Math 8680 Fall 2022
Solutions to selected exercises

Björner-Brenti #1.3

$$\begin{aligned} a_1 &= (12)(34) & \text{have } a_1a_2 &= (345) \text{ of order } 3 \\ a_2 &= (12)(45) & a_1a_3 &= (13)(24) \text{ of order } 2 \\ a_3 &= (14)(23) & a_2a_3 &= (15423) \text{ of order } 5 \end{aligned}$$

and $a_1^2 = a_2^2 = a_3^2 = 1$, so get a well-defined homomorphism

$$W(H_3) = W\big(\underset{s_1}{\circ}\!-\!\!-\!\underset{s_2}{\circ}\!-\!\!-\!\underset{s_3}{\circ}\big) \xrightarrow{\varphi} \langle a_1, a_2, a_3 \rangle \subset \mathfrak{S}_5$$

$$\begin{aligned} s_1 &\longmapsto a_1 \\ s_2 &\longmapsto a_2 \\ s_3 &\longmapsto a_3 \end{aligned}$$

Since each of a_1, a_2, a_3 lies in the alternating group $\mathfrak{A}_5$, which has cardinality $|\mathfrak{A}_5| = \frac{1}{2}|\mathfrak{S}_5| = \frac{5!}{2} = 60$,

$$\begin{aligned} \text{and } \langle a_1, a_2, a_3 \rangle &> \langle a_1 a_2 \rangle \quad \longleftarrow \text{size } 3 \\ &> \langle a_2 a_3 \rangle \quad \longleftarrow \text{size } 5 \\ &> \langle a_1, a_3 \rangle = \{1, a_1, a_3, a_1a_3\} \quad \longleftarrow \text{size } 4, \end{aligned}$$

one concludes $|\langle a_1, a_2, a_3 \rangle| \geq \operatorname{lcm}(3,4,5) = 60$

and hence φ surjects onto $\mathfrak{A}_5$.

On the other hand, recall $\mathfrak{A}_5$ is generated by its 3-cycles $\{(ijk)\}_{1\le i<j<k\le 5}$. So if we pick $\{w_{ijk}\} \in W(H_3)$ having $\varphi(w_{ijk}) = (ijk)$, then $\varphi(w_{ijk}^2) = (ijk)^2 = (ikj)$ also generate $\mathfrak{A}_5$.

Thus φ maps the subgroup $\langle \{w_{ijk}^2\} \rangle \twoheadrightarrow \mathfrak{A}_5$, and this subgroup lies inside the alternating subgroup $\mathfrak{A}(W(H_3))$, because $sgn(w_{ijk}^2) = sgn(w_{ijk})^2 = (\pm 1)^2 = +1$.

However $|\mathfrak{A}(W(H_3))| = \frac{1}{2}|W(H_3)| = \frac{1}{2}(120) = 60 = |\mathfrak{A}_5|$,

so φ maps $\mathfrak{A}(W(H_3)) \xrightarrow{\sim} \mathfrak{A}_5$ isomorphically.

Björner-Brenti #1.6

(a) $T = \{(i,j) : 1 \le i < j \le n\}$ = transpositions in $\mathfrak{S}_n$

Identifying $A \subseteq T$ with a subset of edges in the complete graph K_n on vertex set $\{1,2,\ldots,n\}$, one can see that $\langle A \rangle = \mathfrak{S}_n$ if and only if A connects the vertex set $\{1,2,\ldots,n\}$:

if the connected components of A are $V = \{1,2,\ldots,n\} = V_1 \times \ldots \times V_k$ for some $k \ge 2$,

then $\langle A \rangle \le \mathfrak{S}_{V_1} \times \ldots \times \mathfrak{S}_{V_k} \lneq \mathfrak{S}_n$,

while $\mathfrak{S}_n = \langle \mathfrak{S}_{\{1,2,\ldots,n-1\}}, (in) \rangle$ for any $1 \leq i \leq n-1$ shows that $\langle A \rangle = \mathfrak{S}_n$ if A connects $\{1,2,\ldots,n\}$.
But then the inclusion-minimal subsets A of edges of K_n that connect $\{1,2,\ldots,n\}$ are its spanning trees.

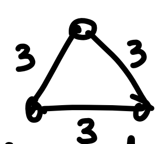
(b) If the tree A is linear, then by re-indexing it looks like $1 - 2 - 3 - 4 - \cdots - n$,

i.e. $A = \{ \underset{s_1}{\underset{\shortparallel}{(12)}}, \underset{s_2}{\underset{\shortparallel}{(23)}}, \ldots, \underset{s_{n-1}}{\underset{\shortparallel}{(n-1,n)}} \}$

which we know are Coxeter system (W,S) for $\mathfrak{S}_n$.

If the tree A is not linear, it has a vertex of degree ≥ 3, and so by re-indexing it contains $(1,2), (1,3), (1,4)$. If A gave a Coxeter system (W,S), then these three generators would give a parabolic subsystem $(W_J, \underset{\{(12),(13),(14)\}}{\underset{\shortparallel}{J}})$. But since

$$(12)(13) = (132)$$
$$(12)(14) = (142)$$
$$(13)(14) = (143)$$

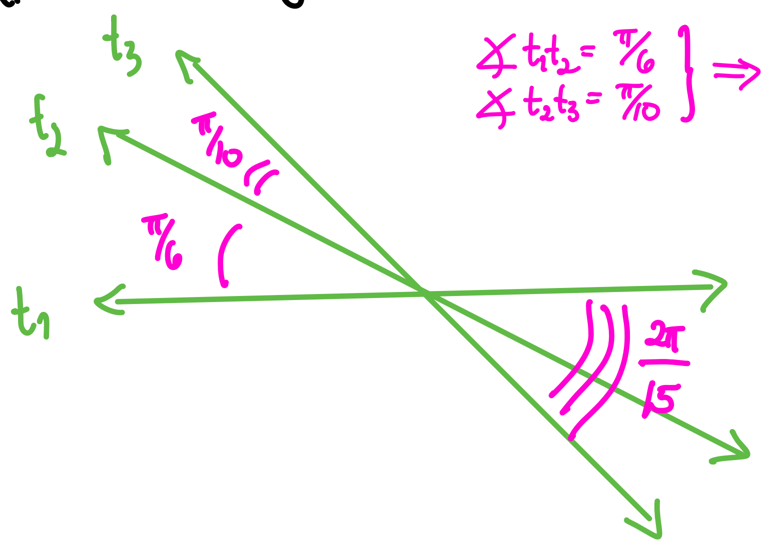
all have order 3, this $W_J = W\Big($  $\Big) = W(\tilde{A}_2)$, an infinite Coxeter group. This contradicts $W_J \leq \mathfrak{S}_n$.

(c) Inside a dihedral group like $I_2(\underbrace{30}_{\substack{\shortparallel \\ 2\cdot 3\cdot 5}}) = W\left(\underset{s_1}{\circ}\overset{30}{\text{——}}\underset{s_2}{\circ}\right)$

we know one can generate it by 2 reflections in T, e.g. s_1, s_2 whose refl'n lines have a dihedral angle of $\frac{\pi}{30}$.

Now take 3 refl'ns t_1, t_2, t_3 whose dihedral angles are as shown here:

$$\left.\begin{array}{l} \measuredangle t_1 t_2 = \pi/6 \\ \measuredangle t_2 t_3 = \pi/10 \end{array}\right\} \Rightarrow \measuredangle t_1 t_3 = \pi/6 + \pi/10 = \frac{5+3}{30}\pi = \frac{2\pi}{15}$$



Then we find that pairwise they all generate dihedral subgroups

$$\begin{aligned} \langle t_1, t_2 \rangle &\cong I_2(6) && \text{of size } 12 \\ \langle t_2, t_3 \rangle &\cong I_2(10) && \text{of size } 20 \\ \langle t_1, t_3 \rangle &\cong I_2(15) && \text{of size } 30 \end{aligned}$$

and so $|\langle t_1, t_2, t_3 \rangle| \geq \operatorname{lcm}(12, 20, 30) = 60 = |I_2(30)|$

$$\Rightarrow \langle t_1, t_2, t_3 \rangle = I_2(30)$$

with $A = \{t_1, t_2, t_3\} \subset T$ an inclusion-minimal generating set of 3 reflections, not 2.

Björner-Brenti #1.10.

Given $t \in T = \bigcup_{\substack{w \in W \\ s \in S}} wsw^{-1}$, find a palindromic reduced expression for t as follows. Start with any reduced expression $t = s_1 s_2 \cdots s_{\ell(t)}$ $(*)$.

Since $\ell(t \cdot t) = \ell(1) = 0 < \ell(t)$, $t \in T_L(t)$ and hence $\exists$ some $k \in \{1, 2, \ldots, \ell(t)\}$ with

$$t = \underbrace{s_1 s_2 \cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1}_{\text{palindromic, } 2k-1 \text{ letters total}} \qquad (**)$$

CASE 1: $k \leq \frac{\ell(t)+1}{2}$ i.e. $2k-1 \leq \ell(t)$

Then one must have equality $2k-1 = \ell(t)$ and $(**)$ must be reduced, so we're done.

CASE 2: $k > \frac{\ell(t)+1}{2}$ i.e. $2k-1 > \ell(t)$

Then $t = t^{-1} = s_{\ell(t)} \cdots s_{k+1} s_k s_{k-1} \cdots s_2 s_1$ from $(*)$

and $t = s_1 s_2 \cdots\cdots s_{k-1} s_k s_{k-1} \cdots s_2 s_1$ from $(**)$

$\Rightarrow s_1 s_2 \cdots s_{k-1} = s_{\ell(t)} \cdots s_{k+1}$

$\Rightarrow t = s_1 s_2 \cdots s_{k-1} s_k s_{k+1} \cdots s_{\ell(t)}$

$= \underbrace{s_{\ell(t)} \cdots s_{k+1} \cdot s_k s_{k+1} \cdots s_{\ell(t)}}_{\# \text{ of letters } = 2(\ell(t)-k)+1 = 2\ell(t)-(2k-1)}$

$< 2\ell(t) - \ell(t) = \ell(t)$

since $2k-1 > \ell(t)$

Contradiction. ↯

CRM-LACIM Spring School

EXERCISE #2

Want to show that a rational function $f(t) = \dfrac{1}{(1-t^{d_1})\cdots(1-t^{d_n})}$

with $d_1 \le d_2 \le \cdots \le d_n$ has the multiset uniquely determined, i.e. if $\dfrac{1}{\prod_{i=1}^{n}(1-t^{d_i})} \overset{(*)}{=} \dfrac{1}{\prod_{j=1}^{m}(1-t^{d_i'})}$

then $m=n$ and if $d_1 \le \cdots \le d_n$, $d_1' \le \cdots \le d_n'$, one has $d_i = d_i'$ $\forall i$.

Show this by induction on $\max\{n,m\}$.

Since $(*)$ implies $\prod_{i=1}^{n}(1-t^{d_i}) \overset{(**)}{=} \prod_{j=1}^{m}(1-t^{d_i'})$ in $\mathbb{Z}[t]$

one can use unique factorization into irreducibles in $\mathbb{Z}[t]$.

Recall the irreducible factorization for $1-t^d$ is

$$1-t^d = \prod_{\substack{\text{divisors} \\ e \text{ of } d}} \Phi_e(t) \qquad \text{where } \Phi_e(t) := e^{th} \text{ cyclotomic polynomial}$$

$$= \prod_{\substack{\text{primitive} \\ e^{th} \text{ roots } \zeta \text{ of } 1 \\ \text{in } \mathbb{C}^\times}} (t-\zeta)$$

Hence $(**)$ implies

$$d_n = d_n' = \max\{e : \Phi_e(t) \text{ divides either side of } (**)\}$$

Furthermore, the multiplicity μ of $d_n = d_n'$ in either list $d_1 \leq \cdots \leq d_n$ or $d_1' \leq \cdots \leq d_n'$ must be the same, since they are both the multiplicity μ of $\Phi_{d_n}(t) = \Phi_{d_n'}(t)$ as a factor on either side of $(**)$.

Now cancel these factors of $(1-t^{d_n})^\mu = (1-t^{d_n'})^\mu$ from both sides of $(**)$, and proceed by induction.