Name: ______________________

Signature: ______________________

Section and TA: ______________________

Math 1271. Lecture 060 (V. Reiner) Midterm Exam I
Thursday, October 1, 2009

This is a 50 minute exam. No books, notes, calculators, cell phones or other elecronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
Total:	

Problem 1. *(30 points total) Compute the following limits, or indicate that they do not exist. It is important that you show your work. The answer alone is not sufficient.*

a. (7 points):

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{10-x}$$

b. (7 points):

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x}$$

c. (7 points):

$$\lim_{x\to 4\pi} \frac{3+\sin x}{1+\cos x}$$

d. (9 points total): *For the function defined by*

$$f(x) = \begin{cases} x^2+1 & \text{for } x \geq 0 \\ x^3 & \text{for } x < 0, \end{cases}$$

what are

(i) (3 points)

$$\lim_{x\to 0^+} f(x)$$

(ii) (3 points)

$$\lim_{x\to 0^-} f(x)$$

(iii) (3 points)

$$\lim_{x\to 0} f(x)$$

Problem 2. *(30 points) Let*

$$f(x) = 3x^2 + 30x.$$

a. (5 points): *Write down a limit that defines $f'(x)$, but do not evaluate it (yet).*

b. (10 points): *Compute $f'(x)$ by evaluating the limit from (a). Use only algebra and limit laws; do not use any derivative short-cuts that you may have learned elsewhere.*

c. (8 points): *Write down the equation for the tangent line to the graph of* $y = f(x)$ *at the point* $(1, f(1))$.

d. (7 points): *Find all values of* x_0 *for which the tangent line to the graph* $y = f(x)$ *at* $(x_0, f(x_0))$ *has horizontal slope.*

Problem 3. (10 points) *Prove (that is, explain convincingly why) the polynomial $f(x) = x^{33} + x^5 + 1$ must have at least one root x lying in the interval $[-1, 0]$, that is, at least one such value of x for which $f(x) = 0$. Do not bother trying to find or approximate such a root.*

Problem 4. (30 points) *Let*

$$f(x) = \frac{2x^2 - 8}{3x^2 - 27}$$

a. (4 points): *What is the natural domain of $f(x)$?*

b. (4 points): *For which values of x is $f(x)$ continuous?*

c. (4 points): *What is the y-intercept for $f(x)$, that is, the y-value for the point where the graph $y = f(x)$ intersects the y-axis?*

d. (4 points): *What are the x-intercepts for $f(x)$, if any? That is, what are the x-values for points where the graph $y = f(x)$ intersects the x-axis?*

e. (4 points): *Describe any lines which are vertical asymptotes for $y = f(x)$.*

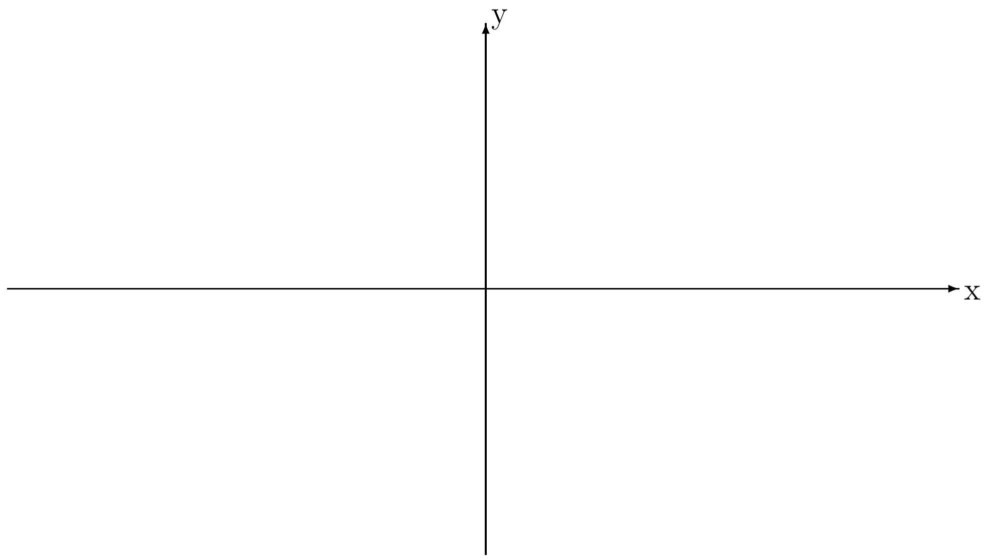

FIGURE 1. Axes for your sketch in part (g) of the graph $y = f(x) = \frac{2x^2-8}{3x^2-27}$

f. (4 points): *Compute* $\lim_{x\to+\infty} f(x)$ *and* $\lim_{x\to-\infty} f(x)$. *Then describe any lines which are horizontal asymptotes for* $y = f(x)$.

g. (6 points): *On the axes shown at the top of the page, draw a rough sketch of the graph* $y = f(x)$*, clearly indicating the features found in parts (c),(d),(e),(f).*

Brief solutions

1.

a. (7 points)

$$\lim_{x\to 9} \frac{\sqrt{x}-3}{10-x} = \frac{\sqrt{9}-3}{10-9} = 0$$

using quotient rule, difference rule, and the fact that x and $\sqrt{x}$ are continuous functions at $x = 9$.

b. (7 points)

$$\begin{aligned}\lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} &= \lim_{x\to 9} \frac{\sqrt{x}-3}{9-x} \cdot \frac{\sqrt{x}+3}{\sqrt{x}+3} \\ &= \lim_{x\to 9} \frac{x-9}{(9-x)\sqrt{x}+3} \\ &= \lim_{x\to 9} \frac{-1}{\sqrt{x}+3} \\ &= \frac{-1}{\sqrt{9}+3} = \frac{-1}{6}\end{aligned}$$

using in the second-to-last step the quotient, sum rules and continuity of $\sqrt{x}$ at $x = 9$.

c. (7 points)

$$\lim_{x\to 4\pi} \frac{3+\sin x}{1+\cos x} = \frac{3+\sin 4\pi}{1+\cos 4\pi} = \frac{3+0}{1+\cos 1} = \frac{3}{2}$$

using quotient and sum rules, along with continuity of $\sin(x)$, $\cos(x)$ at $x = 4\pi$.

d. (9 points total) For the function defined by

$$f(x) = \begin{cases} x^2+1 & \text{for } x \geq 0 \\ x^3 & \text{for } x < 0, \end{cases}$$

what are

(i) (3 points)

$$\lim_{x\to 0^+} f(x) = 0^2 + 1 = 1$$

by continuity of $x^2 + 1$.

(ii) (3 points)

$$\lim_{x\to 0^-} f(x) = 0^3 = 0$$

by continuity of x^3.

(iii) (3 points)

$$\lim_{x\to 0} f(x)$$

does not exist since the right- and left-hand limits don't agree.

2. Let

$$f(x) = 3x^2 + 30x.$$

a. (5 points) Write down a limit that defines $f'(x)$, but do not evaluate it (yet).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(3(x+h)^2 + 30(x+h)) - 3x^2 + 30x}{h} \end{aligned}$$

b. (10 points) Compute $f'(x)$ by evaluating the limit from(a).

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{(3(x+h)^2 + 30(x+h)) - (3x^2 + 30x)}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 + 30x + 30h - 3x^2 - 30x}{h} \\ &= \lim_{h\to 0} \frac{6xh + 3h^2 + 30h}{h} \\ &= \lim_{h\to 0} 6x + 3h + 30 = 6x + 30 \end{aligned}$$

where we've used the sum law and continuity of $3h$ at $h = 0$ in the last step.

c. (8 points) Write down the equation for the tangent line to the graph of $y = f(x)$ at the point $(1, f(1))$. The slope is $f'(1) = 6 \cdot 1 + 30 = 36$, and the point $(1, f(1)) = (1, 3 \cdot 1^2 + 30 \cdot 1) = (1, 33)$, so the point-slope formula says the line has equation

$$\frac{y - 33}{x - 1} = 36$$

or $y - 33 = 36(x - 1)$ or $y = 36x - 3$.

d. (7 points) Find all values of x_0 for which the tangent line to the graph $y = f(x)$ at $(x_0, f(x_0))$ has horizontal slope. Horizontal slope means $0 = f'(x) = 6x + 30$, that is $6x = -30$ or $x = -5$.

3. **(10 points)** Prove (that is, explain convincingly why) the polynomial $f(x) = x^{33} + x^5 + 1$ must have at least one root x lying in the interval $[-1, 0]$, that is, at least one such value of x for which $f(x) = 0$. Do not bother trying to find or approximate such a root.

Since $f(-1) = (-1)^{33} + (-1)^5 + 1 = -1 + (-1) + 1 = -1 < 0$ and $f(0) = 0^3 3 + 0^5 + 1 > 0$, and since $f(x)$ is a polynomial and therefore continuous everywhere, the Intermediate Value Theorem says that there exists at least one x in the interval $(-1, 0)$ for which $f(x) = 0$, that is, at least one root x.

4. **(30 points)** Let

$$f(x) = \frac{2x^2 - 8}{3x^2 - 27}$$

a. (4 points) What is the natural domain of $f(x)$?

Since f is a rational function, the natural domain is all x for which the denominator does not vanish, that is, $3x^2 - 27 \neq 0$. This means $3(x-3)(x+3) \neq 0$, so $x \neq \pm 3$. In interval notation, the domain is $(-\infty, -3) \cup (-3, +3) \cup (+3, +\infty)$.

b. (4 points) For which values of x is $f(x)$ continuous?

Since f is a rational function, it is continuous at all points in its natural domain, that is, $x \neq \pm 3$.

c. (4 points) What is the y-intercept for $f(x)$, that is, the y-value for the point where the graph $y = f(x)$ intersects the y-axis?

This is where $x = 0$, so $y = f(0) = \frac{2 \cdot 0^2 - 8}{3 \cdot 0^2 - 27} = \frac{-8}{-27} = \frac{8}{27}$

d. (4 points) What are the x-intercepts for $f(x)$, if any? That is, what are the x-values for points where the graph $y = f(x)$ intersects the x-axis?

This is where the numerator vanishes, but not the denominator. Here this means $2x^2 - 8 = 0$, i.e. $2(x-2)(x+2) = 0$, so $x = \pm 2$.

e. (4 points) Describe any lines which are vertical asymptotes for $y = f(x)$.

This is where the denominator vanishes, but not the numerator. Here this means $3x^2 - 27 = 0$, i.e. $3(x-3)(x+3) = 0$, so $x = \pm 3$.

f. (4 points) Compute $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. Then describe any lines which are horizontal asymptotes for $y = f(x)$.

$$\lim_{x \to \pm\infty} \frac{2x^2 - 8}{3x^2 - 27} = \lim_{x \to \pm\infty} \frac{2x^2 - 8}{3x^2 - 27} \cdot \frac{1/x^2}{1/x^2} = \lim_{x \to \pm\infty} \frac{2 - 8/x^2}{3 - 27/x^2} = \frac{2 - 0}{3 - 0} = \frac{2}{3}$$

where the second-to-last step used quotient and sum laws.

Since $\lim_{x \to \pm\infty} f(x) = \frac{2}{3}$, the horizontal line $y = \frac{2}{3}$ is the only horizontal asymptote to $y = f(x)$.

g. (6 points) On the axes shown at the top of the page, draw a rough sketch of the graph $y = f(x)$, clearly indicating the features found in parts (c),(d),(e),(f).

Here's what Maple's plotter gives:

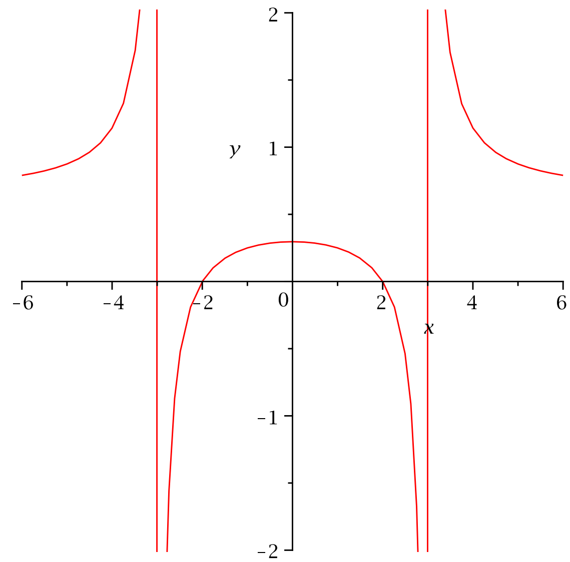

FIGURE 2. The graph of $y = f(x) = \frac{2x^2-8}{3x^2-27}$.