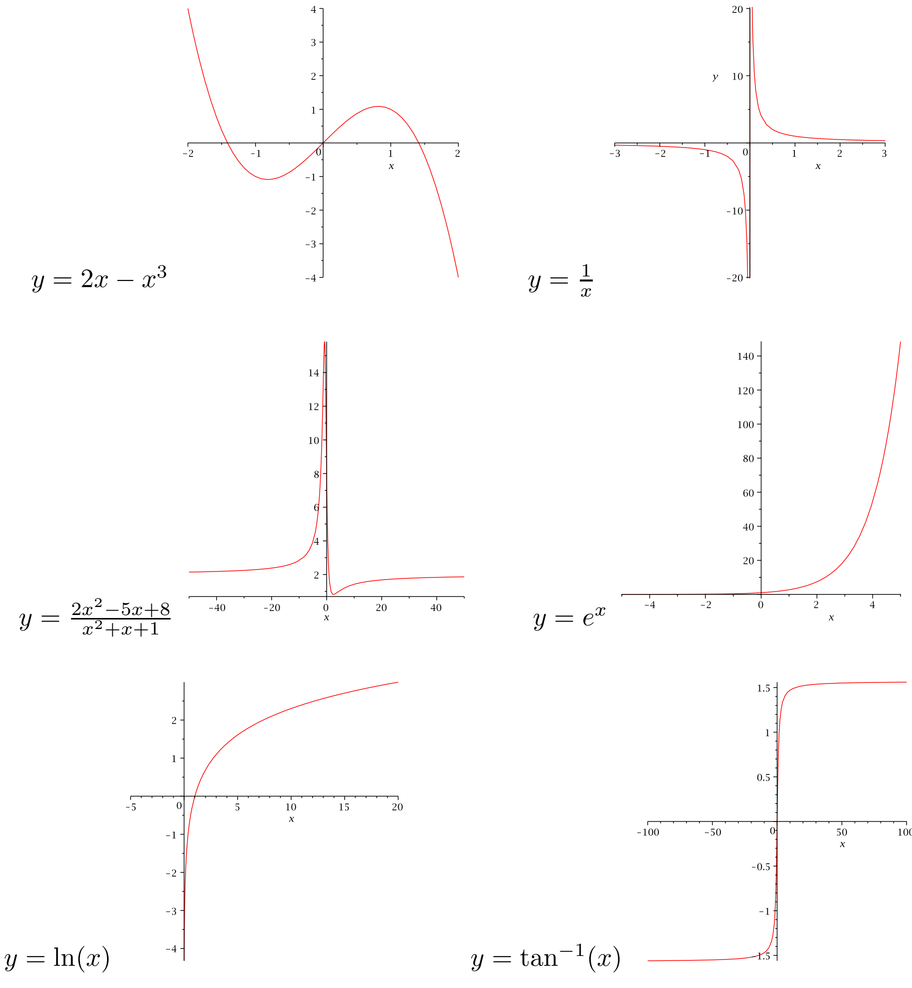

FIGURE 1. Various graphs of $y = f(x)$.

Behavior of functions at infinity:
infinite limits and horizontal asymptotes[1]

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Consider the graphs of $y = f(x)$ shown in Figure 1 for the functions

$$f(x) = 2x - x^3, \quad \frac{1}{x}, \quad \frac{2x^2 - 5x + 8}{x^2 + x + 1}, \quad e^x, \quad \ln(x), \quad \tan^{-1}(x).$$

How would you describe what happens to these functions $f(x)$ when x gets large and positive, that is, as x approaches $+\infty$? What about when x gets large and negative, that is, as x approaches $-\infty$?

We seek some language involving limits to describe this. *Informally*, one might say $\lim_{x\to+\infty} f(x) = +\infty$ to mean that we can ensure that the values of $f(x)$ are arbitrarily large and positive by choosing x sufficiently large and positive. Similarly, one might say *informally* $\lim_{x\to+\infty} f(x) = L$ for some real number L to mean that we can ensure that the values of $f(x)$ are arbitrarily close to L by choosing x sufficiently large and positive. One could suitably modify these descriptions to define informally when

$$\lim_{x\to+\infty} f(x) = \begin{cases} +\infty & \text{as with } f(x) = e^x \text{ or } \ln(x) \\ -\infty & \text{as with } f(x) = 2x - x^3 \\ L & \text{as with } f(x) = \frac{1}{x} \text{ for } L = 0, \\ & \text{or } f(x) = \frac{2x^2-5x+8}{x^2+x+1} \text{ for } L = 2, \\ & \text{or } f(x) = \tan^{-1}(x) \text{ for } L = \frac{\pi}{2}, \end{cases}$$

and

$$\lim_{x\to-\infty} f(x) = \begin{cases} +\infty & \text{as with } f(x) = 2x - x^3 \\ -\infty & \\ L & \text{as with } f(x) = \frac{1}{x} \text{ or } e^x \text{ for } L = 0, \\ & \text{or } f(x) = \frac{2x^2-5x+8}{x^2+x+1} \text{ for } L = 2, \\ & \text{or } f(x) = \tan^{-1}(x) \text{ for } L = -\frac{\pi}{2}, \end{cases}$$

Note that for some functions one might have no limit at all for $f(x)$ as x approaches $\pm\infty$, that is, there is no real number L for which $\lim_{x\to\pm\infty} f(x) = L$, nor does $\lim_{x\to\pm\infty} f(x) = +\infty$, nor does $\lim_{x\to\pm\infty} f(x) = -\infty$. In this case, say that $\lim_{x\to+\infty} \sin(x)$ *does not exist.*

Example. $\lim_{x\to+\infty} \sin(x)$ does not exist. As x gets arbitrarily large and positive, the values of $f(x) = \sin(x)$ do not get arbitrarily large and positive, nor arbitrarily large and negative, nor do they approach closer and closer to any real number L. Rather the values of $f(x)$ forever oscillate, staying bounded between -1 and $+1$.

As with definitions of the usual kinds of limits $\lim_{x\to a} f(x) = L$, one can capture the intuition behind these informal definitions $\lim_{x\to\pm\infty} f(x)$ with something formal.

Definition. Formally, define $\lim_{x\to+\infty} f(x) = +\infty$ to mean that for every $M > 0$, there exists an $N > 0$ such that the inequality $f(x) > M$ holds for all $x > N$.

Definition. Similarly, define formally $\lim_{x\to+\infty} f(x) = L$ for a real number L to mean that for every $\epsilon > 0$, there exists an $N > 0$ such that the inequality $|f(x) - L| < \epsilon$ holds for all $x > N$.

Similar modifications exist to define formally what is meant by the other variations $\lim_{x\to\pm\infty} f(x) = \pm\infty$ or $\lim_{x\to\pm\infty} f(x) = L$.

Definition. When either $\lim_{x\to+\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, one says that the horizontal line $y = L$ is a *horizontal asymptote* for the graph $y = f(x)$. One can also say that the curve $y = f(x)$ *approaches the line $y = L$ asymptotically.*

Example. Let's check formally that $\lim_{x\to+\infty} e^x = +\infty$. To do this, if our adversary names for us some $M > 0$, we must find an N such that $e^x > M$ for all $x > N$.

A little thought, foresight, or experience with such arguments[2] might suggest trying $N = \ln(M)$. And indeed one can check that for $x > N = \ln(M)$ one has

$$f(x) = e^x > e^N = e^{\ln(M)} = M$$

where that inequality in the middle is due to the fact that $f(x) = e^x$ is a monotonically increasing function of x.

The formal definitions can be used to prove *limit laws* similar to the ones we have seen for other limits: in situations where the limits of $f(x), g(x)$ which appear on the right side of these laws are real numbers (not $\pm\infty$), one has

$$\lim_{x\to\pm\infty} (f(x) \pm g(x)) = \lim_{x\to\pm\infty} f(x) \pm \lim_{x\to\pm\infty} g(x)$$
$$\lim_{x\to\pm\infty} f(x)g(x) = \lim_{x\to\pm\infty} f(x) \cdot \lim_{x\to\pm\infty} g(x)$$
$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\pm\infty} f(x)}{\lim_{x\to\pm\infty} g(x)}$$

assuming $\lim_{x\to\pm\infty} g(x) \neq 0$ in this last law.

Another such law says that when $f(x)$ is a *continuous function* for all values x in the range of $g(x)$, and $\lim_{x\to\pm\infty} g(x) = L$ for some real number L, then $\lim_{x\to\pm\infty} f(g(x)) = f(L)$.

This does not exhaust all the possible such limit laws. Also, some of these limit laws still apply even when $f(x), g(x)$ do not have finite limits.

Example. If $\lim_{x\to+\infty} f(x) = L$ and $\lim_{x\to+\infty} g(x) = +\infty$, then

$$\lim_{x\to+\infty} (f(x) + g(x)) = +\infty.$$

We sometimes abbreviate this law informally by saying "$L + \infty = +\infty$". Similarly, one has "$\frac{0}{\infty} = 0$".

However, one has to be careful, as some cases where one would like to apply a limit law are *indeterminate forms*, like

$$\frac{\pm\infty}{\pm\infty}, \quad \frac{0}{0}, \quad 0 \cdot (\pm\infty), \quad \infty \pm \infty.$$

Sometimes in these cases, algebraic manipulation and/or L'Hôpital's rule (a later calculus topic) comes to our aid.

Example. Starting from the fact (which one can justify from the formal definition) that integer powers x^n of x have

$$\lim_{x\to\pm\infty} x^n = \begin{cases} \pm\infty & \text{if } n = 1, 2, 3, \dots \\ 1 & \text{if } n = 0 \\ 0 & \text{if } n = -1, -2, -3, \dots \end{cases}$$

it's not hard to analyze the behavior at infinity for any *rational function*. Recall that a rational function is $h(x) = \frac{f(x)}{g(x)}$ where

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r$$
$$g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_sx^s$$

[2] The sometimes subtle art of how to pick the N correctly is not something we will emphasize in our version of Math 1271, as we won't often ask students to prove a limit is correct via the formal definition!

are polynomials, say of degrees r and s, so that $a_r, b_s \neq 0$.

If one tries to analyze $\lim_{x\to\pm\infty} h(x)$ by immediately using the quotient rule for limits it often leads to the indeterminate form $\frac{\infty}{\infty}$. However, a useful algebraic trick comes from realizing that if x^N is the *highest power of* x appearing anywhere in either $f(x)$ or $g(x)$ (so N is just maximum of the two degrees r and s), then these terms x^N wherever they occur will dominate the behavior of $f(x)$ when $x \to \pm\infty$. And we can "scale them away" by multiplying by $\frac{1/x^N}{1/x^N}$, leaving an equivalent limit, for which the quotient limit law will now work[3] For example,

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x^2-1}{x+1} &= \lim_{x\to\pm\infty} \frac{2x^2-1}{x+1} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\pm\infty} \frac{2-1/x^2}{1/x+1/x^2} \\
&= \frac{\lim_{x\to\pm\infty}(2-1/x^2)}{\lim_{x\to\pm\infty}(1/x+1/x^2)} \quad = \frac{2}{\pm\infty} = \pm\infty
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x^2-1}{x^2+1} &= \lim_{x\to\pm\infty} \frac{2x^2-1}{x^2+1} \cdot \frac{1/x^2}{1/x^2} \\
&= \lim_{x\to\pm\infty} \frac{2-1/x^2}{1+1/x^2} \\
&= \frac{\lim_{x\to\pm\infty}(2-1/x^2)}{\lim_{x\to\pm\infty}(1+1/x^2)} \quad = \frac{2}{1} = 2
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\pm\infty} \frac{2x^2-1}{x^3+1} &= \lim_{x\to\pm\infty} \frac{2x^2-1}{x^3+1} \cdot \frac{1/x^3}{1/x^3} \\
&= \lim_{x\to\pm\infty} \frac{2/x-1/x^3}{1+1/x^3} \\
&= \frac{\lim_{x\to\pm\infty}(2/x-1/x^3)}{\lim_{x\to\pm\infty}(1+1/x^3)} \quad = \frac{0}{1} = 0
\end{aligned}$$

The graphs of these three rational functions are shown in Figure 2.

Doing this analysis in general for the rational function $\frac{f}{g}$ where f, g have degrees r, s and leading coefficients f_r, g_s shows the following[4]:

$$\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \begin{cases} 0 & \text{if } r < s, \\ \frac{f_r}{g_s} & \text{if } r = s, \\ (-1)^{r-s}\operatorname{sign}(\frac{f_r}{g_s}) \cdot (\pm\infty) & \text{if } r > s \end{cases}$$

where $\operatorname{sign}(x) = \frac{|x|}{x}$ give the sign ± 1 of a nonzero number x.

[3]Alternatively, one can do this same trick but multiply by $\frac{1/x^s}{1/x^s}$, which will also provide an illuminating scaling, and still work with the limit laws.

[4]... which is not really worth memorizing; the trick of multiplying $\frac{1/x^N}{1/x^N}$ is more important.

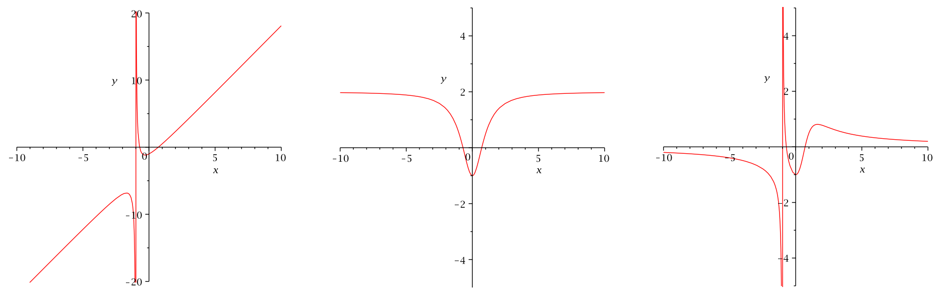

FIGURE 2. The graphs of $y = h_1(x), h_2(x), h_3(x)$.

Exercises.

(a) Say whether $\lim_{x\to+\infty} f(x)$ is some real number L, or $+\infty$ or $-\infty$ or nonexistent for each of the following functions $f(x)$. Remember to give some justification for your answer.

(b) Do the same for $\lim_{x\to-\infty} f(x)$.

(c) Then list any horizontal asymptotes for the graph $y = f(x)$.

1. $f(x) = \cos(x)$
2. $f(x) = x\cos(x)$
3. $f(x) = \frac{1}{x(2+\cos(x))}$
4. $f(x) = x^2 \sin(x)$
5. $f(x) = \frac{\sin(x)}{x^2}$
6. $f(x) = e^{-2x}\sin(x)$
7. $f(x) = e^{2x}\sin(x)$
8. $f(x) = \frac{x^{100}+x^3+x}{3x^{50}+x^4+x-7}$
9. $f(x) = \frac{x^{100}+x^3+x}{3x^{100}+x^4+x-7}$
10. $f(x) = \frac{x^{100}+x^3+x}{3x^{101}+x^4+x-7}$
11. $f(x) = \tan^{-1}(3x^3 + 4x - 100)$
12. $f(x) = \tan^{-1}(-3x^3 + 4x - 100)$