Name: ______________________

Signature: ______________________

Section and TA: ______________________

Math 1271. Lecture 060 (V. Reiner) Midterm Exam II
Thursday, October 29, 2009

This is a 50 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	______
2.	______
3.	______
4.	______
Total:	______

Problem 1. *(24 points total) Compute the following derivatives. Do not worry about simplifying your answer after the derivative has been computed.*

a. (8 points) $\frac{d}{dx}\left(\sqrt[3]{x^5} \cdot e^{x^2}\right)$

b. (8 points) $\frac{d}{dx}\left(\arcsin(x)^{2x}\right)$

c. (8 points total; 2 points each) *Assuming that*

$$f(10) = 1,\ g(10) = 2,\ f'(10) = 3,\ g'(10) = 4,\ f'(2) = 5,$$

compute $h'(10)$ *for each of these functions* $h(x)$*:*

$$h(x) = f(x) + 100g(x)$$

$$h(x) = f(x) \cdot g(x)$$

$$h(x) = \frac{f(x)}{g(x)}$$

$$h(x) = f(g(x))$$

Problem 2. *(25 points) The sides of a cube are growing in such a way that it always maintains a cubical shape (so all sidelengths are equal), but its volume is increasing by 2 cubic centimeters per second. At what rate is its sidelength growing when the volume is 1000 cubic centimeters?*

Problem 3. *(26 points) Consider the curve in the plane defined by the equation*

$$(x-1)^4 + y^4 = 16.$$

a. (10 points) *Write an expression for the slope* $\frac{dy}{dx}$ *of the tangent line to a point* (x, y) *on this curve, as a function of* x *and* y.

b. (8 points) *Find the* (x, y) *coordinates of both points on this curve where the tangent line to the curve is horizontal.*

c. (8 points) *Find the equation of the tangent line to the curve at the point* $(x, y) = (1 + \sqrt[4]{8}, \sqrt[4]{8})$.

Problem 4. *(25 points) Carbon-16 is radioactive, and decays at a rate proportional to how much is present, with a half-life of about 5730 years. How many kilograms will remain after a 1 kilogram sample has been allowed to decay for 1000 years?*

Note*: Since you are not allowed to use a calculator, do not provide an answer in decimals; rather you may leave functions like exponentials and logarithms in your answer.*

Brief solutions.

1. a. (8 points)

$$\frac{d}{dx}\left(\sqrt[3]{x^5}\cdot e^{x^2}\right) = \frac{d}{dx}\left(x^{\frac{5}{3}}\right)\cdot e^{x^2} + x^{\frac{5}{3}}\cdot\frac{d}{dx}e^{x^2} = \frac{5}{3}x^{\frac{2}{3}}\cdot e^{x^2} + x^{\frac{5}{3}}\cdot e^{x^2}\cdot 2x.$$

b. (8 points) In computing $\frac{d}{dx}\left(\arcsin(x)^{2x}\right)$, there were two ways that people interpreted $\arcsin(x)^{2x}$, and I accepted either answer, if they differentiated it correctly.

The first one is the way I intended:

$$\begin{aligned}
y &= (\arcsin(x))^{2x} \\
\ln(y) &= \ln(\arcsin(x))^{2x}) = 2x\ln(\arcsin(x)) \\
\frac{d}{dx}\ln(y) &= \frac{d}{dx}\left(2x\ln(\arcsin(x))\right) \\
\frac{1}{y}\frac{dy}{dx} &= 2\ln(\arcsin(x)) + 2x\cdot\frac{1}{\arcsin(x)}\frac{1}{\sqrt{1-x^2}} \\
\frac{dy}{dx} &= y\left(2\ln(\arcsin(x)) + 2x\cdot\frac{1}{\arcsin(x)}\frac{1}{\sqrt{1-x^2}}\right) \\
&= (\arcsin(x))^{2x}\left(2\ln(\arcsin(x)) + 2x\cdot\frac{1}{\arcsin(x)}\frac{1}{\sqrt{1-x^2}}\right)
\end{aligned}$$

The second way went like this:

$$\frac{d}{dx}\arcsin(x^{2x}) = \frac{1}{\sqrt{1-(x^{2x})^2}}\cdot\frac{d}{dx}(x^{2x})$$

and one uses logarithmic differentiation to compute that for $y = x^{2x}$, one has

$$\begin{aligned}
\ln(y) &= \ln(x^{2x}) = 2x\ln(x) \\
\frac{d}{dx}\ln(y) &= \frac{d}{dx}\left(2x\ln(x)\right) \\
\frac{1}{y}\frac{dy}{dx} &= 2\ln(x) + 2x\cdot\frac{1}{x} = 2\ln(x) + 2 \\
\frac{dy}{dx} &= y\left(2\ln(x) + 2\right) \\
\frac{dy}{dx} &= 2x^{2x}\left(\ln(x) + 1\right)
\end{aligned}$$

so plugging into the above calculation gives

$$\frac{d}{dx}\arcsin(x^{2x}) = \frac{1}{\sqrt{1-(x^{2x})^2}}\cdot 2x^{2x}\left(\ln(x)+1\right).$$

c. (8 points total; 2 points each)] Assuming that

$$f(10) = 1,\ g(10) = 2,\ f'(10) = 3,\ g'(10) = 4,\ f'(2) = 5,$$

compute $h'(10)$ for each of these functions $h(x)$:

$$\begin{aligned} h(x) &= f(x) + 100g(x) \\ h'(10) &= f'(10) + 100g'(10) = 3 + 100 \cdot 4 = 403 \end{aligned}$$

$$\begin{aligned} h(x) &= f(x) \cdot g(x) \\ h'(10) &= f'(10)g(10) + f(10)g'(10) = 3 \cdot 2 + 1 \cdot 4 = 10 \end{aligned}$$

$$\begin{aligned} h(x) &= \frac{f(x)}{g(x)} \\ h'(10) &= \frac{g(10)f'(10) - f(10)g'(10)}{g(10)^2} = \frac{2 \cdot 3 - 1 \cdot 4}{2^2} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} h(x) &= f(g(x)) \\ h'(10) &= f'(g(10))g'(10) = f'(2)g'(10) = 5 \cdot 4 = 20 \end{aligned}$$

2. (25 points) The sides of a cube are growing in such a way that it always maintains a cubical shape (so all sidelengths are equal), but its volume is increasing by 2 cubic centimeters per second. At what rate is its sidelength growing when the volume is 1000 cubic centimeters?

Letting $V(t)$ be the volume of the cube, and $s(t)$ its sidelength, one is given that $\frac{dV}{dt} = 2cm^3/sec$ and they are asking for $\frac{ds}{dt}$ at the moment when $V = 1000cm^3$ (so it must be that $s = \sqrt[3]{1000cm^3} = 10cm$ at that moment).

Starting with $V = s^3$ and implicitly taking $\frac{d}{dt}$ gives

$$\begin{aligned} \frac{dV}{dt} &= 3s^2 \frac{ds}{dt} \text{ at all times, so} \\ 2cm^3/sec &= 3(10cm)^2 \frac{ds}{dt} \text{ at that moment, and} \\ \frac{ds}{dt} &= \frac{2cm^3/sec}{3(10cm)^2} \\ &= \frac{1}{150} cm/sec \end{aligned}$$

3. (26 points total) Consider the curve in the plane defined by the equation

$$(x-1)^4 + y^4 = 16.$$

a. (10 points) Write an expression for the slope $\frac{dy}{dx}$ of the tangent line to a point (x, y) on this curve, as a function of x and y.

Taking $\frac{d}{dx}$ of the above equation implicitly gives

$$4(x-1)^3 + 4y^3\frac{dy}{dx} = 0$$

and solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = -\frac{(x-1)^3}{y^3}$$

b. (8 points) Find the (x, y) coordinates of both points on this curve where the tangent line to the curve is horizontal.

A horizontal tangent means that

$$0 = \frac{dy}{dx} = -\frac{(x-1)^3}{y^3}$$

which forces the $(x-1)^3 = 0$, so that $x = 1$. The points on the curve having $x = 1$ are found by plugging $x = 1$ back into the equation, and solving for their y coordinates:

$$(1-1)^4 + y^4 = 16$$

so $y^4 = 16$ and $y = \sqrt[4]{16} = \pm 2$. Thus the two points are $(x, y) = (1, 2)$ and $(1, -2)$.

c. (8 points)] Find the equation of the tangent line to the curve at the point $(x, y) = (1 + \sqrt[4]{8}, \sqrt[4]{8})$.

The slope will be

$$\left[\frac{dy}{dx}\right]_{(x,y)=(1+\sqrt[4]{8},\sqrt[4]{8})} = -\frac{(1+\sqrt[4]{8}-1)^3}{(\sqrt[4]{8})^3} = -1.$$

Hence the equation of the line is given by

$$\frac{y - \sqrt[4]{8}}{x - (1+\sqrt[4]{8})} = -1$$

or

$$y - \sqrt[4]{8} = -(x - (1 + \sqrt[4]{8})).$$

4. (25 points) Carbon-16 is radioactive, and decays at a rate proportional to how much is present, with a half-life of about 5730 years. How

many kilograms will remain after a 1 kilogram sample has been allowed to decay for 1000 years?

Since we know the half-life $t_{half} = 5730yrs$, we can get the rate constant k for the decay $\frac{dy}{dt} = ky$ from the all-important solution equation

$$y(t) = y_0 e^{kt}$$

by noting that

$$\begin{aligned} \frac{1}{2}y_0 &= y(t_{half}) = y_0 e^{k \cdot t_{half}} \\ \frac{1}{2} &= e^{k \cdot 5730} \\ \ln(\frac{1}{2}) &= 5730k \\ k &= \frac{-\ln(2)}{5730} \end{aligned}$$

Since we are also given $y_0 = 1kg$, we now know

$$y(t) = (1kg) \cdot e^{\frac{-\ln(2)}{5730} \cdot t}.$$

Hence at time $t = 1000$ years, one has

$$y(1000) = (1kg) \cdot e^{\frac{-\ln(2)}{5730} \cdot 1000} = (1kg) \cdot (e^{-\ln(2)})^{\frac{1000}{5730}} = (1kg) \left(\frac{1}{2}\right)^{\frac{1000}{5730}}.$$