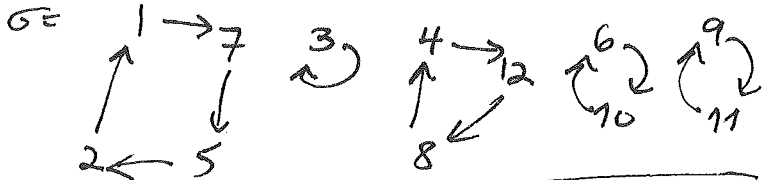
(20) 

# Permutations and cycles (Stanley §1.3)

Recall $\mathfrak{S}_n$ = symmetric group on $n$ letters
= permutations of $[n]$

**Notations:**

- ~~two~~-line: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 7 & 1 & 3 & 12 & 2 & 10 & 5 & 4 & 11 & 6 & 9 & 8 \end{pmatrix}$
- one-line: $\sigma = (7, 1, 3, 12, 2, 10, 5, 4, 11, 6, 9, 8)$
- functional digraph (directed graph): $\sigma =$




- cycle notation: $\sigma = (1\ 7\ 5\ 2)(3)(4\ 12\ 8)(6\ 10)(9\ 11)$

$= (8\ 4\ 12)(10\ 6)(5\ 2\ 1\ 7)(3)(11\ 9)$

$= (8\ 4\ 12)(10\ 6)(11\ 9)(7\ 5\ 2\ 1)$

$= (\underline{3})(\underline{7}\ 5\ 2\ 1)(\underline{10}\ 6)(\underline{11}\ 9)(\underline{12}\ 8\ 4)$

↖ standard form:
- each cycle has its biggest element first,
- and cycles appear with these biggest elements increasing left-to-right

Q: How many $\sigma \in S_n$ of cycle type $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$?

$= 1^{a_1} 2^{a_2} 3^{a_3} \ldots$

↖ multiplicity notation for $\lambda$

e.g. $\lambda = (5\,5\,5\,3\,2\,2\,2\,2\,2\,1\,1)$

$= 1^2 2^5 3^1 4^0 5^3$

$c_1=2\ c_2=5\ c_3=1\ c_4=0\ c_5=3$
$c_6=c_7=\ldots=0$

e.g. $n=4$

| | | |
|---|---|---|
| $\lambda = 1^4$ | (a)(b)(c)(d) | 1 |
| $2^1 1^2$ | (ab)(c)(d) | $\binom{4}{2}=6$ |
| $2^2$ | (ab)(cd) | 3 |
| $3^1 1^1$ | (abc)(d) | 8 |
| $4^1$ | (abcd) | $6=\frac{4!}{4}$ |

PROP: There are $\dfrac{n!}{1^{c_1} c_1! \, 2^{c_2} c_2! \, 3^{c_3} c_3! \cdots}$ perms in $S_n$ of cycle type $\lambda = 1^{c_1} 2^{c_2} 3^{c_3} \ldots$

proof: Recall $\mathfrak{S}_n$ acts on the set of them transitively via conjugation:

$$\underbrace{\begin{pmatrix} 1&2&3&4&5&6&7 \\ a&b&c&d&e&f&g \end{pmatrix}}_{\sigma} (1\,2\,3\,4)(5\,6\,7) \underbrace{\begin{pmatrix} a&b&c&d&e&f&g \\ 1&2&3&4&5&6&7 \end{pmatrix}}_{\sigma^{-1}}$$

$$= (a\,b\,c\,d)(e\,f\,g)$$

So the size of the orbit is $|\mathcal{O}| = \dfrac{|S_n|}{|Z_{S_n}(\sigma_\lambda)|}$ if $\sigma_\lambda$ has cycle type $\lambda$

where $Z_{S_n}(\sigma) := \{ \tau \in S_n : \tau\sigma_\lambda = \sigma_\lambda \tau \}$ the centralizer of $\sigma_\lambda$ in $\mathfrak{S}_n$
i.e. $\tau \sigma_\lambda \tau^{-1} = \sigma_\lambda$

Why is this equivalent to Stanley's argument counting # of $n!$ placements of $[n]$ into $\underbrace{(\cdot)(\cdot)\cdots}_{c_1} \underbrace{(\cdot\cdot)(\cdot\cdot)\cdots}_{c_2}$ that give same standard form?

Who centralizes $\sigma_\lambda = \underbrace{(a)(b)\cdots}_{c_1 \text{ 1-cycles}} \underbrace{(cd)(ef)\cdots}_{c_2 \text{ 2-cycles}}, \cdots$ ?

- Products of powers of each cycle – there are $1^{c_1} 2^{c_2} 3^{c_3} \ldots$ of these
- Swapping two cycles of same size, preserving cyclic orders – there are $c_1! \, c_2! \, c_3! \ldots$ of these

e.g. $(1\,2\,3\,4)(5\,6\,7)(8\,9\,10)$

is centralized by $\tau = \underbrace{(4\,3\,2\,1)}_{=(1234)^3} \cdot \begin{pmatrix} 5&6&7 \\ 9&10&8 \end{pmatrix}$

Thus $|\mathcal{O}| = \dfrac{n!}{\prod_{j \geq 1} j^{c_j} \cdot \prod_{j \geq 1} c_j!} = \dfrac{n!}{\prod_{j \geq 1} j^{c_j} \cdot c_j!}$ ∎

DEF'N: For a subgroup $G$ of $\mathfrak{S}_n$, define its cycle ~~indicator~~ index polynomial

$$Z_G(t_1, t_2, \ldots) := \frac{1}{|G|} \sum_{\sigma \in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots \quad \text{where } c_j(\sigma) = \# j\text{-cycles of } \sigma.$$

**COR (Touchard):** The cycle indicators $Z_{\mathfrak{S}_n}$ have generating function

$$\sum_{n=0}^{\infty} Z_{\mathfrak{S}_n}(\underline{t})\, x^n = e^{t_1\frac{x}{1} + t_2\frac{x^2}{2} + t_3\frac{x^3}{3} + \cdots} = e^{\sum_{j\geq 1} t_j \frac{x^j}{j}}$$

*proof:* (Direct but mysterious, we'll see a better one later...)

$$e^{t_1\frac{x^1}{1} + t_2\frac{x^2}{2} + \cdots} = e^{t_1\frac{x^1}{1}}\, e^{t_2\frac{x^2}{2}} \cdots$$

$$= \left(\sum_{c_1\geq 0} \frac{\left(\frac{t_1 x^1}{1}\right)^{c_1}}{c_1!}\right)\left(\sum_{c_2\geq 0} \frac{\left(\frac{t_2 x^2}{2}\right)^{c_2}}{c_2!}\right)\cdots$$

$$= \sum_{(c_1,c_2,\ldots)} x^{1\cdot c_1 + 2\cdot c_2 + \cdots}\, \frac{t_1^{c_1} t_2^{c_2}\cdots}{1^{c_1}c_1!\, 2^{c_2}c_2!\cdots}$$

$$= \sum_{n\geq 0} x^n \frac{1}{n!} \sum_{\substack{(c_1,c_2,\ldots)\\ \sum_j j c_j = n}} \boxed{\frac{n!}{1^{c_1}c_1!\,2^{c_2}c_2!\cdots}}\; t_1^{c_1} t_2^{c_2}\cdots$$

where the boxed quantity $= \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ has } c_j \text{ } j\text{-cycles}\}$

$$= \sum_{n\geq 0} x^n \underbrace{\frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots}_{Z_{\mathfrak{S}_n}(\underline{t})} \qquad \blacksquare$$

This has lots of consequences...

**EXAMPLES**

① **DEF'N:** $\sum_{k=0}^{n} c(n,k)\, t^k := \sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{cycles}(\sigma)}$ , i.e. $c(n,k) = \#\{\sigma\in\mathfrak{S}_n : \sigma \text{ has } k \text{ cycles}\}$

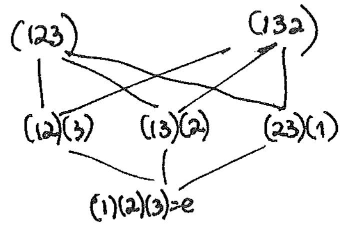
$c(n,k)$: (signless) Stirling number of 1st kind

e.g. $\boxed{n=3}$

$t^3 + 3t^2 + 2t^1$ $\quad c(3,1) = 2$

$= t(t+1)(t+2)$ $\quad c(3,2) = 3$

$c(3,3) = 1$

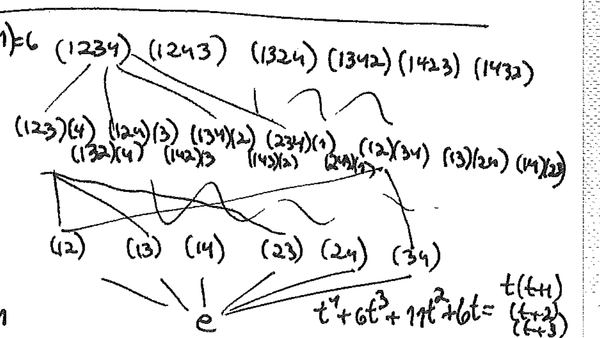



$\boxed{n=4}$ $c(4,1)=6$

$c(4,2)=11$

$c(4,3)=6$

$c(4,4)=1$




$t^4 + 6t^3 + 11t^2 + 6t = t(t+1)(t+2)(t+3)$

The partial order on $\mathfrak{S}_3, \mathfrak{S}_4$ depicted is $(\mathfrak{S}_n, \leq_{abs})$
<br>($\leq_{abs}$ = absolute order)

defined by $\sigma <_{abs} \tau$ if $\tau = \sigma \circ (i,j)$ for some $i,j$
and $\#\text{cycles}(\tau) = \#\text{cycles}(\sigma) - 1$
(as opposed to $\#\text{cycles}(\tau) = \#\text{cycles}(\sigma) + 1$)

a ranked poset with $\hat{0} = e$, and rank sizes $c(n,k)$

COR (to Touchard): $\displaystyle\sum_{k=1}^{n} c(n,k) = t(t+1)(t+2)\cdots(t+(n-1))$

proof: Set $t_1 = t_2 = \cdots = t$ in

$$\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots = e^{t_1\frac{x^1}{1} + t_2\frac{x^2}{2} + \cdots}$$

$$\rightsquigarrow$$

$$\sum_{n\geq 0} \frac{x^n}{n!} \underbrace{\sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{cycles}(\sigma)}}_{\sum_{k=1}^{n} c(n,k) t^k} = e^{t\left(\frac{x^1}{1} + \frac{x^2}{2} + \cdots\right)}$$

$$= e^{t(-\log(1-x))}$$

$$= (1-x)^{-t}$$

$$= \sum_{n\geq 0} \binom{-t}{n} (-x)^n$$

$$= \sum_{n\geq 0} \underbrace{\binom{t+n-1}{n}}_{\frac{t(t+1)(t+2)\cdots(t+n-1)}{n!}} x^n$$

Now compare coeffs of $\frac{x^n}{n!}$ ∎

RMK: PROP: The map $\mathfrak{S}_n \longrightarrow \mathfrak{S}_n$
$\sigma \longmapsto \hat{\sigma}$ = put $\sigma$ in standard cycle form and erase parentheses

is a bijection, with $\#\text{cycles}(\sigma) = \#$ L-to-R-maxima in $\hat{\sigma}$.

Hence $\displaystyle\sum_{\sigma\in\mathfrak{S}_n} t^{\#\text{L-to-R-maxes}(\sigma)} = t(t+1)\cdots(t+(n-1))$

e.g. $n=3$

| $\sigma$ | # L-to-R-maxes($\sigma$) |
|---|---|
| 123 | 3 |
| 132 | 2 |
| 213 | 2 |
| 231 | 2 |
| 312 | 1 |
| 321 | 1 |

$t^3 + 3t^2 + 2t = t(t+1)(t+2)$

proof: $\sigma \longmapsto \hat{\sigma}$ is reversible: the L-to-R maxes in $\hat{\sigma}$ tell you where to put the left parens "(", and then add in right parens ")" just before them (not at beginning, one extra at end)

$\sigma = (\underline{3})(\underline{7}521)(\underline{10}\,6)(\underline{11}\,9)(\underline{12}\,8\,4)$

$\underline{3}\ \underline{7}\ 5\ 2\ 1\ \underline{10}\ 6\ \underline{11}\ 9\ \underline{12}\ 8\ 4$

② Can we compute $E_k(n)$ := expected number of $k$ cycles in a random $\sigma \in \mathfrak{S}_n$ (using uniform distribution, i.e. all $\sigma \in \mathfrak{S}_n$ have same probability $\frac{1}{n!}$)

$$= \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} c_k(\sigma)$$

$$= \frac{1}{n!} \left[ \frac{\partial}{\partial t_k} \sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots \right]_{t_1 = t_2 = \cdots = 1}$$

Thus
$$\sum_{n \geq 0} E_k(n) x^n = \left[ \frac{\partial}{\partial t_k} \sum_{n \geq 0} \frac{x^n}{n!} \sum_{\sigma \in \mathfrak{S}_n} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots \right]_{t_1 = t_2 = \cdots = 1}$$

$$= \left[ \frac{\partial}{\partial t_k} e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + \cdots} \right]_{t_i = 1}$$

$$= \left[ \frac{x^k}{k} e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + \cdots} \right]_{t_i = 1}$$

$$= \frac{x^k}{k} e^{\frac{x^1}{1} + \frac{x^2}{2} + \cdots} = \frac{x^{k-1}}{k} e^{-\log(1-x)} = \frac{x^k}{k(1-x)}$$

$$= \sum_{n \geq k} \frac{1}{k} \cdot x^n \quad \text{i.e. } E_k(n) = \begin{cases} \frac{1}{k} & \text{for } n \geq k, \\ 0 & \text{else.} \end{cases}$$

So $E_k(n)$ is eventually constant in $n$ (!)

RMK: In fact one can show #k-cycles($\sigma$) for $\sigma \in \mathfrak{S}_n$ approaches (as $n \to \infty$) a Poisson random variable with expectation $\lambda = \frac{1}{k}$ i.e. $\text{Prob}(\#k\text{-cycles} = m) \xrightarrow{n \to \infty} e^{-\lambda} \frac{\lambda^m}{m!}$ for $\lambda = \frac{1}{k}$

③ There are special classes of permutations defined by restrictions on their cycle sizes, so all have nice (exponential) gen. fns.

e.g. <u>no large cycles</u>: $\sigma \in \mathfrak{S}_n$ is an <u>involution</u> $\sigma^2 = e$

$\Leftrightarrow$ $\sigma$ has only 1-cycles and 2-cycles

$\sigma = (ab)(cd)\cdots(x)(y)(z)$

Hence
$$\sum_{n\geq 0} \frac{x^n}{n!} \#\{\text{involutions in } \mathfrak{S}_n\} = \left[ e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + \cdots} \right]_{t_3 = t_4 = t_5 = \cdots = 0}$$
$$= e^{x + \frac{x^2}{2}}$$

or even
$$\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\substack{\text{involutions} \\ \sigma \in \mathfrak{S}_n}} t^{\#1\text{-cycles}(\sigma)} = e^{tx + \frac{x^2}{2}} \quad \text{similarly}$$

What about <u>no small cycles</u>?

<u>DEF'N</u>: A <u>derangement</u> $\sigma \in \mathfrak{S}_n$ is a permutation with <u>no fixed points</u> i.e. $c_1(\sigma) = 0$.

<u>Q</u>: (~~Hat-check~~ Derangement problem) What is the probability that 100 (= $n$) people with hats all get the wrong hat back from the hat-check attendant?

i.e. what is $\frac{d_n}{n!}$ where $d_n = \#\{\sigma \in \mathfrak{S}_n : \sigma \text{ a derangement}\}$?

$$\sum_{n\geq 0} \frac{x^n}{n!} d_n = \left[ e^{t_1 \frac{x^1}{1} + t_2 \frac{x^2}{2} + \cdots} \right]_{t_1 = 0,\ t_2 = t_3 = \cdots = 1}$$
$$= e^{\frac{x^2}{2} + \frac{x^3}{3} + \cdots}$$
$$= e^{-\log(1-x) - \frac{x^1}{1}} = \boxed{\frac{e^{-x}}{1-x}}$$
$$= (1 + x + x^2 + \cdots)\left(1 - \frac{x^1}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)$$
$$= \sum_{n\geq 0} x^n \underbrace{\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right)}_{\frac{d_n}{n!}}$$

$$\frac{d_n}{n!} \to e^{-1} = \frac{1}{e}$$

(consistent with $c_1(\sigma) \to$ Poisson with mean 1)

(26) <u>A digression</u>

Why the cycle index $Z_G(\underline{t}) := \frac{1}{|G|} \sum_{\sigma \in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots$ for perm. groups $G \leq \mathfrak{S}_n$ ??

Polya theory – counts the <u>G-orbits of colorings</u> of a finite set $X$

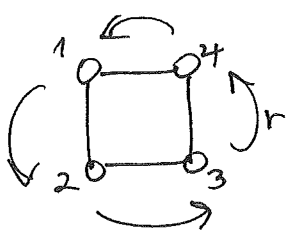
with $k$ colors $a_1, a_2, \ldots, a_k$ ,

and more refined, counts the <u>pattern inventory</u> ($a_1 = a_2 = \cdots = a_k = 1$)

$$\sum_{\substack{G\text{-orbits } \mathcal{O} \\ \text{of } k\text{-colorings of } X}} a_1^{\#\text{times color 1 is used}} \, a_2^{\#\text{times color 2 is used}} \cdots$$

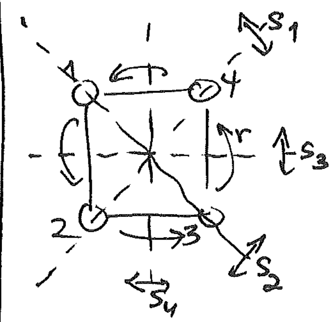
<u>EXAMPLES:</u>

$G = C_4 =$ cyclic group of order 4 $= \langle (1234) \rangle$



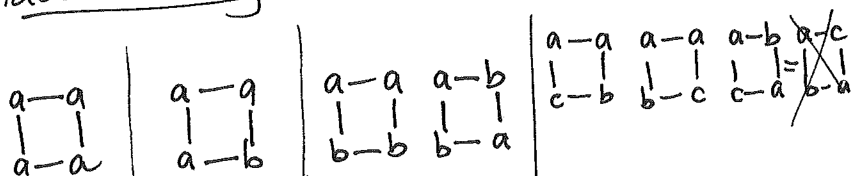

acting on $X$ = vertices of square and 3-colorings ($k$) via colors $\{a,b,c\}$

$G = D_8 =$ dihedral group of order 8



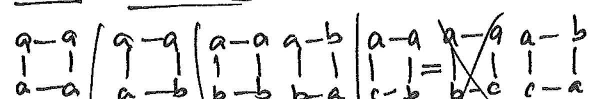

with $X$, colors $\{a,b,c\}$ as before

Pattern inventory:



$a^4+b^4+c^4 + a^3b + a^3c + ab^3 + ac^3 + b^3c + bc^3 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2 + 3a^2bc + 3ab^2c + 3abc^2$

Pattern inventory

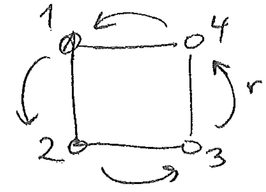


$a^4+b^4+c^4+\cdots$ (same) $+ 2a^2bc + 2ab^2c + 2abc^2$

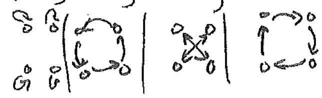
THM (Polya) The # of G-orbits of $k$-colorings of $X$ is $\frac{1}{|G|} \sum_{\sigma \in G} k^{\#\text{cycles}(\sigma)}$

and the pattern inventory is $\left[ \underbrace{\frac{1}{|G|} \sum_{\sigma \in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)} \cdots}_{Z_G(\underline{t})} \right]_{t_j = a_1^j + a_2^j + \cdots + a_k^j}$

EXAMPLES: ① $G = C_4 =$




$= \{ e, r, r^2, r^3 \}$ with $e = (1)(2)(3)(4)$, $r = (1234)$, $r^2 = (13)(24)$, $r^3 = (1432)$



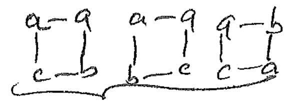
$$Z_{C_4}(\underline{t}) = \frac{1}{4}\left( t_1^4 + t_4^1 + t_2^2 + t_4^1 \right)$$

$$= t_1^4 + t_2^2 + 2t_4^1$$

$$\Big\downarrow \; t_j = a^j + b^j + c^j$$

$$\frac{1}{4}\left( (a+b+c)^4 + (a^2+b^2+c^2)^2 + 2(a^4+b^4+c^4) \right)$$

$$\Big\downarrow \; [a^2 b^1 c^1]$$




$$\frac{1}{4}\left( \binom{4}{2\,1\,1} + 0 + 0 \right) = \frac{1}{4}\,\frac{4!}{2!\,1!\,1!} = 3 \;\checkmark$$

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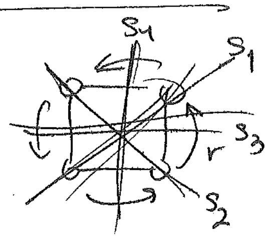
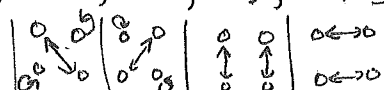
② $G = D_4 =$




$= \{ e, r, r^2, r^3, s_1, s_2, s_3, s_4 \}$



$$Z_{D_8}(\underline{t}) = \frac{1}{8}\left( t_1^4 + t_4^1 + t_2^2 + t_4^1 + t_2 t_1^2 + t_2 t_1^2 + t_2^2 + t_2^2 \right)$$

$$= \frac{1}{8}\left( t_1^4 + 3t_2^2 + 2t_4 + 2t_2 t_1^2 \right)$$

$$\Big\downarrow \; t_j = a^j + b^j + c^j$$

$$\frac{1}{8}\left( (a+b+c)^4 + 3(a^2+b^2+c^2)^2 + 2(a^4+b^4+c^4) + 2(a^2+b^2+c^2)(a+b+c)^2 \right)$$

$$\Big\downarrow \; [a^2 b^1 c^1]$$

$$\frac{1}{8}\left( \binom{4}{2\,1\,1} + 0 + 0 + 2 \cdot 2 \right)$$


$$= \frac{1}{8}\left( \frac{4!}{2!\,1!\,1!} + 4 \right) = \frac{1}{8}(12+4) = \frac{16}{8} = 2 \;\checkmark$$

(28)

## proof of Polya's Thm.

The engine driving it is...

Burnside's Lemma: For a group $G$ of permutations of a finite set $X$, $\#\, G\text{-orbits } \mathcal{O} \text{ on } X = \frac{1}{|G|}\sum_{\sigma\in G} \#\{x\in X : \sigma(x)=x\}$

proof: $\sum_{\sigma\in G} \#\{x\in X : \sigma(x)=x\} = \#\{(\sigma,x)\in G\times X : \sigma(x)=x\}$

$= \sum_{x\in X} \underbrace{\#\{\sigma\in G : \sigma(x)=x\}}_{G_x := G\text{-stabilizer of } x}$

$= \sum_{G\text{-orbits } \mathcal{O} \text{ on } X} \underbrace{\sum_{x\in \mathcal{O}} |G_x|}_{|\mathcal{O}||G_x| = |G|}$ ← orbit-stabilizer lemma, i.e. $|\mathcal{O}| = \frac{|G|}{|G_x|} = [G:G_x] = |G/G_x|$

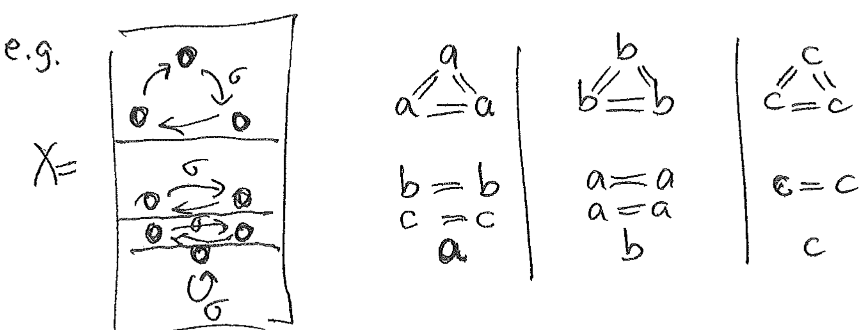
$= |\mathcal{O}|\cdot|G|$ ∎

9/28/2015 > When $G$ permutes $X$,

it also permutes $k$-colorings of $X$

and $\sigma\in G$ fixes a $k$-coloring $\Leftrightarrow$ the $k$-coloring is constant within cycles of $\sigma$

e.g.




$\cdots = (a^3+b^3+c^3)^1(a^2+b^2+c^2)^2(a+b+c)^1$

Hence $\sum_{\substack{k\text{-colorings}\\ \text{fixed by } \sigma}} a_1^{\#\text{color 1 used}} a_2^{\#\text{color 2 used}} \cdots = \prod_{\substack{\text{cycles}\\ C \text{ of } \sigma}} \left(a_1^{|C|}+a_2^{|C|}+\dots+a_k^{|C|}\right) = \left[t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j + a_2^j + \dots + a_k^j}$

Hence pattern inventory $= \sum_{G\text{-orbits } \mathcal{O}} \underline{a}^{\text{colors in } \mathcal{O}} = \sum_{\underline{c}=(c_1,c_2,\dots,c_k)} \underline{a}^{\underline{c}}\, \#\left\{\begin{matrix} G\text{-orbits } \mathcal{O} \\ \text{using } \underline{c}\end{matrix}\right\} = \sum_{\underline{c}} \underline{a}^{\underline{c}} \frac{1}{|G|}\sum_{\sigma\in G} \underbrace{\#\left\{\begin{matrix} k\text{-colorings} \\ \text{using } \underline{c} \\ \text{fixed by } \sigma\end{matrix}\right\}}_{[\underline{a}^{\underline{c}}]\left[t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j + a_2^j + \cdots}}$

$= \left[\frac{1}{|G|}\sum_{\sigma\in G} t_1^{c_1(\sigma)} t_2^{c_2(\sigma)}\cdots\right]_{t_j = a_1^j + a_2^j + \dots + a_k^j}$ ∎

# Some theory of ordinary generating functions (Ardila §2.2)

Roughly speaking, if $\mathcal{A}$ is some class of combinatorial structures, with $a_n = \#$ (weighted?) $\mathcal{A}$-structures of weight/size $n$ $\in R$ (a ring with 1)

then we can form the ordinary gen. fn. (o.g.f.) $A(x) = \sum_{n \geq 0} a_n x^n \in R[[x]]$.

PROP:

- If $\mathcal{C}$-structures of size $n$ are a choice of either an $\mathcal{A}$- or $\mathcal{B}$-structure (i.e. $c_n = a_n + b_n$) ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$")

  then $C(x) = A(x) + B(x)$

- If $\mathcal{C}$-structures of size $n$ are a choice of
  an $\mathcal{A}$-structure of size $i$
  $\mathcal{B}$-structure of size $j$
  for some $i + j = n$

  (i.e. $c_n = \sum_{\substack{i+j=n \\ i,j \geq 0}} a_i b_j$) ("$\mathcal{C} = \mathcal{A} \times \mathcal{B}$")

  then $C(x) = A(x)B(x)$

- If $\mathcal{C}$-structures of size $n$ are a choice of
  $\mathcal{B}$-structures of sizes $i_1, i_2, \ldots, i_k$ for some $i_1 + \ldots + i_k = n$, $i_j \geq 0$

  (i.e. $c_n = \sum_{\substack{(i_1, i_2, \ldots, i_k): \sum_j i_j = n \\ i_j \geq 0}} b_{i_1} b_{i_2} \cdots b_{i_k}$) ("$\mathcal{C} = \text{Seq}(\mathcal{B})$")

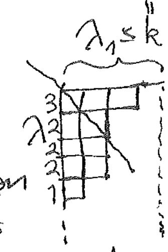
  then $C(x) = \frac{1}{1 - B(x)}$

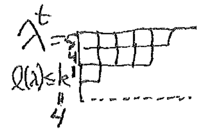
## EXAMPLES (see also Ardila §2.2.2)

① Let $p_k(n) := \#\{\text{partitions } \lambda \vdash n \text{ with } \lambda_1 \leq k \text{ (so } \lambda_i \leq k\ \forall i)\}$
$(\lambda_1 \geq \lambda_2 \geq \dots)$

$\Updownarrow$ in bijection via conjugation of partitions $\lambda \leftrightarrow \lambda^t$ = reflect Ferrers diagram across diagonal

$= \#\{\text{partitions } \lambda \vdash n \text{ with } \ell(\lambda) \leq k\}$




Then $P_k(q) := \sum_{n \geq 0} p_k(n) q^n = \sum_{\substack{\lambda: \\ \lambda_1 \leq k}} q^{|\lambda|} = \sum_{\substack{\lambda: \\ \ell(\lambda) \leq k}} q^{|\lambda|}$

$$= \underbrace{\frac{1}{1-q}}_{\substack{\text{ogf for } \lambda \\ \text{with only} \\ \text{parts of size 1}}} \cdot \underbrace{\frac{1}{1-q^2}}_{\substack{\text{ogf for } \lambda \\ \text{with only} \\ \text{parts of size 2}}} \cdots \underbrace{\frac{1}{1-q^k}}_{\substack{\text{(same)} \\ \dots\text{of size } k}} = \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}$$

i.e. $\left\{\begin{matrix} \lambda \text{ with} \\ \lambda_1 \leq k \end{matrix}\right\} = \mathrm{Seq}(\text{Ones}) \times \mathrm{Seq}(\text{Twos}) \times \dots \times \mathrm{Seq}(k\text{'s})$

Similarly, $\sum_{\substack{\lambda: \\ \lambda_1 \leq k}} q^{|\lambda|} t^{\ell(\lambda)} = \frac{1}{(1-tq)(1-tq^2)\cdots(1-tq^k)} = \sum_{\substack{\lambda: \\ \ell(\lambda) \leq k}} q^{|\lambda|} t^{\lambda_1}$

② (Ardila §2.2.2 #5)

Let $a_n := \#\{\text{of compositions } \alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell) \models n\}$ ← "$\alpha$ is a composition of $n$"
of any length $\ell$, $\alpha_i \in \mathbb{P}$

we saw before $= \begin{cases} 2^{n-1} & \text{for } n \geq 1 \\ 1 & \text{for } n = 0 \end{cases}$

but seen another way, $A(x) = \sum_{n \geq 0} a_n x^n = \dfrac{1}{1 - \underbrace{(x + x^2 + x^3 + \dots)}_{\substack{\text{o.g.f. for} \\ \text{compositions of } n \\ \text{with one part}}}} = \dfrac{1}{1 - \frac{x}{1-x}} = \dfrac{1-x}{1-x-x}$

$= \dfrac{1-x}{1-2x} = 1 + \dfrac{x}{1-2x}$

$= 1 + \sum_{n \geq 1} 2^{n-1} x^n$ ✓

③ (Ardila §2.2.2 #6) More interestingly,

what about $a_n := \{\#\text{ compositions } \alpha \models n \text{ with only odd parts}\}$?

| $n$ | $\alpha \vdash n$ with odd parts | |
|---|---|---|
| 0 | () | 1 |
| 1 | 1 | 1 |
| 2 | 1+1 | 1 |
| 3 | 3, 1+1 | 2 |
| 4 | 3+1, 1+3, 1+1+1+1 | 3 |
| 5 | 5, 3+1+1, 1+3+1, 1+1+3, 1+1+1+1+1 | 5 |
| 6 | . — | 8 |

Guess $a_n = \begin{cases} F_{n-1} & \text{for } n \geq 1 \\ 1 & \text{for } n = 0 \end{cases}$

and indeed,

$$A(x) = \sum_{n\geq 0} a_n x^n = \frac{1}{1-(x^1+x^3+x^5+\dots)}$$

$$= \frac{1}{1-\frac{x}{1-x^2}} = \frac{1-x^2}{1-x^2-x} = 1 + \underbrace{\frac{x}{1-x-x^2}}_{\overset{?}{=}} \overset{?}{=} 1 + \sum_{n\geq 1} F_{n-1} x^n \checkmark$$

seen earlier

## ④ Stirling numbers of the 2nd kind

$S(n,k) :=$ # set-partitions of $[n]$ into exactly $k$ (nonempty) blocks

for $1 \leq k \leq n$

$=$ rank numbers for the poset $(\Pi_n, \leq)$, where $\Pi_n = \{$all set partitions of $[n]\}$ and $\leq$ is refinement, i.e. $\pi_1 \leq \pi_2$ if $\pi_1$ refines $\pi_2$

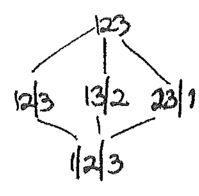
e.g. $n=1$: $S(1,1)=1$ trivial

$n=2$:


$S(2,1)=1$

$S(2,2)=1$

$n=3$:

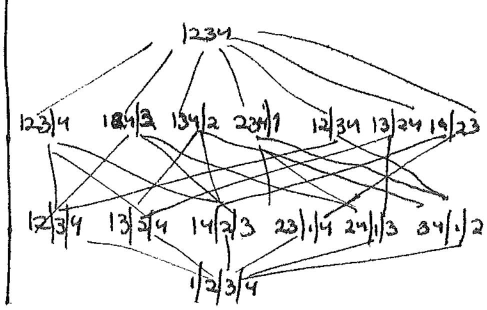



$S(3,1)=1$

$S(3,2)=3$

$S(3,3)=1$

$n=4$:




$S(4,1)=1$

$S(4,2)=7$

$S(4,3)=6$

$S(4,4)=1$

| $n \backslash k$ | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | . | . | |
| 2 | 1 | 1 | 0 | - | - | |
| 3 | 1 | 3 | 1 | 0 | - - | |
| 4 | 1 | 7 | 6 | 1 | 0 | . . |
| 5 | 1 | | | 10 | 1 | 0 ... |

(Pascal-like)
Recurrence:

$$S(n,k) = \underbrace{S(n-1,k-1)}_{n \text{ is a singleton block}} + \underbrace{k\,S(n-1,k)}_{n \text{ goes in one of the } k \text{ other blocks}} \quad \text{for } k \geq 2$$

and $S(n,1) = 1 \;\forall n$

$S(0,0)=1$

$S(n,k)=0$ if $k>n$

(32)

Let's get $F_k(x) := \sum_{\substack{\text{set partitions } \pi \\ \text{with } k \text{ blocks}}} x^{|\pi|} = \sum_{\substack{n \geq k \\ (\text{or } n \geq 0)}} S(n,k) x^n$ in 2 ways.

(a) Solve recurrence: For $k \geq 2$

$$\sum_{n \geq 0} S(n,k) x^n = \sum_{n \geq 0} S(n-1,k-1) x^n + \sum_{n \geq 0} k S(n-1,k) x^n$$

$$F_k(x) = x F_{k-1}(x) + kx F_k(x)$$

$$(1-kx) F_k(x) = x F_{k-1}(x)$$

$$\boxed{F_k(x) = \frac{x}{1-kx} F_{k-1}(x)}$$

and for $k=1$, $F_1(x) = \sum_{n \geq 0} S(n,1) x = x + x^2 + x^3 + \dots = \frac{x}{1-x}$

$$\Rightarrow F_k(x) = \frac{x}{1-kx} \cdot \frac{x}{1-(k-1)x} \cdots \frac{x}{1-2x} \cdot \frac{x}{1-x}$$

$$\boxed{\sum_{n \geq k} S(n,k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}}$$

(b) (Ardila 2.2.2 #13) Let $A_m$ := the structure strings of letters from $[m]$ that start with an $m$ whose weight is their length, e.g. for $m=3$

$\underbrace{31312}_{\text{weight 5}}$ or $\underbrace{3311}_{\text{weight 4}}$

and $A_m(x) = \frac{x}{1-mx} = x + mx^2 + m^2x^3 + \dots$

PROP: $\left\{ \begin{array}{l} \text{Set partitions} \\ \pi \text{ of } [n] \\ \text{with } k \text{ blocks} \end{array} \right\} \overset{\text{bijection}}{\longleftrightarrow} \left\{ \begin{array}{l} \text{total weight } n \text{ structures in} \\ A_1 \times A_2 \times \dots \times A_k \end{array} \right\}$

$\pi \longmapsto$ the restricted growth function $[n] \xrightarrow{f} [k]$ associated to $\pi$

$n=16$, $k=4$

$\pi = \underline{1},2,4,\overset{①}{5},8,12 \mid \underline{3},6,\overset{②}{9},10,14 \mid \underline{7},\overset{③}{11},16 \mid \overset{④}{\underline{13}},15$

number the blocks ①, ②, ..., ⓚ according to increasing smallest elements

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(i)$ | ① | ① | ② | ① | ① | ② | ③ | ① | ② | ② | ③ | ① | ④ | ② | ④ | ③ |
| | $\in A_1$ | | $\in A_2$ | | | | $\in A_3$ | | | | | | $\in A_4$ | | | |

$f(i)$ := block number of $i$

proof: EXERCISE ∎

COR: $\sum S(n,k) x^k = \frac{x}{1-x} \frac{x}{1-2x} \cdots \frac{x}{1-kx} = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}$
$= A_1(x) A_2(x) \cdots A_k(x)$

(33)

$\#\{\sigma \in \mathfrak{S}_n : \#\text{cycles}(\sigma) = k\}$

How are $S(n,k)$ and $c(n,k)$ related?

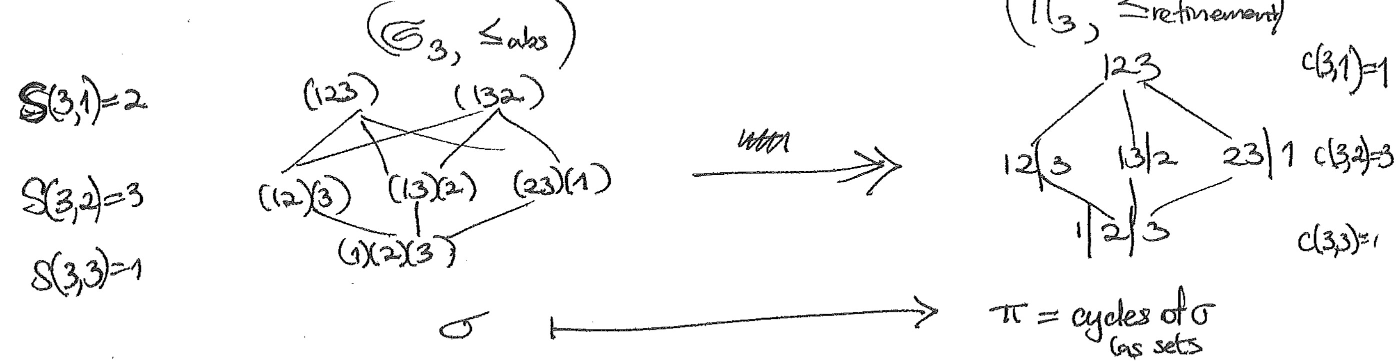
$S(n,k)$: Stirling #'s of 2nd kind; $c(n,k)$: (Signless) Stirling #'s of 1st kind

| $n \backslash k$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 0 | - | - | |
| 2 | 1 | 1 | 0 | - | |
| 3 | 2 | 3 | 1 | 0 | - |
| 4 | 6 | 11 | 6 | 1 | 0 |
| 5 | 24 | 50 | 35 | 10 | 1 |

More Superficial: ① The $c(n,k)$ have a similar recurrence

$$c(n,k) = \underbrace{c(n-1,k-1)}_{n \text{ goes in a 1-cycle}} + \underbrace{(n-1)\,c(n-1,k)}_{n \text{ maps to some } i \text{ in } [n-1]}$$

② They are rank #'s for posets with an order & rank-preserving surjection relating them:




$\sigma \longmapsto \pi$ = cycles of $\sigma$ (as sets)

9/30/2015 > ③ The real reason comes from this...

**PROP**: (a) $x^n = \sum_{k=1}^{n} S(n,k)\,(x)_k$ where $(x)_k := x(x-1)(x-2)\cdots(x-(k-1))$

while (b) $(x)_n = \sum_{k=1}^{n} s(n,k)\, x^k$

where $s(n,k) \overset{DEF}{=} (-1)^{n-k} c(n,k)$ = (signed) Stirling # of 1st kind

and hence (c) the infinite matrices $(S(n,k))_{\substack{n=1,2,\ldots \\ k=1,2,\ldots}}$, $(s(n,k))_{\substack{n=1,2,\ldots \\ k=1,2,\ldots}}$

give the inverse change-of-basis matrices between the ordered bases $\{x^n\}_{n=0,1,2,\ldots}$, $\{(x)_n\}_{n=0,1,2,\ldots}$ of $\mathbb{C}[x]$

(d) In particular, $\sum_{k=1}^{m} S(n,k)\,s(k,m) = \delta_{n,m} = \sum_{k=1}^{n} s(n,k)\,S(k,m)$.

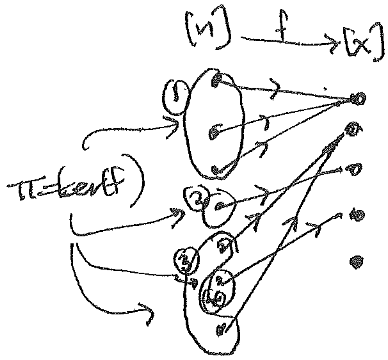
proof: For (a), note both sides lie in $\mathbb{C}[x]$ (of degree $n$),

so it is enough to prove it holds for $x = 1, 2, 3, \ldots$

(a polynomial $f(x) \in \mathbb{C}[x]$ that vanishes for $x = 1, 2, 3, \ldots$ must have $f \equiv 0$ ) — A useful general proof principle!

($f(x)$ = LHS − RHS)

For $x \in \mathbb{P}$, $x^n = \#\left\{ \begin{matrix} \text{functions} \\ [n] \xrightarrow{f} [x] \end{matrix} \right\} = \sum_{\substack{\text{set partitions} \\ \pi \text{ of } [n]}} \#\left\{ \begin{matrix} f : [n] \to [x] \\ \text{having } \pi \\ \text{as its fibers } \{f^{-1}(i)\}_{i \in [x]} \end{matrix} \right\}$




$$= \sum_{k=1}^{n} S(n,k) \underbrace{x(x-1)(x-2)\cdots(x-(k-1))}_{(x)_k}$$

For (b), recall $x(x+1)(x+2)\cdots(x+(n-1)) = \sum_{k=1}^{n} c(n,k) x^k$

$\wr$ $x \mapsto -x$, then multiply by $(-1)^n$

$$x(x-1)(x-2)\cdots(x-(n-1)) = \sum_{k=1}^{n} \underbrace{(-1)^{n-k} c(n,k)}_{s(n,k)} x^k$$

Parts (c), (d) then follow. ∎

---

Back to o.g.f. theory examples...

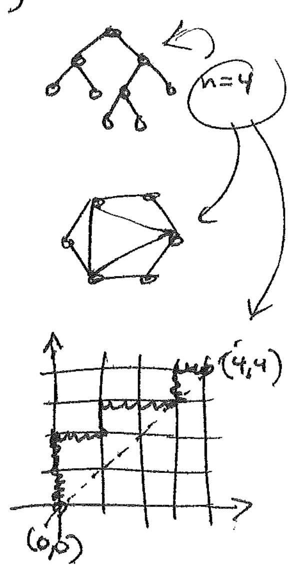
(Ardila 2.2.12 #14)

⑤ The Catalan family (see Stanley's book(!) on this topic)

$C_n$ = Catalan number $:= \#\left\{ \begin{matrix} \text{plane binary trees} \\ \text{with } n+1 \text{ leaves} \end{matrix} \right\}$ (or $n$ internal vertices, each having a left/right child)

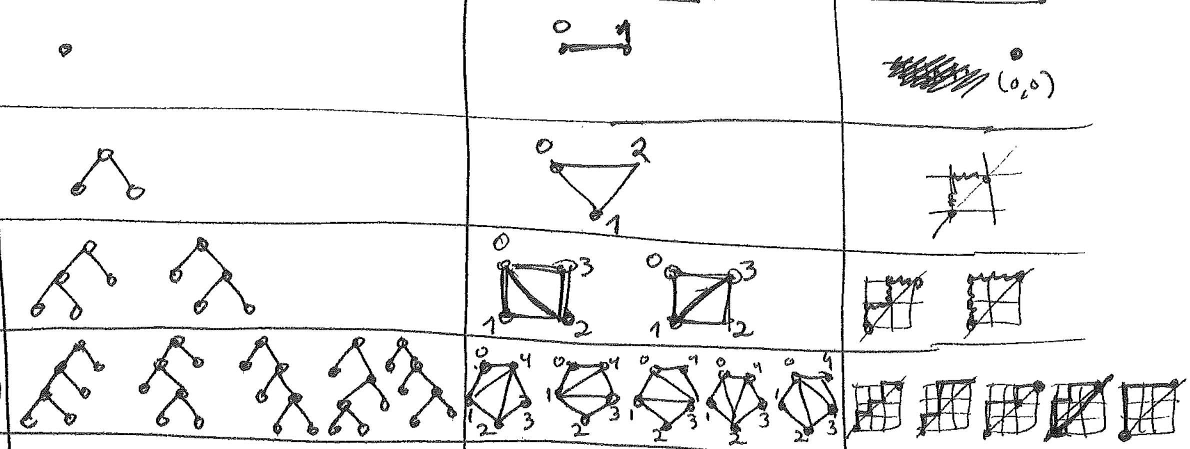
$= \#\{\text{triangulations of an } (n+2)\text{-gon}\}$

$= \#\left\{ \begin{matrix} \text{lattice paths taking } N, E \text{ steps} \\ (0,0) \to (n,n) \\ \text{staying (weakly) above } y=x \end{matrix} \right\}$




THEOREM: $C_n = \frac{1}{n+1}\binom{2n}{n} \left( = \frac{(2n)!}{(n+1)!\,n!} = \frac{1}{2n+1}\binom{2n+1}{n} \right)$

| $n$ | $C_n$ | plane binary trees | triangulations | lattice paths |
|---|---|---|---|---|
| 0 | $1=\frac{1}{1}\binom{0}{0}$ | | | |
| 1 | $1=\frac{1}{2}\binom{2}{1}$ | | | |
| 2 | $2=\frac{1}{3}\binom{4}{2}$ | | | |
| 3 | $5=\frac{1}{4}\binom{6}{3}$ | | | |
| 4 | $14=\frac{1}{5}\binom{8}{4}$ | ... | ... | ... |

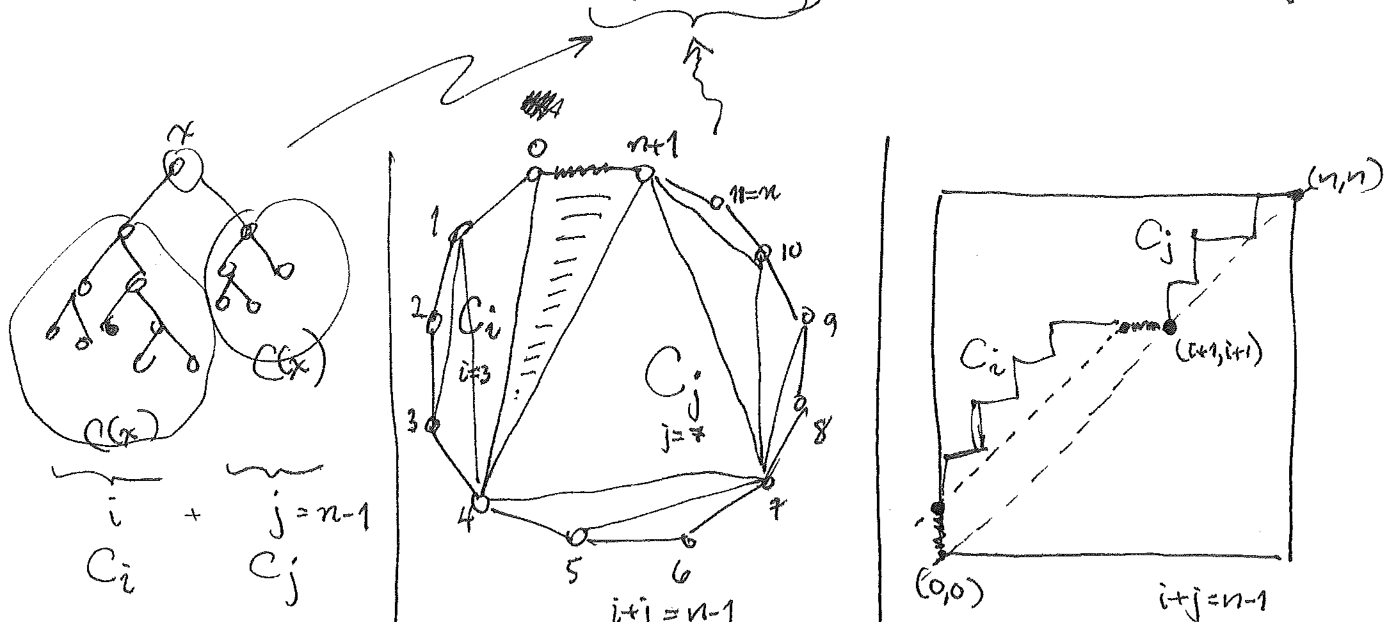



(1st) <u>proof of THM1:</u>

$$C(x) := \sum_{n\geq 0} C_n x^n = 1 + \sum_{n\geq 1} C_n x^n$$

$$= 1 + \underbrace{C(x)\cdot x\cdot C(x)}$$

i.e. for $n\geq 1$, $C_n = \sum_{i+j=n-1} C_i C_j$




Consequently,
$$C(x) = 1 + x\,C(x)^2$$
$$0 = x\,C(x)^2 - C(x) + 1$$
$$C(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$
⟸ let's expand $\sqrt{1-4x}$ to figure out the +/− choice...

(36)

$$\sqrt{1-4x} = (1-4x)^{\frac{1}{2}} = \sum_{n\geq 0} \binom{\frac{1}{2}}{n}(-4x)^n = \sum_{n\geq 0}{}' \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{(2n-3)}{2})}{n!}(-1)^n 4^n x^n$$

$$= 1 - 2\sum_{n\geq 1}{}' \frac{2^{n-1}(1)(3)\cdots(2n-3)}{n!} x^n$$

$$= 1 - 2x \sum_{n\geq 1} \frac{(2)(4)\cdots(2n-2)\cdot(1)(3)\cdots(2n-3)}{(n-1)!\, n!} x^{n-1}$$

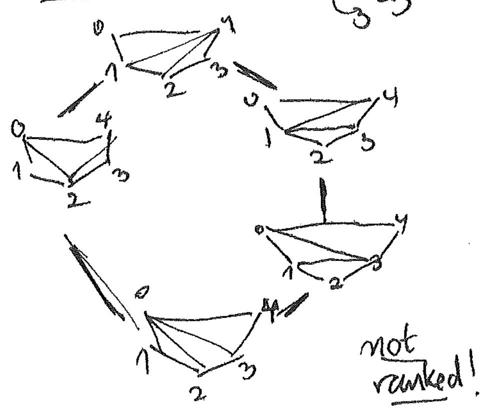
$$= 1 - 2x \sum_{n\geq 1} \frac{1}{n}\binom{2n-1}{n-1} x^{n-1}$$

$$\Rightarrow C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 1} \frac{1}{n}\binom{2n-1}{n-1} x^{n-1} = \sum_{n\geq 0} \frac{1}{n+1}\binom{2n}{n} x^n \quad \blacksquare$$

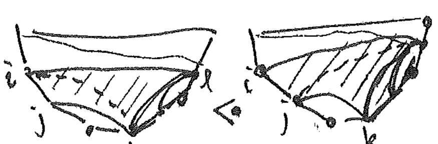
There are (at least) 3 different interesting poset ~~structures~~ on $C_n$ objects:

**Tamari lattice** on triangulations of $(n+2)$-gon

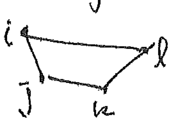
$n=3$: $C_3 = 5$




not ranked!

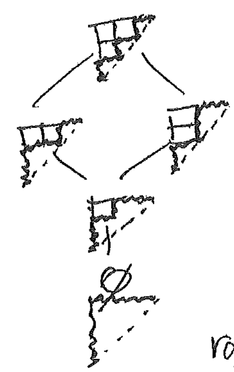



switch ik diagonal to jl diagonal in




**The interval** $[\varnothing, \text{staircase}]$ in Young's lattice $\mathbb{Y}$

rank #'s: 1, 2, 1, 1

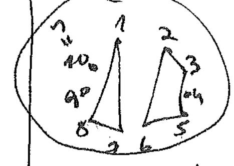


ranked;

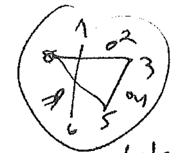
Q: Are its rank numbers unimodal? (Stanton)

$1 \leq 1 \leq 2 \geq 1$

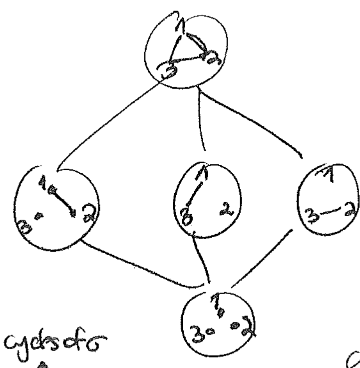
**NC(n)** := noncrossing set partitions of $[n]$



17|2356|4|9|10 is noncrossing



16|2|358|4|7 is crossing



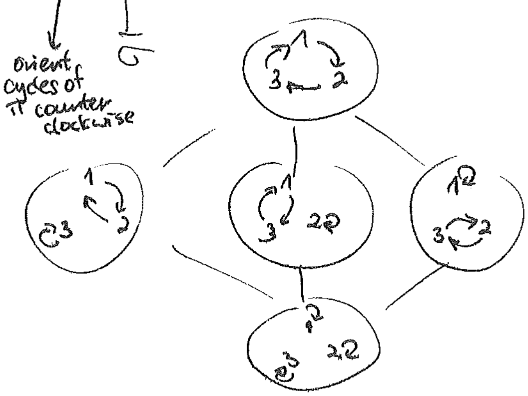

ranked; rank #'s are called Narayana numbers.

≅

$[e, (1\,2\,3\cdots n)]_{\text{abs order on } \mathfrak{S}_n}$

$\pi$ → cycles of $\sigma$

orient cycles of $\pi$ counterclockwise → $\sigma$



⑥ Let $a_n := \#\{\sigma \in \mathfrak{S}_n : \sigma$ is indecomposable/irreducible, $\}$ for $n \geq 1$
i.e. it can't be factored as $\sigma = \sigma_1 \sigma_2$, $\sigma_1 \in \mathfrak{S}_{\{1,2,\ldots,k\}}$, $\sigma_2 \in \mathfrak{S}_{\{k+1,k+2,\ldots,n\}}$ for some $1 \leq k < n$.

e.g. $\sigma = (1\,3\,5)(2)(4) \in \mathfrak{S}_5$ is irreducible

$\sigma = \underbrace{(1\,3)(2)}_{\in \mathfrak{S}_{\{1,2,3\}}}\underbrace{(4\,5)}_{\in \mathfrak{S}_{\{4,5\}}}$ is not

| $n$ | irreducibles in $\mathfrak{S}_n$ | $a_n$ |
|---|---|---|
| ~~0~~ | ~~$\emptyset$~~ | ~~0~~ |
| 1 | $e$ | 1 |
| 2 | $(12)$ | 1 |
| 3 | $(123)$, $(132)$, $(13)(2)$, ~~$(1)(2)(3)$~~, ~~$(12)(3)$~~, ~~$(1)(23)$~~ | 3 |
| 4 | ... | 13 |

Q: How to compute $a_n$?

Note permutations = Seq((non-$\emptyset$) irreducible permutations)

So if we let $A(x) = \sum_{n \geq 0} a_n x^n$

and $B(x) = \sum_{n \geq 0} \underbrace{n!}_{|\mathfrak{S}_n|} x^n$ ($\in \mathbb{C}[[x]]$ but its usual radius of convergence is 0!)

then $$B(x) = \frac{1}{1 - A(x)}$$

so $$A(x) = 1 - \frac{1}{B(x)} = \frac{1}{1 - \sum_{n \geq 0} n!\, x^n}$$ $\xrightarrow{\text{computer algebra packages}}$ $x + x^2 + 3x^3 + 13x^4 + 71x^5 + 461x^6 + \ldots$

10/5/2015

Exponential generating functions (Ardila §2.3)

$\mathcal{A}$ = structure one can place on $n$ labelled objects like $[n]$

$a_n$ = # of such structures

$\rightsquigarrow A(x) := \sum_{n \geq 0} a_n \frac{x^n}{n!}$ =: exponential gen. fn. for $\mathcal{A}$

PROP:
- If $\mathcal{C}$-structures are choice of $\mathcal{A}$ or $\mathcal{B}$-structure ("$\mathcal{C} = \mathcal{A} + \mathcal{B}$") then $C(x) = A(x) + B(x)$
- If $\mathcal{C}$-structures on $[n]$ are a choice of a partition $[n] = S_1 \sqcup S_2$ with an $\mathcal{A}$-structure on $S_1$, $\mathcal{B}$-structure on $S_2$ ("$\mathcal{C} = \mathcal{A} * \mathcal{B}$") so that $c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$, then $C(x) = A(x) B(x)$.
- If $\mathcal{C}$-structures are a choice of (unordered) set partition $\pi$ of $[n]$ and then an $\mathcal{A}$-structure on each block of $\pi$, ("$\mathcal{C} = \mathrm{Set}(\mathcal{A})$") then $$\boxed{C(x) = e^{A(x)}}$$ ← The exponential formula

proof: • $\mathcal{C} = \mathcal{A} + \mathcal{B}$ is obvious

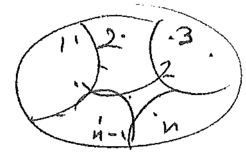
• For $\mathcal{C} = \mathcal{A} * \mathcal{B}$, note

$$c_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

$$\Leftrightarrow \frac{c_n}{n!} = \sum_{i+j=n} \frac{a_i}{i!} \frac{b_j}{j!}$$

$j := n-i$ so $\binom{n}{i} = \frac{n!}{i!\,j!}$

$$\Leftrightarrow C(x) = A(x) B(x).$$

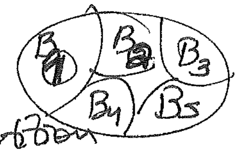



• For $\mathcal{C} = \mathrm{Set}(\mathcal{A})$, note

$$\mathcal{C} = \dot{\bigsqcup_{k=1}^{\infty}} \mathcal{A}^{(k)}$$

where $\mathcal{A}^{(k)} = \{$pick a set partition $\pi$ into exactly $k$ unlabeled blocks and put $\mathcal{A}$-structures on each block$\}$

so $C(x) = \sum_{k=1}^{\infty} A^{(k)}(x)$.




But $k!A^{(k)}(x) = A(x)^k$ = e.g.f. for $\underbrace{\mathcal{A} \times \mathcal{A} \times \ldots \times \mathcal{A}}_{k \text{ times}}$

$= \{$pick a set partition $\pi = B_1 \sqcup \ldots \sqcup B_k$ into $k$ labeled blocks, and put $\mathcal{A}$-structures on each block$\}$.

Hence $A^{(k)}(x) = \frac{A(x)^k}{k!}$

and $C(x) = \sum_{k=1}^{\infty} \frac{A(x)^k}{k!} = e^{A(x)}$ ∎

EXAMPLES:

① Recall $d_n = \#\{\text{derangements in } \mathfrak{S}_n\}$, $D(x) := \sum_{n \geq 0} \frac{d_n}{n!} x^n$

(hat check problem, probability of all wrong hats)

$$\{\text{all permutations}\} = \{\text{fixed point only perms, i.e. identity perms}\} * \{\text{derangements (fixed point free perms)}\}$$

so

$$\sum_{n \geq 0} n! \cdot \frac{x^n}{n!} = \left( \sum_{n \geq 0} 1 \cdot \frac{x^n}{n!} \right) \cdot D(x)$$

$$\frac{1}{1-x} = e^x \cdot D(x)$$

i.e. $D(x) = \frac{e^{-x}}{1-x}$, as we saw.

② $\left\{\underset{\sigma^2=1}{\text{Involutions}}\right\} = \text{Set}\left(\left\{\begin{matrix}\text{Involutions}\\ \text{with exactly}\\ \text{one cycle}\end{matrix}\right\}\right)$

Hence
$$\sum_{n\geq 0} \#\{\sigma\in\mathfrak{S}_n : \sigma^2=1\}\frac{x^n}{n!} = e^{\sum_{n\geq 0}\#\left\{\begin{matrix}\sigma\in\mathfrak{S}_n:\\ \sigma^2=1,\ \sigma\text{ has}\\ \text{exactly one cycle}\end{matrix}\right\}\frac{x^n}{n!}}$$
$$= e^{0\cdot\frac{x^0}{0!}+1\cdot\frac{x^1}{1!}+1\cdot\frac{x^2}{2!}+\frac{0\cdot x^3}{3!}+\frac{0\cdot x^4}{4!}+\cdots}$$
$$= e^{x+\frac{x^2}{2}} \text{ as we saw.}$$

③ More generally, Touchard's THM comes from this:
$$\{\text{permutations}\} = \text{Set}\left(\left\{\begin{matrix}\text{perms}\\ \text{with exactly one cycle}\end{matrix}\right\}\right)$$

and if we weight $\sigma$ by $t_1^{c_1(\sigma)}t_2^{c_2(\sigma)}\cdots$, then the weights are multiplicative for these decompositions

so
$$\sum_{n\geq 0}\frac{x^n}{n!}\left(\sum_{\sigma\in\mathfrak{S}_n} t_1^{c_1(\sigma)}t_2^{c_2(\sigma)}\cdots\right) = e^{\sum_{n\geq 0}\frac{x^n}{n!}\left(\sum_{\substack{\sigma\in\mathfrak{S}_n:\\ \sigma\text{ has exactly}\\ \text{one cycle}}} t_1^{c_1(\sigma)}t_2^{c_2(\sigma)}\cdots\right)}$$
$$= e^{\sum_{n\geq 1}\frac{x^n}{n!}\cdot t_n\underbrace{(n-1)!}}$$

↖ there are $(n-1)!$ $n$-cycles in $\mathfrak{S}_n$ $(1\ \underbrace{a_1\ a_2\cdots a_{n-1}})$ — an arbitrary sequence

$$= e^{\sum_{n\geq 1} t_n\frac{x^n}{n}}$$
$$= e^{t_1\frac{x^1}{1}+t_2\frac{x^2}{2}+t_3\frac{x^3}{3}+\cdots}$$

④ Bell numbers $B_n := \#\{\text{set partitions } \pi \text{ of } [n]\}$

Bell polynomials $B_n(y) := \sum_{\substack{\text{set partitions}\\ \pi\text{ of }[n]}} y^{\#\text{blocks}(\pi)} = \sum_{k=1}^{n} S(n,k)y^k$

Since $\{\text{set partitions}\} = \text{Set}\left(\left\{\begin{matrix}\text{single (non-empty)}\\ \text{block partitions}\end{matrix}\right\}\right)$

$$\sum_{n\geq 0} B_n\frac{x^n}{n!} = e^{1\cdot\frac{x^1}{1!}+1\cdot\frac{x^2}{2!}+1\cdot\frac{x^3}{3!}+\cdots}$$
$$= e^{(e^x-1)}$$

and
$$\sum_{n\geq 0} B_n(y)\frac{x^n}{n!} = e^{1\cdot y\cdot\frac{x^1}{1!}+1\cdot y\frac{x^2}{2!}+1\cdot y\frac{x^3}{3!}+\cdots} = e^{y(e^x-1)}$$

↑ ordinary in $y$ ↖ exponential in $x$

$[y^k] \longrightarrow$ COR: Fix $k$. $\sum_{k=1}^{n} S(n,k)\frac{x^n}{n!} = \frac{(e^x-1)^k}{k!}$

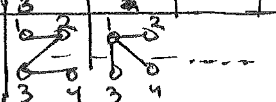
⑤ Let's count connected, simple graphs $G = (V, E)$ with $V = [n]$, $E \subseteq \binom{[n]}{2}$

weighted by their number of edges i.e. $y^{|E|}$:

$y=1$ ↗ $c_n = \#\{\text{conn. graphs on } [n]\}$

$C_n(y) = \sum_{\substack{\text{conn. graphs} \\ G \text{ on } [n]}} y^{\#\text{edges}(G)}$

Can we get at $\mathrm{Conn}(x,y) := \sum_{n\geq 1} \frac{x^n}{n!} C_n(y)$ ?

| $n$ | | $c_n$ | $C_n(y)$ |
|---|---|---|---|
| 1 | 1 o | 1 | 1 |
| 2 | 1 2 o–o | 1 | $y$ |
| 3 |  | 4 | $3y^2+y^3$ |
| 4 |  ...... | 38 | $16y^3+15y^4+6y^5+y^6$ |

$All := \{\text{all simple graphs}\} = \mathrm{Set}(\{\text{connected simple graphs}\})$

so $All(x,y) = e^{\mathrm{Conn}(x,y)}$

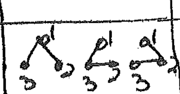
$$\mathrm{Conn}(x,y) = \log(All(x,y)) = \log\Big(\sum_{n\geq 0} \frac{x^n}{n!} \sum_{\substack{\text{simple graphs} \\ G \text{ on } [n]}} y^{\#\text{edges}(G)}\Big)$$

$$= \log\Big(1 + \sum_{n\geq 1} \frac{x^n (1+y)^{\binom{n}{2}}}{n!}\Big)$$

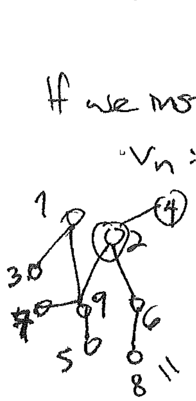
$$= x + \frac{x^2}{2!} 1y + \frac{x^3}{3!}(3y^2+y^3) + \frac{x^4}{4!}(16y^3+15y^4+6y^5+y^6)$$

$$+ \frac{x^5}{5!}(125y^4+222y^5+205y^6+120y^7+45y^8+10y^9+y^{10}) + \dots$$

⑥ Let's try to get some information about $t_n := \#\{\text{trees on } [n]\}$ and $T(x)$ its e.g.f.

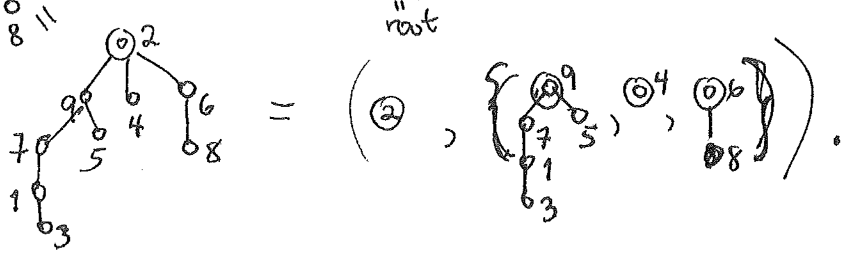
| $n$ | trees | $t_n$ |
|---|---|---|
| 1 | 1 o | 1 |
| 2 | 1 2 o–o | 1 |
| 3 |  | 3 |
| 4 |  ... | 16 |
| 5 | ... | 125 |

If we instead look at $v_n := \#\{\text{vertex-rooted trees on } [n]\}$,



we see that $v_n = n \cdot t_n$

and $V = \{\text{single-vertex trees} = \text{root}\} * \mathrm{Set}(V)$



Hence $\boxed{V(x) = x e^{V(x)}}$, where $V(x) = \sum_{n\geq 1} v_n \frac{x^n}{n!}$

We could rephrase this is $\frac{V(x)}{e^{V(x)}} = x$

or $V(x)$ is the compositional inverse to $A(x) = xe^{-x}$ within $\mathbb{C}[[x]]$

( (easy) PROP: If $A(x) = a_1 x + a_2 x^2 + \cdots \in R[[x]]$ has no constant term ($a_0 = 0$) so that $B(A(x))$ is well-defined, then $A$ has a compositional inverse $B = A^{\langle -1 \rangle}$ satisfying $B(A(x)) = x$ ($\Rightarrow A(B(x)) = x$ by associativity of $A \circ B$)

$\Leftrightarrow a_1 \in R^{\times}$ )

---

Does knowing $V(x) = A^{\langle -1 \rangle}(x)$ for $A(x) = xe^{-x}$ help us?

In this case, it does, via...

Lagrange Inversion Thm:

If $B(x) = A^{\langle -1 \rangle}(x)$, that is, $B(A(x)) = x$ for some $A(x), B(x) \in x\mathbb{C}[[x]]$

then $[x^n]B(x) = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right) \quad \left( = \frac{1}{n}[x^{n-1}]\left(\frac{x}{A(x)}\right)^n \right)$

---

Before we prove it, let's do two examples...

EXAMPLE:

① $V(x) = \sum_{n \geq 0} v_n \frac{x^n}{n!}$ where $v_n$ = #vertex-rooted trees on $[n]$ $= n t_n$

has $V(x) = A^{\langle -1 \rangle}(x)$ for $A(x) = xe^{-x}$

so $\frac{v_n}{n!} = [x^n]V(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n = \frac{1}{n}[x^{n-1}]e^{nx} = \frac{1}{n} \cdot \frac{n^{n-1}}{(n-1)!} = \frac{n^{n-2}}{(n-1)!}$

$\Rightarrow v_n = n^{n-1}$

$\boxed{t_n = \frac{v_n}{n} = n^{n-2}}$ Cayley's THM

② (Ardila 2.2.2 #15)

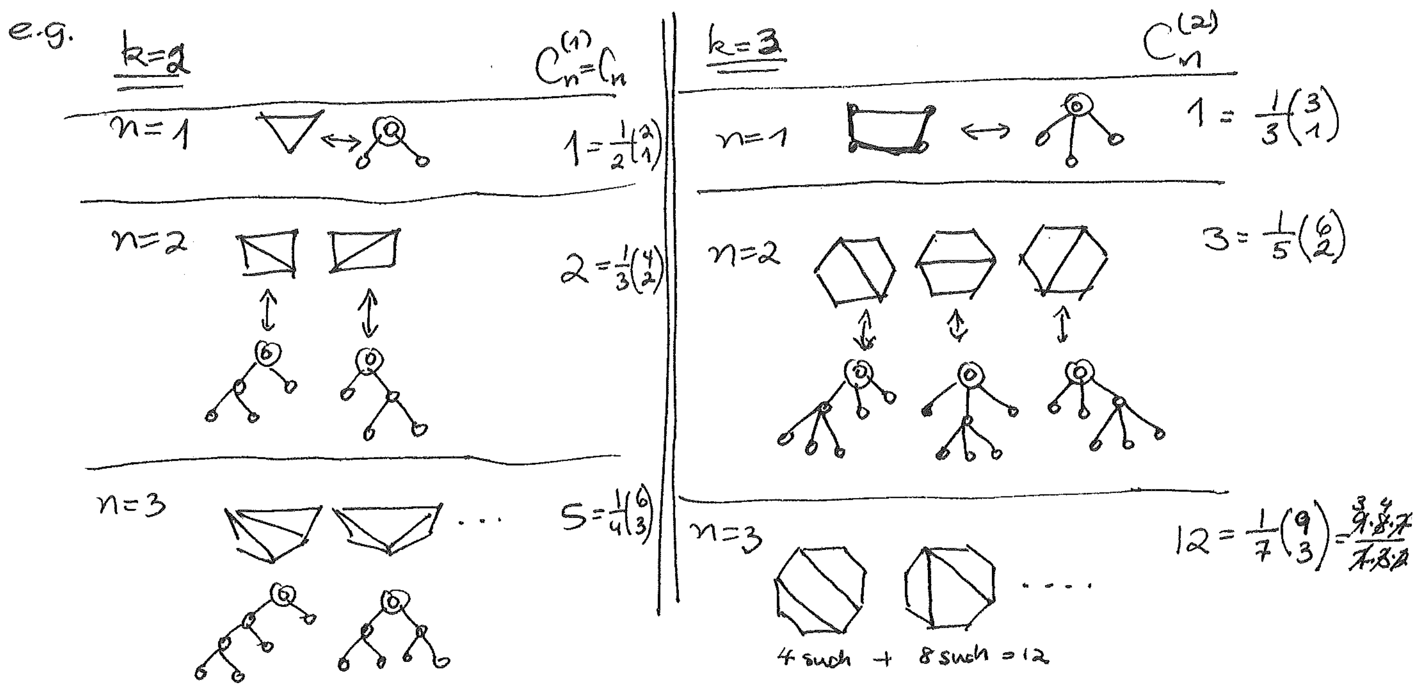
Generalizing Catalan #'s $C_n$, let's define the

**Fuss–Catalan #** $C_n^{(k)} := \#\{$ k-ary rooted plane trees with n internal vertices $\}$

(internal vertices: each having k children ordered left-to-right)

$C_n \overset{k=1}{\leftarrow}$ $= \#\{ (k+1)\text{-angulations of a } ((k-1)n+2)\text{-gon} \}$

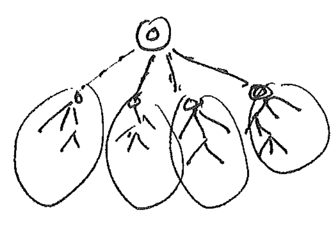
e.g.




THM: $C_n^{(k)} = \frac{1}{(k-1)n+1}\binom{kn}{n}$ $\left( \overset{k=2}{\rightsquigarrow} \frac{1}{n+1}\binom{2n}{n} = C_n \right)$

proof:

$C(x) := \sum_{n\geq 0} C_n^{(k)} \cdot x^n$ satisfies $C(x) = 1 + x\,C(x)^k$ :



So $B(x) := C(x) - 1$ satisfies $B(x) = x\,(B(x)+1)^k$
$= \sum_{n\geq 1} C_n^{(k)} x^n$

$$\frac{B(x)}{(B(x)+1)^k} = x \quad \text{i.e. } B(x) = A^{\langle -1 \rangle}(x) \text{ for } A(x) = \frac{x}{(x+1)^k}$$

Hence Lagrange inversion says

$$C_n^{(k)} = [x^n]\,B(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{\frac{x}{(x+1)^k}}\right)^n = \frac{1}{n}[x^{n-1}](x+1)^{kn} = \frac{1}{n}\binom{kn}{n-1} = \frac{(kn)!}{n!\,(kn-(n-1))!} = \frac{(kn)!}{n!\,((k-1)n+1)!}$$

$$= \frac{1}{(k-1)n+1}\binom{kn}{n} \quad \blacksquare$$

## proof of Lagrange Inversion Thm:

Let $B(x) = \sum_{n\geq 1} b_n x^n$ and assuming $x = B(A(x)) = \sum_{m\geq 1} b_m A(x)^m$,

we want to show $b_n = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right)$:

$\Big\downarrow \frac{d}{dx}$

$$1 = \sum_{m\geq 1} m b_m A(x)^{m-1} A'(x)$$

$\Big\downarrow$ divide by $A(x)^n$

$$\frac{1}{A(x)^n} = \sum_{m\geq 1} m b_m A(x)^{m-n-1} A'(x)$$

(working in $x^{-n}\mathbb{C}[[x]]$ or $\bigcup_{n\geq 0} x^{-n}\mathbb{C}[[x]]$ Laurent series about 0)

$$\frac{1}{A(x)^n} = \underbrace{n b_n \frac{A'(x)}{A(x)}}_{\text{the } m=n \text{ term}} + \underbrace{\sum_{\substack{m\geq 1\\ m\neq n}} m b_m \frac{d}{dx}\left(\frac{A(x)^{m-n}}{m-n}\right)}_{\text{all other terms}}$$

$\Big\downarrow$ take $[x^{-1}]$

$$[x^{-1}]\left(\frac{1}{A(x)^n}\right) = n b_n [x^{-1}] \frac{a_1 x^0 + 2a_2 x^1 + 3a_3 x^2 + \ldots}{a_1 x^1 + a_2 x^2 + a_3 x^3 + \ldots} + \sum_{\substack{m\geq 1\\ m\neq 0}} (0)$$

$$= n b_n$$

LEMMA: since any Laurent series $f(x) = c_{-n}x^{-n} + c_{-n+1}x^{-n+1} + \ldots = \sum_{k\geq -n} c_k x^k$ has $[x^{-1}] f'(x) = 0$

i.e. $b_n = \frac{1}{n}[x^{-1}]\left(\frac{1}{A(x)^n}\right)$

∎

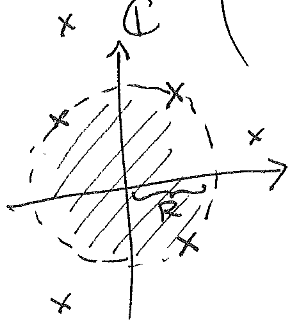
(44) <u>A quick peek at asymptotic coefficient estimation</u>

(see Wilf §2.4 & Ch.5,
Flajolet & Sedgewick "Analytic Combinatorics")

<u>THM</u> (Wilf 2.4.3)

If $f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[[x]]$ has <u>radius of convergence</u> $R$ in $\mathbb{C}$,

(i.e. it is analytic for $|z| < R$ but has one or more singularities $z_0$ with $|z_0| = R$)



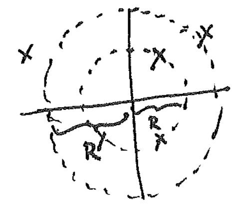

then $\forall \epsilon > 0$, $\exists N$ that $\forall n > N$, $|a_n| < \left(\frac{1}{R} + \epsilon\right)^n$

and for infinitely many $n$, $|a_n| > \left(\frac{1}{R} - \epsilon\right)^n$

(i.e. roughly $|a_n| \approx \frac{1}{R^n}$)

proof: (Complex analysis — see Wilf) ∎

But then if the singularities of $f(x)$ at $z_0$ with $|z_0| = R$ are tame enough, we can subtract off something we understand, and get errors that grow like $\frac{1}{(R')^n}$ where the next further out singularities $z_0'$ have $|z_0'| = R' > R$.




10/12/2015 →

EXAMPLE: (Wilf p.175) Let $\tilde{B}_n :=$ <u>ordered</u> Bell #

$:= \#\{\text{ordered set partitions } \pi = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k \text{ of } [n]\}$

$= \sum_{k=1}^{n} \underbrace{k!\, S(n,k)}_{\#\{\text{ordered set partitions with } k \text{ labelled blocks } B_1, \ldots, B_k\}}$

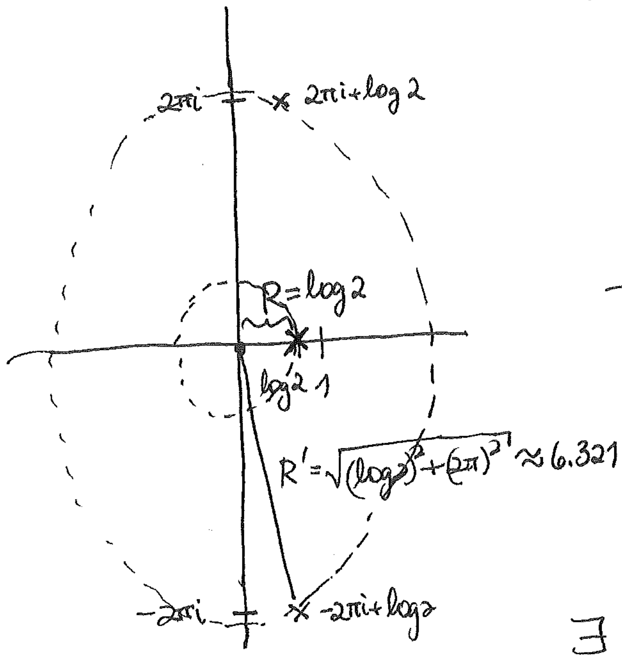
Then $f(x) = \sum_{n\geq 0} \tilde{B}_n \frac{x^n}{n!} = 1 + \sum_{k\geq 1} \sum_{n\geq 0} k!\, S(n,k) \frac{x^n}{n!}$

$= 1 + \sum_{k\geq 1} (e^x - 1)^k = \frac{1}{1-(e^x-1)} = \frac{1}{2-e^x}$

(45)

Note $f(x) = \dfrac{1}{2-e^x}$ has singularities only when $g(x) = 2 - e^x$ has zeroes, i.e. $e^x = 2$

so $x = \log 2 + 2\pi i \cdot k$ for $k \in \mathbb{Z}$.




Hence we expect $\dfrac{\tilde{B}_n}{n!} \approx \left(\dfrac{1}{\log 2}\right)^n$

But note $g(x) = 2 - e^x$ has $g(\log 2) = 0$

$g'(x) = -e^x$ has $g'(\log 2) = -e^{\log 2} = -2 \neq 0$

so the pole in $f(x)$ at $x = \log 2$ is <u>simple</u>, and

$\exists$ a constant $c$ (the <u>residue</u> of $f(x)$ at $x = \log 2$)

so that $h(x) = f(x) - \dfrac{c}{x - \log 2}$ is analytic in $|z| < R'$ ($R' = |2\pi i + \log 2|$)

$\Big\}$ mult. by $x - \log 2$

$$(x - \log 2)\, h(x) = (x - \log 2) f(x) - c$$

$\Big\}$ take $\lim_{x \to \log 2}$

$$0 = \lim_{x \to \log 2} \frac{x - \log 2}{2 - e^x} - c$$

L'Hôpital

$$c = \lim_{x \to \log 2} \frac{1}{-e^x} = \frac{1}{-2}$$

Hence $h(x) = \dfrac{1}{2 - e^x} - \underbrace{\dfrac{-\frac{1}{2}}{x - \log 2}}_{\text{analytic everywhere but } x = \log 2}$ has coefficients $\approx \left(\dfrac{1}{R'}\right)^{n+1}$

since it is analytic in $|z'| < R'$ with 1st poles on $|z'| = R'$

~~i.e. $f(x) = \dfrac{1}{2 - e^x} = \dfrac{-\frac{1}{2}}{x - \log 2} + h(x)$~~

~~$= \dfrac{\frac{1}{2}}{\log 2 - x} + h(x) = \dfrac{\log 2}{2\left(1 - \frac{x}{\log 2}\right)} + h(x)$~~

~~$= \sum_{n \geq 0} \frac{1}{2} \ldots$~~

(46) In other words,

$$\sum_{n\geq 0} \frac{\tilde{B}_n}{n!} x^n = f(x) = \frac{1}{2-e^x} = \frac{-\frac{1}{2}}{x-\log 2} + h(x)$$

$$= \frac{1}{2\log 2\left(1-\frac{x}{\log 2}\right)} + h(x)$$

$$= \sum_{n\geq 0} \frac{x^n}{2(\log 2)^{n+1}} + h(x)$$

so $\dfrac{\tilde{B}_n}{n!} \approx \dfrac{1}{2(\log 2)^{n+1}} + O\left(\left(\dfrac{1}{R'}\right)^n\right)$

$$\boxed{\tilde{B}_n \approx \frac{n!}{2(\log 2)^{n+1}}}$$

| $n$ | $\tilde{B}_n$ | $\frac{n!}{2(\log 2)^{n+1}}$ |
|---|---|---|
| 1 | 1 | 1.04 |
| 2 | 3 | 3.002 |
| 3 | 13 | 12.997 |
| ⋮ | | |
| 5 | 541 | 541.002 |
| ⋮ | | |
| 10 | 102247563 | 102247563 |