof: Re-cast it as a recurrence on $\#S$:

$$\sum_{J \subseteq S} (-1)^{\#J} \frac{W(q)}{W_J(q)} = f_W(q) := \begin{cases} q^{|T|} & W \text{ finite} \\ 0 & W \text{ infinite} \end{cases}$$

$$\Updownarrow$$

$$\sum_{\emptyset \subseteq J \subsetneq S} (-1)^{\#J} \frac{1}{W_J(q)} = \frac{f_W(q) - (-1)^{\#S}}{W(q)}$$

i.e. $$W(q) = \frac{f_W(q) - (-1)^{\#S}}{\displaystyle\sum_{\emptyset \subseteq J \subsetneq S} \frac{1}{W_J(q)}} \Rightarrow W(q) \text{ rational} \quad \blacksquare$$

rational by induction on $\#J$

EXAMPLES

(1) $W = I_2(m) = W\left(\overset{}{\underset{s_1}{\circ}} \overset{m}{\text{—}} \underset{s_2}{\circ}\right)$ has

$$\frac{1}{W_\emptyset(q)} - \frac{1}{W_{\{s_1\}}(q)} - \frac{1}{W_{\{s_2\}}(q)} = \begin{cases} \dfrac{q^m - 1}{W(q)} & \text{if } m < \infty \\[2ex] \dfrac{-1}{W(q)} & \text{if } m = \infty \end{cases}$$

$$= \frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q}$$

$$= 1 - \frac{2}{1+q}$$

$$= \frac{1+q-2}{1+q} = \frac{q-1}{1+q}$$

So if $m<\infty$, get $W(q) = \dfrac{q^m - 1}{\frac{q-1}{1+q}} = [2]_q [m]_q$ ✓

and if $m = \infty$, get $W(q) = \dfrac{-1}{\frac{q-1}{1+q}} = \dfrac{1+q}{1-q}$ ✓

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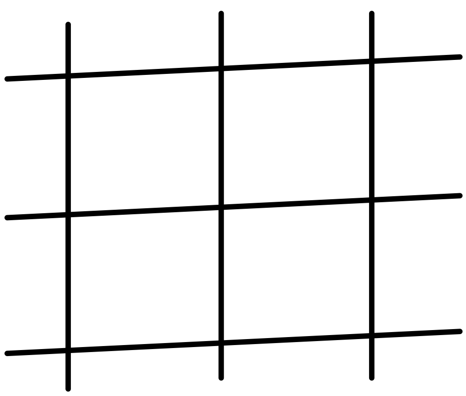
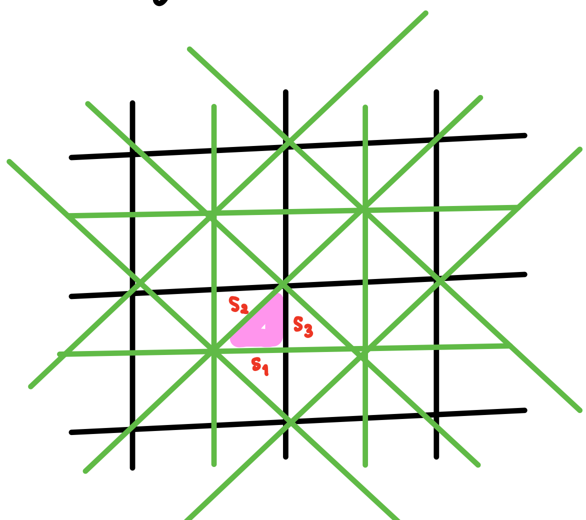
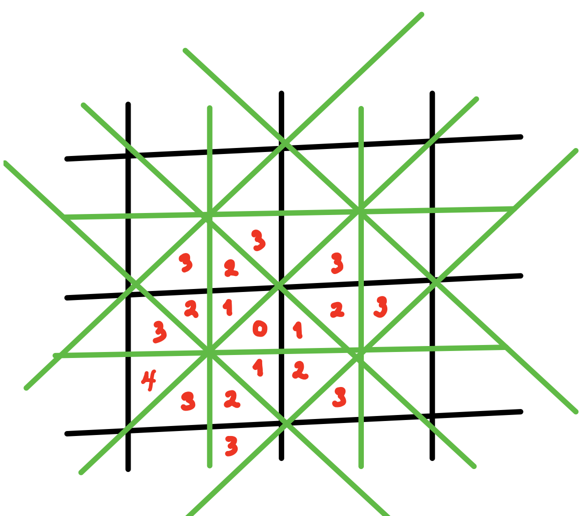
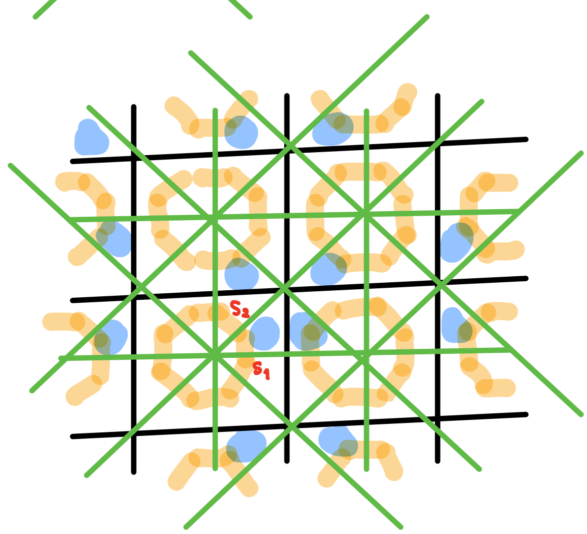
(2) $W =$ (affine) symmetries of square tessellation of $\mathbb{R}^2$
$= W(\overset{4}{\underset{s_1}{\circ}\!-\!\underset{s_2}{\circ}}\!\overset{4}{-}\!\underset{s_3}{\circ})$







$W(q) = 1 + 3q + 5q^2 + 8q^3 + \ldots = W_{\{s_1,s_2\}}(q) \cdot W^{\{s_1,s_2\}}(q)$
$= [2]_q [4]_q \cdot (?)$



= $W^{\{s_1,s_2\}}$ elements

= cosets $wW_{\{s_1,s_2\}}$

$$\frac{1}{W_{\emptyset}(q)} - \left(\frac{1}{W_{\{s_1\}}(q)} + \frac{1}{W_{\{s_2\}}(q)} + \frac{1}{W_{\{s_3\}}(q)}\right) + \left(\frac{1}{W_{\{s_1,s_2\}}(q)} + \frac{1}{W_{\{s_1,s_3\}}(q)} + \frac{1}{W_{\{s_2,s_3\}}(q)}\right) = \frac{(-1)^2}{W(q)}$$

$$1 - \frac{3}{1+q} + \frac{1}{[2]_q[4]_q} + \frac{1}{[2]_q[2]_q} + \frac{1}{[2]_q[4]_q} = \frac{1}{W(q)}$$

$$1 - \frac{3}{[2]_q} + \frac{2}{[2]_q[4]_q} + \frac{1}{[2]_q[2]_q} = \frac{1}{W(q)}$$

$$\frac{1}{[2]_q[4]_q}\left(\underbrace{[2]_q[4]_q - 3[4]_q + 2 + (1+q^2)}\right) = \frac{1}{W(q)}$$

$$\begin{array}{l}
1+2q+2q^2+2q^3+q^4 \\
-3-3q-3q^2-3q^3 \\
+2 \\
+1 \qquad +q^2 \\
\hline
1 - q - q^3 + q^4 = (1-q)(1-q^3)
\end{array}$$

$$\frac{(1-q)(1-q^3)}{[2]_q[4]_q} = \frac{1}{W(q)}$$

$$W(q) = \frac{[2]_q[4]_q}{(1-q)(1-q^3)}$$

REMARK: In our overview, we mentioned that if W has fundamental degrees $d_1, \ldots, d_n$ of $f_1, \ldots, f_n$ in $\mathbb{C}[x_1, \ldots, x_n]^W = \mathbb{C}[f_1, \ldots, f_n]$, then...

THEOREM (Chevalley, Solomon)

$$W(q) = [d_1]_q [d_2]_q \cdots [d_n]_q$$

which we'll prove by comparing the above recurrence for $W(q)$ with some invariant theory.

But there is also ...

THEOREM (Bott) If $\widetilde{W}$ is the affine Coxeter group associated to W, then

$$\widetilde{W}(q) = \frac{[d_1]_q \cdots [d_n]_q}{(1-q^{d_1-1}) \cdots (1-q^{d_n-1})}$$

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e.g. $\widetilde{W} = I_2(\infty)$ if $W = \mathfrak{S}_2$

$s_1 \overset{\infty}{—} s_2$ $\qquad$ s_1

$$\widetilde{W}(q) = \frac{1+q}{1-q} = \frac{[2]_q}{1-q^1}$$

$\widetilde{W} = s_1 \overset{4}{—} s_2 \overset{4}{—} s_3$ if $W = s_1 \overset{4}{—} s_2$

$$\widetilde{W}(q) = \frac{[2]_q [4]_q}{(1-q^1)(1-q^3)}$$

We won't prove it, but I linked notes on it to syllabus page.

One can refine this calculation of $W(q)$ to something much finer, that also records the right descents $D_R(w)$.

DEF'N: Given (W, S), $S = \{s_1, \ldots, s_n\}$, introduce variables $\underline{t} = (t_1, \ldots, t_n)$

and the length/descent set generating function

$$W(\underline{t}, q) \overset{\text{DEF}}{:=} \sum_{w \in W} q^{\ell(w)} \underbrace{\prod_{s_i \in D_R(w)} t_i}_{\text{call this } \underline{t}^{D_R(w)}} \in \mathbb{Z}[[q]][\underline{t}]$$

THEOREM:

$$W(\underline{t}, q) = \sum_{J \subseteq S} \underbrace{\underline{t}^J}_{\prod_{i \in J} t_i} \underbrace{(1-\underline{t})^{S \setminus J}}_{\prod_{i \in S \setminus J}(1-t_i)} \frac{W(q)}{W_{S \setminus J}(q)}$$

$$(\in \mathbb{Q}(q)[\underline{t}])$$

EXAMPLE $W = I_2(m)$, $m < \infty$

w	$\ell(w)$	$D_R(w)$
1	0	$\varnothing$
s_1	1	$\{s_1\}$
s_2	1	$\{s_2\}$
$s_2 s_1$	2	$\{s_1\}$
$s_1 s_2$	2	$\{s_2\}$
$\vdots$	$\vdots$	$\vdots$
$\cdots s_1 s_2 s_1$	$m-1$	$\{s_1\}$
$\cdots s_2 s_1 s_2$	$m-1$	$\{s_2\}$
w_0	m	$\{s_1, s_2\}$

$$W(\underline{t}, q) = 1 + (t_1 + t_2)(q + q^2 + \cdots + q^{m-1}) + t_1 t_2 q^m$$

According to the THEOREM, $W(\underline{t},q)$

$$= (1-t_1)(1-t_2)\frac{W(q)}{W_{\{s_1,s_2\}}(q)} + t_1(1-t_2)\frac{W(q)}{W_{\{s_2\}}(q)} + t_2(1-t_1)\frac{W(q)}{W_{\{s_1\}}(q)} + t_1t_2\frac{W(q)}{W_\phi(q)}$$

$J=\phi$ $\quad J=\{s_1\}$ $\quad J=\{s_2\}$ $\quad J=\{s_1,s_2\}$

$$= (1-t_1)(1-t_2)\cdot 1 + t_1(1-t_2)\frac{[2]_q[m]_q}{[2]_q} + t_2(1-t_1)\frac{[2]_q[m]_q}{[2]_q} + t_1t_2\frac{[2]_q[m]_q}{1}$$

$$= (1-t_1)(1-t_2) + \left(t_1(1-t_2)+t_2(1-t_1)\right)[m]_q + t_1t_2[2]_q[m]_q$$

$$= 1-(t_1+t_2)+t_1t_2 + (t_1+t_2-2t_1t_2)[m]_q + t_1t_2[2]_q[m]_q$$

$$= 1 + (t_1+t_2)(-1+[m]_q) + t_1t_2\left(1-2[m]_q + (1+q)[m]_q\right)$$

$$= 1 + (t_1+t_2)(q+q^2+\dots+q^{m-1}) + t_1t_2(\underbrace{1-[m]_q+q[m]_q}_{=q^m}) \quad ✓$$

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proof of THEOREM: Expand the RHS ...

$$\sum_{J\subseteq S}\underline{t}^J(1-\underline{t})^{S\setminus J}\frac{W(q)}{W_{S\setminus J}(q)} = \sum_{J\subseteq S}\underline{t}^J(1-\underline{t})^{S\setminus J}W^{S\setminus J}(q)$$

$$= \sum_{J\subseteq S}\underline{t}^J(1-\underline{t})^{S\setminus J}\sum_{\substack{w\in W:\\ D_R(w)\subseteq J}} q^{\ell(w)} = \sum_{w\in W} q^{\ell(w)}\sum_{J:D_R(w)\subseteq J\subseteq S}\underline{t}^J(1-\underline{t})^J$$

$$= \sum_{w\in W} q^{\ell(w)}\underline{t}^{D_R(w)}\prod_{i\in S\setminus D_R(w)}(t_i+(1-t_i))$$

$$= \sum_{w\in W} q^{\ell(w)}\underline{t}^{D_R(w)} = W(\underline{t},q) \quad ▨$$