Math 4707
Inclusion-Exclusion and Derangements

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Suppose we have a set S of objects and a list of subsets of S, $A_1, A_2, \ldots, A_m$. We would like to count the number of elements of S that do not belong to any of these subsets. Sometimes the subsets are called *properties*, and we wish to count the elements of S with none of the properties. In set language, we wish to find

$$|A_1^c \cap A_2^c \cap \cdots \cap A_m^c| = |S| - |A_1 \cup A_2 \cup \cdots \cup A_m|$$

where A^c denotes the set complement (inside S) of A. (You should verify that the two sides of this equation are the same!)

Let's now introduce some notation. Suppose $T \subseteq \{1, 2, \ldots, m\}$. We think of T as a collection of properties. Define

$$A_T = \bigcap_{i \in T} A_i$$

That is, A_T is the set of elements in S having all the properties listed in T (and perhaps others). Also, define $A_\emptyset = S$.

Now let

$$S_k = \sum_{T:|T|=k} |A_T| \,.$$

Note that $S_0 = |S|$.

Theorem 1.

$$|A_1^c \cap A_2^c \cap \cdots \cap A_m^c| = \sum_T (-1)^{|T|} |A_T| = \sum_{j=0}^{m} (-1)^j S_j \,.$$

This theorem is called the *Principle of Inclusion-Exclusion* (PIE). It is the general form for the material in section 2.3 of the textbook.

Proof. Typical proofs of PIE use that fact that the alternating sum of binomial coefficients is 0. See, for instance, the Wikipedia article on the principle.

The proof given here relies on a device called a *sign-reversion involution.* Suppose X is a finite set of objects (not the S we've been talking about) and suppose each element of X has a "sign," that is, has been assigned the value $+1$ or -1 through a function ϵ. Now suppose ρ is a one-to-one, onto function from the elements of X to the elements of X. (Such functions are also known as permutations, but we wish to view them as functions here.) Furthermore, we suppose ρ has two important properties.

First, when ρ is repeated, the original element of X returns. That is, $\rho(\rho(x)) = x$ for any $x \in X$. Such permutations are called *involutions.* An involution will break the elements of X into two sets: the elements x such that $\rho(x) \neq x$ and the elements of x such that $\rho(x) = x$. Let's call the former X_M and the latter X_F. The latter are called the *fixed points* of the involution.

Second, if $x \in X_M$, then $\epsilon(\rho(x)) = -\epsilon(x)$. That is, ρ reverses the sign of any $x \in X_M$.

The following lemma should be immediately clear.

Lemma 1.

$$\sum_{x \in X_F} \epsilon(x) = \sum_{x \in X} \epsilon(x)\,.$$

In effect, the sign-reversing involution "kills off" all the elements of X_M.

As one example of a sign-reversiong involution, and an application of the preceding lemma, let X be all the subsets of $\{1, 2, \ldots, n\}$ $(n \geq 1)$ and let $\epsilon(A) = (-1)^{|A|}$. Let $\rho(A) = A'$ where $A' = A \cup \{1\}$ if $1 \notin A$ and $A' = A - \{1\}$ if $1 \in A$. It is easy to see that ρ is an involution–it puts 1 into the set if 1 is not in it and takes it out if it is in it. Doing this twice will return the original set. It is also easy to see that ρ is sign-reversing, since it changes the size of the subset by 1. Furthermore, ρ has no fixed points. Therefore the left hand side of the lemma is 0 and the right hand side is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\,.$$

Therefore it follows that the alternating sum of the binomial coefficients is 0.

In the case of the PIE, we need only dress this previous example up a bit. Let X be the set of pairs (T, a) where $T \subseteq \{1, 2, \ldots, m\}$ and $a \in A_T$. That is, T is a list of properties and a is an element of S that has all the properties in T. Let $\epsilon(T, a) = (-1)^{|T|}$. Notice that the sign only depends on T, not a. The RHS of the above lemma gives

$$\sum_{\substack{T \subseteq \{1,2,\ldots,m\} \\ a \in A_T}} \epsilon(T, a) = \sum_{T \subseteq \{1,2,\ldots,m\}} (-1)^{|T|} |A_T|\,,$$

which is the RHS of the PIE.

We now introduce a sign-reversing involution, ρ. Suppose $(T, a) \in X$ and suppose a has at least one property, that is, $a \in A_i$ for some i. Then define $\rho(T, a) = (T', a)$ where T' is constructed as follows. Let i be the largest index such that $a \in A_i$. That is, in the list of all the properties, $A_1, A_2, \ldots, A_m$, i is largest index such that a has property A_i.

There are then two cases: either $i \in T$ or $i \notin T$. In the former case, let $T' = T - \{i\}$. In the latter case, let $T' = T \cup \{i\}$. That is, if i is in T, take it out, and if it is not in T, put it in. Since $a \in A_i$, the resulting pair (T', a) is still in X. Furthermore, repeating this process results in the original element of X, (T, a). This is because a is unchanged, so the same i will obviously be selected. Finally, $\epsilon(\rho(T', a)) = -\epsilon(T, a)$ because T' will have either one more or one fewer element than T.

This defines ρ when a has at least one property. Now suppose $a \notin A_i$ for all i. Then if $a \in A_T$, it must be that $T = \emptyset$. For such pairs $(\emptyset, a)$, define $\rho(\emptyset, a) = (\emptyset, a)$, i. e., $(\emptyset, a)$ is a fixed point. Also $\epsilon(\emptyset, a) = +1$. But since a has none of the properties, the LHS of the lemma above is then

$$|A_1^c \cap A_2^c \cap \cdots \cap A_m^c|$$

which is the LHS of the PIE. □

The PIE has many beautiful applications. One is the problem of derangements.

Nine friends are having dinner at a local restaurant. Each orders something different. The server brings the dishes back to the table, and gives each friend one of the orders. But the server gets every order wrong! The 9 dishes are correct, but he gives an incorrect order to each patron.

What are the odds?

The enumeration problem here is counting *derangements*. Derangements are permutations where nothing is where it is supposed to be. Let's set up some notation. Let's let $1, 2, \ldots, n$ represent the friends at the restaurant (in the example, $n = 9$). Let's also let $1, 2, \ldots, n$ represent the dishes they ordered. Then let

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ a_1 & a_2 & a_3 & \ldots & a_n \end{pmatrix}$$

represent the server's distribution of the dishes to the patrons: friend 1 gets dish a_1, etc. The second row in this matrix will be a permutation of the numbers $1, 2, \ldots, n$. The entire matrix is also called a permutation *in two line notation.* A *fixed point* is a patron who gets the dish she ordered, that is, an i such that $a_i = i$. Although they are related, do not confuse these fixed points with the fixed points in the sign-reversing involutions discussed earlier.

A derangement has no fixed points. Let D_n be the number of such derangements.

Since we want to compute the probability that the server has created a derangement, we need to select an appropriate probability model. If we assume that the server assigns the dishes to the patrons completely at random, then each of the $n!$ permutations of the n dishes is equally likely. Our sample space has $n!$ equally likely sample points and the event we are interested in has D_n sample points. Therefore, the probability that the server gets all the dishes wrong is $D_n/n!$. We therefore concentrate on computing D_n.

As is typical with inclusion-exclusion problems, we decide what we want to eliminate. In this case, we wish to eliminate fixed points. Let S be the set of all $n!$ permutations and let A_i denote the permutations that fix i. We wish to count permutations that are not in any of the A_i, that is,

$$|A_1^c \cap A_2^c \cap \cdots \cap A_n^c| .$$

This is the LHS of the PIE. All that we need to do is compute S_k for each k. But if $T \subseteq \{1, 2, \ldots, n\}$, then $|A_T| = (n - |T|)!$ since any permutation in A_T must fix all of T. The number of subsets T of size k is $\binom{n}{k}$, so

$$S_k = \binom{n}{k}(n-k)!\,.$$

This formula holds even when $k = 0$ (check it out!). Therefore we have

Theorem 2.

$$D_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!$$

Then

$$D_n/n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(n-k)!/n! = \sum_{k=0}^{n} (-1)^k/k!\,.$$

This sum is the first $n+1$ terms of the Taylor series expansion of $1/e$. This series is known to converge quite rapidly.

Corollary 1. *The probability of a derangement is approximately* $1/e$*.*

Exercise 1. Using just the definition of derangements, show $D_{n+1} = n(D_n + D_{n-1})$, for $n > 0$.

Exercise 2. How many permutations of $1, 2, \ldots, n$ are there in which only the odd integers must be deranged (even integers may be in their own positions)?

Exercise 3. How many permutations of $1, 2, \ldots, n$ are there in which i is never immediately followed by $i+1$, $i = 1, 2, \ldots, n-1$? Show your answer is equal to $D_n + D_{n-1}$.

Exercise 4. How many ways are there to distribute r distinct coins into 5 distinct piggy banks such that no piggy bank is empty? Hint: Let A_i be the number of distributions in which piggy bank i is empty.

Exercise 5. How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 20\,,$$

where each x_i is an integer, $0 \le x_i \le 8$? Hint: Let A_i be the number of integer solutions such that $x_i > 8$.