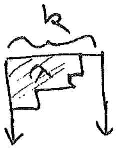
q-binomial coefficients (Stanley §1.7)

Recall $\sum_{\text{all }\lambda} q^{|\lambda|} = \sum_{n\ge 0} p(n)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$

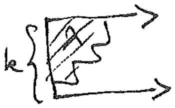


and $\sum_{\substack{\lambda:\\ \lambda_1\le k}} q^{|\lambda|} = \sum_{n\ge 0} \underbrace{p_{\le k}(n)}_{\text{Stanley's notation}} q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots(1-q^k)}$

$\lambda \updownarrow \lambda^t$

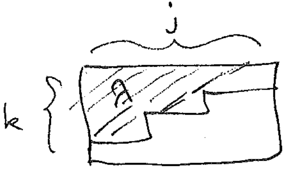
$\qquad\qquad \| $

$\sum_{\substack{\lambda:\\ \ell(\lambda)\le k}} q^{|\lambda|}$

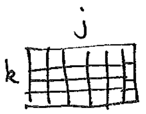


Q: What about $\sum_{\substack{\lambda:\\ \lambda_1\le j\\ \ell(\lambda)\le k}} q^{|\lambda|} \overset{\text{DEF}}{=:} \begin{bmatrix} j+k \\ k \end{bmatrix}_q$?

q-binomial coefficient

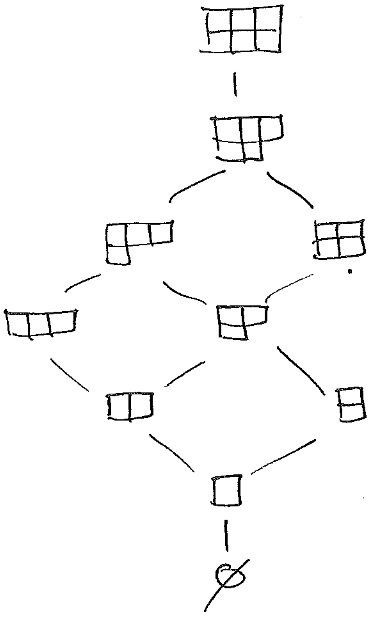


$\|$

rank generating function for $[\emptyset,$  $]_Y$

Y = Young's lattice

e.g. k=2
j=3



q^6

q^5

$2q^4$

$2q^3$

$2q^2$

$1\cdot q^1$

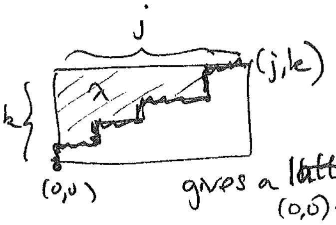
$1\cdot q^0$

$$\begin{bmatrix}5\\2\end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6$$
$$\left(=(1+q+q^2+q^3+q^4)(1+q^2)\right)$$

(48) Let's collect some properties of $\begin{bmatrix} j+k \\ k \end{bmatrix}_q \ldots$

PROP:

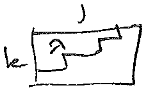
(a) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q \overset{q=1}{\rightsquigarrow} \binom{j+k}{k}$ (since k



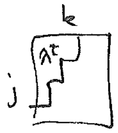
gives a ~~lattice~~ lattice path $(0,0) \to (j,k)$)

(b) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k \\ j \end{bmatrix}_q$

(since k



$\longleftrightarrow$



)

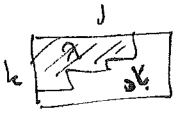
(c) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q =: \sum_{n=0}^{jk} p(j,k,n)\, q^n$ has symmetric coefficients:

$p(j,k,n) = p(j,k,jk-n)$

e.g. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = 1+q+2q^2+2q^3+2q^4+q^5+q^6$

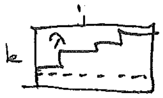
$(1,1,2,2,2,1,1)$

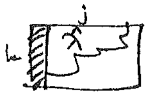
(since k



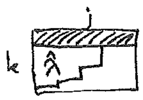
have $|\lambda| + |\lambda^c| = jk$)

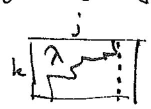
(d) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (1^{st} q-Pascal recurrence)





$= q^j \begin{bmatrix} j+k-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} j+k-1 \\ k \end{bmatrix}_q$ (2^{nd} q-Pascal recurrence)





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(e) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \sum_{\substack{\text{rearrangements} \\ w \text{ of } 0^j 1^k}} q^{\mathrm{inv}(w)}$

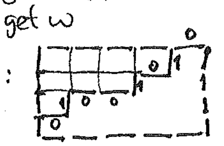
$w = (w_1, \ldots, w_{j+k})$, $0^j 1^k = \underbrace{00\cdots0}_{j}\underbrace{11\cdots1}_{k}$

where $\mathrm{inv}(w) := \#\{(a,b) : 1 \leq a < b \leq j+k, \; w_a > w_b\}$

# of inversions in w

e.g. $\mathrm{inv}(01010010) = 4+3+1 = 8$

(since can read boundary of λ backwards as 0=west, 1=south to get w and $|\lambda| = \mathrm{inv}(w)$:



)

(f) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \#\{k\text{-dimensional subspaces of } (\mathbb{F}_q)^{j+k}\}$ if $q = p^d$ is a prime power, so $q = |\mathbb{F}_q|$.

(g) $\begin{bmatrix} j+k \\ k \end{bmatrix}_q = \dfrac{[j+k]!_q}{[j]!_q [k]!_q}$ where $[n]!_q := [1]_q [2]_q \cdots [n]_q$

$[n]_q := 1+q+q^2+\ldots+q^{n-1} = \dfrac{1-q^n}{1-q} \left(= \dfrac{q^n-1}{q-1} \right)$

e.g. $\begin{bmatrix} 5 \\ 2 \end{bmatrix}_q = \dfrac{[5]!_q}{[3]!_q [2]!_q} = \dfrac{[5]_q[4]_q\cancel{[3]_q}\cancel{[2]_q}\cancel{[1]_q}}{\cancel{[3]_q}\cancel{[2]_q}\cancel{[1]_q} \cdot [2]_q[1]_q} = (1+q+q^2+q^3+q^4)\dfrac{(1+q+q^2+q^3)}{(1+q)}$

$= (1+q+q^2+q^3+q^4)(1+q^2)$ ✓

(49)

proof: (a)✓, (b)✓, (c)✓, (d)✓, (e)✓ proved already.

For (f), we claim that one has q bijection

(We could prove (f), (g) fairly easily using (d) and induction, but we won't.)

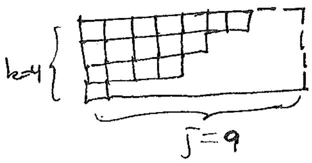
$$\begin{tikzcd}
 & \left\{\begin{matrix}\text{k-dim'l subspaces}\\ V\subset(\mathbb{F}_q)^{j+k}\end{matrix}\right\} \arrow[d, leftrightarrow, "\text{see LEMMA below}"] & \mathrm{RowSpace}(A) \\
\begin{matrix}\pi^{-1}(\lambda)\\ \text{has } |\pi^{-1}(\lambda)|=q^{|\lambda|}\end{matrix} \arrow[d, mapsto] & \left\{\begin{matrix}\text{matrices } A\in\mathbb{F}_q^{k\times(j+k)}\\ \text{of (full) rank } k \text{ in row-reduced}\\ \text{echelon form}\end{matrix}\right\} \arrow[d, two heads, "\pi"] & A \arrow[u, mapsto] \arrow[d, mapsto] \\
\lambda & \left\{\text{partitions } \lambda\subset_k \overset{j+k}{\boxed{\lambda}}\right. & \begin{matrix}\text{shape of the *'}\\ \text{(=nonzero entries}\\ \text{in nonpivot columns)}\\ \text{read backwards}\end{matrix}
\end{tikzcd}$$

e.g. $k=4$ $\underbrace{\qquad\qquad}_{}$ $j+k=13,\ j=9$

$$\begin{bmatrix}
0&0&1&*&*&0&*&0&*&*&*&0&*\\
0&0&0&0&0&1&*&0&*&*&*&0&*\\
0&0&0&0&0&0&0&1&*&*&*&0&*\\
0&0&0&0&0&0&0&0&0&0&0&1&*
\end{bmatrix}$$

$$\begin{matrix}
&&*&*&*&*&*\\
&&*&*&*&*&*\\
&&&*&*&*&*\\
&&&&&&*
\end{matrix}$$

$\updownarrow$



LEMMA: If $A,B\in\mathbb{F}_q^{k\times(j+k)}$ are both in RREF and have same row space, then $A=B$.

proof: $\mathrm{RowSpace}\begin{bmatrix}-A-\end{bmatrix}=\mathrm{RowSpace}\begin{bmatrix}-B-\end{bmatrix}$

$\iff PA=B$ for some $P\in GL_k(\mathbb{F}_q)$

$\xrightarrow[\text{pivot columns}]{\text{think about}} P=\begin{bmatrix}1&&0\\&\ddots&\\0&&1\end{bmatrix}=I_{k\times k}\Rightarrow A=B$ ∎

Once you believe $|\pi^{-1}(\lambda)|=q^{|\lambda|}$, then $\left|\left\{\begin{matrix}\text{k-dim'l}\\ \text{subspace}\end{matrix}\ V\subset\mathbb{F}_q^{j+k}\right\}\right|=\sum_{\lambda\subset\boxed{j}_k}\overbrace{|\pi^{-1}(\lambda)|}^{q^{|\lambda|}}=\begin{bmatrix}j+k\\k\end{bmatrix}_q$.

For (e), it suffices to check $\#\left\{\begin{matrix}\text{k-dim'l}\\ \text{subspaces}\end{matrix}\ V\subset\mathbb{F}_q^{j+k}\right\}\overset{?}{=}\frac{[j+k]!_q}{[k]!_q[j]!_q}$ since there are infinitely many $q=p^d$

$$\| $$

$$\frac{\#\{\text{ordered bases } (v_1,v_2,\dots,v_k) \text{ for } \overset{\text{all}}{\vee}\, k\text{-subspaces in } \mathbb{F}_q^{j+k}\}}{\#\{\text{ordered bases } (v_1,v_2,\dots,v_k) \text{ for } \underline{\text{one}}\ \underline{\text{particular}}\ k\text{-subspace, like } \mathbb{F}_q^k\}}=\frac{(\overset{\text{pick } v_1}{q^{j+k}-1})(\overset{\text{pick } v_2 \text{ not in } \mathbb{F}_q v_1}{q^{j+k}-q})(q^{j+k}-q^2)\cdots(q^{j+k}-q^{k-1})}{(q^k-1)(q^k-q)(q^k-q^2)\cdots(q^k-q^{k-1})}$$

$$=\frac{(q^{j+k}-1)(q^{j+k-1}-1)(q^{j+k-2}-1)\cdots(q^{j+1}-1)}{(q^k-1)(q^{k-1}-1)(q^{k-2}-1)\cdots(q^1-1)}=\frac{[j+k]_q[j+k-1]_q\cdots[j+1]_q}{[k]_q[k-1]_q\cdots[1]_q}$$ ✓ ∎

(50)

More generally, one can define

the q-multinomial coefficient $\left[{n \atop k_1,k_2,\dots,k_\ell}\right]_q \overset{\text{DEF'N}}{:=} \dfrac{[n]!_q}{[k_1]!_q[k_2]!_q\cdots[k_\ell]!_q}$ if $\sum_{i=1}^{\ell} k_i = n$

$\ell=2$, $(k_1,k_2)=(k,j)$

$$\left[{j+k \atop k}\right]_q = \left[{j+k \atop j}\right]_q = \left[{j+k \atop k,j}\right]_q$$

$q=1$

$$\binom{n}{k_1\ k_2 \cdots k_\ell}$$

PROP: (a) $\left[{n \atop k_1\ k_2 \cdots k_\ell}\right]_q = \displaystyle\sum_{\substack{\text{rearrangements}\\ w=(w_1,\dots,w_n)\\ \text{of } k_1\ 1\text{'s}\\ k_2\ 2\text{'s}\\ \vdots\\ k_\ell\ \ell\text{'s}}} q^{\mathrm{inv}(w)}$. In particular, $\left[{n \atop 1,\dots,1}\right]_q = [n]!_q = \displaystyle\sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)}$.

(b) $\left[{n \atop k_1,\dots,k_\ell}\right]_q = \#\{$partial flags of subspaces $\{0\}\subset V_{k_1}\subset V_{k_1+k_2}\subset\cdots\subset V_{k_1+k_2+\cdots+k_{\ell-1}}\subset \mathbb{F}_q^n\}$ with $\dim_{\mathbb{F}_q} V_i = i$

In particular, $[n]!_q = \#\{$complete flags $\{0\}\subset V_1\subset V_2\subset\cdots\subset V_{n-1}\subset\mathbb{F}_q^n\}$

proof: For both, use

$$\left[{n \atop k_1 k_2 \cdots k_\ell}\right]_q \overset{\text{easy!}}{=} \left[{n \atop k_1}\right]_q \cdot \left[{n-k_1 \atop k_2\ k_3 \cdots k_\ell}\right]_q$$

to prove it by induction on ℓ, with $\ell=1$ trivial
$\ell=2$ already proven in our previous PROP

and in the inductive step

- for (a), note that $\mathrm{inv}(w) = \#\{$inversions between 1's & all of 2's, 3's, ..., ℓ's$\}$ $+\#\{$inversions ~~between~~ among 2's, 3's, ..., ℓ's$\}$

  e.g. $w = 124213241$

  $\mathrm{inv}(w) = \mathrm{inv}(122212221)$
  $+ \mathrm{inv}(242\ 324)$

- for (b), note that after fixing V_{k_1}, $\{$flags $\{0\}\subset V_{k_1}\subset V_{k_1+k_2}\subset\cdots\subset\mathbb{F}_q^n\}$
  $\updownarrow$
  $\{$flags $\{0\}\subset V_{k_1+k_2}/V_{k_1}\subset V_{k_1+k_2+k_3}/V_{k_1}\subset\cdots\subset \mathbb{F}_q^n/V_{k_1}\}$ ■

(51) RMK:

(A geometry/topology digression)

$[n]_q$

For any field $\mathbb{F}$ (e.g. $\mathbb{R}, \mathbb{C}, \mathbb{F}_q, \dots$) one defines $\xrightarrow{k=1}$ $\mathbb{P}^{n-1}_{\mathbb{F}} := \{$projective space of lines in $\mathbb{F}^n\}$

$\left[{n \atop k}\right]_q$ $Gr(k, \mathbb{F}^n) := \{$Grassmannian of k-dim'l $\mathbb{F}$-subspaces in $\mathbb{F}^n\}$

$[n]!_q$ $Fl(n) := \{$complete flags $\{0\} \subset V_1 \subset \dots \subset V_{n-1} \subset \mathbb{F}^n\}$ flag manifold

$\left[{n \atop k_1 \dots k_\ell}\right]_q$ $Fl_{k_1,\dots,k_\ell}(n) := \{$partial flags $\{0\} \subset V_{k_1} \subset \dots \subset V_{k_1+\dots+k_{\ell-1}} \subset \mathbb{F}^n\}$ partial flag manifold

and they turn out to be smooth projective varieties $\forall \mathbb{F}$ (embeddable into $\mathbb{P}^N_{\mathbb{F}}$ for various N)

and (smooth) manifolds for $\mathbb{F} = \mathbb{R}, \mathbb{C}$

with a Schubert/Bruhat cell decomposition for

$$Fl_{k_1,\dots,k_\ell}(n) = \underset{\substack{\text{rearrangements} \\ w \text{ of } 1^{k_1} 2^{k_2} \dots \ell^{k_\ell}}}{\bigsqcup} X_w \quad \text{with } X_w \cong \mathbb{F}^{\mathrm{inv}(w)}$$

a cell of $\mathbb{F}$-dimension inv(w)

whose closures $\overline{X}_w$ are called Schubert subvarieties.

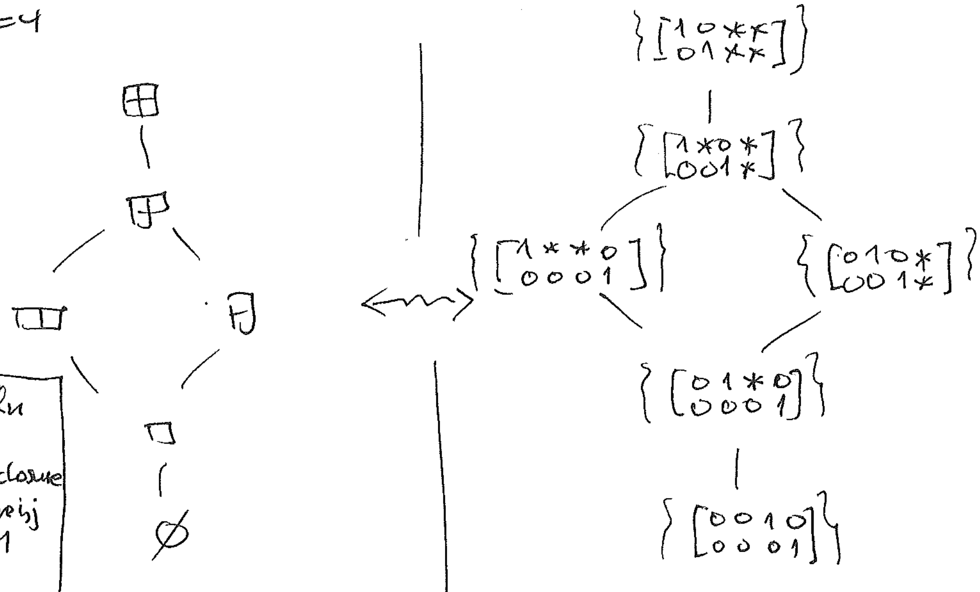
They help not only count $|Fl_{k_1,\dots,k_\ell}(n)| = \left[{n \atop k_1,\dots,k_\ell}\right]_q$ for $\mathbb{F} = \mathbb{F}_q$

but compute the (co-)homology when $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

The poset of cells ordered by containment of closures $\left({w_1 < w_2 \atop \text{if } \overline{X}_{w_1} \subset \overline{X}_{w_2}} \right)$ is $[\varnothing, ▦]_Y$

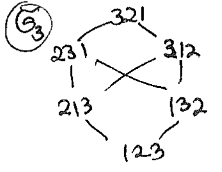
e.g. k=2
n=4

$[\varnothing, ⊞]_Y$



$Gr(2, \mathbb{F}^4)$
$\|$
$Fl_{2,2}(4)$

Bruhat order on cells of Fl_n
$\Leftrightarrow (\mathfrak{S}_n, \leq_{Bruhat})$
where $\leq_{Bruhat}$ is transitive closure of $x \lessdot y$ if $y = x\,(i,j)$ for some i,j and $\mathrm{inv}(y) = \mathrm{inv}(x)+1$



(52) Descents (Stanley §1.4)

DEF'N: For $w = \begin{pmatrix} 1 & 2 & \cdots & n \\ w_1 & w_2 & \cdots & w_n \end{pmatrix} \in \mathfrak{S}_n$

its descent set $D(w) \overset{DEF}{:=} \{ i : 1 \leq i \leq n-1,\ w_i > w_{i+1} \}$

$des(w) := |D(w)|$ descent number

$maj(w) := \sum_{i \in D(w)} i$. major index (considered by P.A. MacMahon)

Eulerian polynomial $A_n(x) := \sum_{w \in \mathfrak{S}_n} x^{1+des(w)}$

Mahonian polynomial $\sum_{w \in \mathfrak{S}_n} q^{maj(w)} =: Mahon(q)$

EXAMPLES:

$\underline{n=1}$: $A_1(x) = x^1 = x$ ~~$\sum_{w \in \mathfrak{S}_1} q^{maj(w)} = 1$~~

$Mahon(q) = q^0 = 1 = [1]!_q$

$\underline{n=2}$: $A_2(x) = \underset{12}{x^1} + \underset{2.1}{x^2}$

$Mahon(q) = q^0 + q^1 = 1 + q = [2]!_q$

$\underline{n=3}$:

w	$des(w)$	$maj(w)$
123	0	0
13.2	1	2
2.13	1	1
23.1	1	2
3.12	1	1
3.2.1	2	3

$A_3(x) = x^1 + 4x^2 + x^3$

$Mahon(q) = q^0 + 2q^1 + 2q^2 + q^3$

$= (1+q)(1+q+q^2)$

$= [3]!_q$

$\underline{n=4}$: $A_4(x) = x + 11x^2 + 11x^3 + x^4$

$Mahon(q) = [4]!_q$

THM 1: $\mathrm{Mahon}(q) = [n]!_q$

i.e. $$\sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)} = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} = [n]!_q$$

THM 2: $$\sum_{m\geq 0} m^n x^m \overset{(a)}{=} \frac{A_n(x)}{(1-x)^{n+1}}$$

and consequently $$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} \overset{(b)}{=} \frac{1-x}{1-xe^{t(1-x)}}$$

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(why does (a) $\Rightarrow$ (b)? (a) gives $$\sum_{n\geq 0} \frac{A_n(x)}{(1-x)^{n+1}}\frac{t^n}{n!} = \sum_{\substack{n\geq 0,\\ m\geq 0}} x^m \frac{m^n t^n}{n!}$$

$$= \sum_{m\geq 0} \underbrace{x^m \underbrace{e^{mt}}_{(e^t)^m}}_{(xe^t)^m} = \frac{1}{1-xe^t}$$

so $$\sum_{n\geq 0} A_n(x)\frac{(t/(1-x))^n}{n!} = \frac{1-x}{1-xe^t}$$

$$\Big\downarrow \text{ replace } t \text{ by } t(1-x)$$

$$\sum_{n\geq 0} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}$$).

Let's deduce them from this:

THM: (a) $$\left(\frac{1}{1-q}\right)^n = \frac{\sum_{w\in\mathfrak{S}_n} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)--(1-q^n)}$$ ($\Rightarrow$ THM 1 by clearing denominator)

(b) $$\sum_{m\geq 0} ([m]_q)^n x^m = \frac{\sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)}}{(1-x)(1-xq)(1-xq^2)\cdots(1-xq^n)}$$ ($\Rightarrow$ THM 2 by $\lim_{q\to 1}$ in $\mathbb{C}[q][[x]]$)

Proof: For (a), note

$$\text{LHS} = \left(\frac{1}{1-q}\right)^n = \sum_{\substack{f:[n]\to\mathbb{N} \\ \parallel \\ (f_1,\dots,f_n)}}' q^{\overbrace{f_1+\dots+f_n}^{|f|}}$$

(obvious) LEMMA: Every $f:[n]\to\mathbb{N}$ has a ! permutation $w\in\mathfrak{S}_n$ such that f is *w-compatible* in the sense that

$$f_{w_1}\geq f_{w_2}\geq\dots\geq f_{w_n} \text{ and } f_{w_i}>f_{w_{i+1}} \text{ if } i\in D(w) \atop (w_i>w_{i+1})$$

proof: e.g. $f=(\overset{1}{2},\overset{2}{0},\overset{3}{5},\overset{4}{0},\overset{5}{3},\overset{6}{3},\overset{7}{2},\overset{8}{0})$ has $\overset{5}{f_3}\geq\overset{3}{f_5}\geq\overset{3}{f_6}>\overset{2}{f_1}\geq\overset{2}{f_7}>\overset{0}{f_2}\geq\overset{0}{f_4}\geq\overset{0}{f_8}$

so is w-compatible for $w=(3,5,6,\cdot 1,7,\cdot 2,4,8)\in\mathfrak{S}_8$

$\blacksquare$

Thus $$\text{LHS} = \sum_{w\in\mathfrak{S}_n}\ \sum_{\substack{f:[n]\to\mathbb{N}\\ w\text{-compatible}}}' q^{|f|}$$

subtract off the smallest w-compatible f_0 from f to get λ

$$\begin{array}{rl} (5,3,3,2,2,0,0,0) & f \\ -\ \underline{(2,2,2,1,1,0,0,0)} & f_0 \quad \underline{\underline{|f_0|=\text{maj}(w)}} \\ (3,1,1,1,1,0,0,0)=\lambda & \end{array}$$

$$= \sum_{w\in\mathfrak{S}_n}\ \sum_{\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n)\geq 0}' q^{\text{maj}(w)+|\lambda|}$$

$$= \sum_{w\in\mathfrak{S}_n} q^{\text{maj}(w)} \sum_{\lambda:} q^{|\lambda|}$$

$$= \sum_{w\in\mathfrak{S}_n} q^{\text{maj}(w)} \frac{\overbrace{\quad 1 \quad}^{\ell(\lambda)\leq n}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

For (b), we'll do something similar, showing

$$(1-x)\sum_{m\geq 0}[m]_q^n x^m \overset{(*)}{\underset{?}{=}} \frac{\sum_{w\in\mathfrak{S}_n} x^{\text{des}(w)+1} q^{\text{maj}(w)}}{(1-xq)(1-xq^2)\cdots(1-xq^n)}$$

Re-interpret

$$\text{LHS} = (1-x)\sum_{m\geq 0}' x^m \sum_{f:[n]\to\{0,1,\dots,m-1\}} q^{|f|} = (1-x)\sum_{m\geq 0} x^m \sum_{\substack{f:[n]\to\mathbb{N} \text{ s.t.}\\ \max(f)\leq m-1}} q^{|f|} = \sum_{m\geq 0} x^m \sum_{\substack{f:[n]\to\mathbb{N}\text{ s.t.}\\ \max(f)=m-1}} q^{|f|}$$

$$= \sum_{f:[n]\to\mathbb{N}} x^{\max(f)+1} q^{|f|}$$

(55)

$$\text{LHS} = \sum_{f:[n]\to\mathbb{N}} x^{\max(f)+1} q^{|f|}$$

$$= \sum_{w\in\mathfrak{S}_n} \sum_{\substack{f:[n]\to\mathbb{N}\\ f\ w\text{-compatible}}} x^{\max(f)+1} q^{|f|}$$

subtract off the smallest w-compatible f_0 from f to get λ

$$= \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)} q^{\mathrm{maj}(w)} \underbrace{\sum_{\lambda=(\lambda_1\geq\cdots\geq\lambda_n)\geq 0} x^{\overbrace{\max(\lambda)}^{\lambda_1}} q^{|\lambda|}}$$

same as $\displaystyle\sum_{\substack{\lambda:\\ \lambda_1\leq n}} x^{\ell(\lambda)} q^{|\lambda|}$ via $\lambda \leftrightarrow \lambda^t$

$$= \sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)+1} q^{\mathrm{maj}(w)} \frac{1}{(1-xq)(1-xq^2)\cdots(1-xq^n)}$$

■

REMARKS

① $\displaystyle\sum_{w\in\mathfrak{S}_n} x^{\mathrm{des}(w)} = \sum_{w\in\mathfrak{S}_n} x^{\mathrm{asc}(w)}$ where $\mathrm{asc}(w) = \#$ascents of w

$= \{1\leq i\leq n-1 : w_i < w_{i+1}\}$

$= n-1-\mathrm{des}(w))$

and they have symmetric coefficient sequences

e.g. $\displaystyle\sum_{w\in\mathfrak{S}_4} x^{\mathrm{des}(w)} = 1+11x+11x^2+x^3$

$(1, 11, 11, 1)$

since $\mathrm{des}(\underset{w_1\cdots w_n}{\overset{w}{\shortparallel}}) = \mathrm{asc}(\underbrace{(n+1-w_1, \ldots, n+1-w_n)}_{\overset{\shortparallel}{w_0 w}}) = \mathrm{asc}(\underbrace{(w_n, w_{n-1}, \ldots, w_2, w_1)}_{w w_0})$

where $w_0 = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & & 2 & 1 \end{pmatrix}$

② The map $w \longmapsto \hat{w}$ that sent $\#\mathrm{cyc}(w) = \#\text{L-to-R-max}(\hat{w})$)

$\shortparallel$

$(2_A)(7\,1_A6_A)(8_A)(9\,4\,3_A5_A)$ $\qquad$ $2_A7\,1_A6_A8_A9\;\;4\;\;3_A5$

has the property that $1+\mathrm{asc}(\hat{w}) = \#\{1\leq i\leq n : \underbrace{i\leq w(i)}_{\text{called a weak excedance of } w}\}$

$n-(-)$

$\downarrow$

$$\mathrm{des}(\hat{w}) = n - \#\{1\leq i\leq n : i\leq w(i)\}$$
$$= \#\{1\leq i\leq n : \underbrace{i > w(i)}_{\text{called a non-excedance of } w}\}$$

Hence $$\sum_{w\in\mathfrak{S}_n} x^{des(w)} = \sum_{w\in\mathfrak{S}_n} x^{\text{non-exc}(w)} \Big\updownarrow w\leftrightarrow w^{-1}$$
$$= \sum_{w\in\mathfrak{S}_n} x^{exc(w)}$$

where $exc(w) = \{1\le i\le n : w(i)>i\}$

e.g. $n=3$

w	$exc(w)$	$des(w)$
$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	0	0
$\begin{pmatrix}1&\enclose{circle}{\begin{matrix}2\\3\end{matrix}}&3\\1&\cdot&2\end{pmatrix}$	1	1
$\begin{pmatrix}\enclose{circle}{\begin{matrix}1\\2\end{matrix}}\cdot&2&3\\&1&3\end{pmatrix}$	1	1
$\begin{pmatrix}\enclose{circle}{\begin{matrix}1\\2\end{matrix}}&\enclose{circle}{\begin{matrix}2\\3\end{matrix}}\cdot&3\\&&1\end{pmatrix}$	2	2
$\begin{pmatrix}\enclose{circle}{\begin{matrix}1\\3\end{matrix}}\cdot&2&3\\&1&2\end{pmatrix}$	1	1
$\begin{pmatrix}\enclose{circle}{\begin{matrix}1\\3\end{matrix}}\cdot&2&3\\&2\cdot&1\end{pmatrix}$	1	2

③ Can we count $\beta(S) := \#\{w : D(w)=S\}$?
for $S\subset[n-1]$ $\quad\Big\uparrow q=1$

Or even better, $\beta(S,q) := \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)=S}} q^{inv(w)}$?

e.g. $n=4$, $S=\{2\}$

$\{w\in\mathfrak{S}_n : D(w)=\{2\}\}$	$inv(w)$
13·24	1
14·23	2
23·14	2
24·13	3
34·12	4

$q+2q^2+q^3q^4$
$\overset{q=1}{\rightsquigarrow}$ 5

It turns out to be easier to reinterpret $\alpha(S) := \#\{w : D(w)\subseteq S\}$
$\Big\uparrow q=1$

and $$\alpha(S,q) := \sum_{\substack{w\in\mathfrak{S}_n:\\ D(w)\subseteq S}} q^{inv(w)}$$ since $inv(w^{-1}) = inv(w)$

$$= \sum_{\substack{w\in\mathfrak{S}_n\\ D(w^{-1})\subseteq S}} q^{inv(w)} \quad \Big(\underset{\substack{\S 1.4\\ \text{Stanley}}}{=} \sum$$

$$= \sum_{\substack{\text{rearrangements}\\ w=(w_1,\dots,w_n)\\ \text{of } 1^{k_1}2^{k_2}\cdots\ell^{k_\ell}}} q^{inv(w)} \qquad = \begin{bmatrix} n \\ k_1,\dots,k_\ell\end{bmatrix}_q$$

if S = partial sums $\{k_1, k_1+k_2, \dots, k_1+\dots+k_{\ell-1}\}$ of $\underline{k}=(k_1,\dots,k_\ell)\models n$

because
$\{w\in\mathfrak{S}_n : D(w^{-1})\subset S\}$ = ~~s~~ <u>shuffles</u> of $1<2<\dots<k_1$
$k_1+1<\dots<k_1+k_2$
$\vdots$

e.g. $S=\{3,5\}\subset 8$
$\quad\Big\updownarrow$
$\underline{k}=(3,2,3)\models 8$

$$1\,1\,1\;2\,2\;3\,3\,3 \longleftrightarrow 1\,2\,3\,4\,5\,6\,7\,8$$
$$2\,3\,1\;3\,3\,2\,1\,1 \longleftrightarrow \underline{\underline{4}}\,\underline{\underline{\underline{6}}}\,\underline{1}\;\underline{\underline{\underline{7}}}\,\underline{\underline{\underline{8}}}\,\underline{\underline{5}}\;\underline{2}\,\underline{3}$$

(57)

So how do we recover $\beta(S)$ from $\alpha(S) = \sum_{T \subseteq S} \beta(S)$?

$\beta(S,q)$ from $\alpha(S,q)$

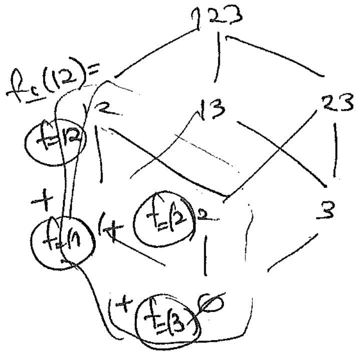
PROP (Principle of Inclusion-Exclusion)

Given two functions $f_{\subseteq}, f_{=} : 2^{[n]} \to \underbrace{R}_{\text{an abelian group}}$

$$S \mapsto f_{\subseteq}(S)$$
$$S \mapsto f_{=}(S)$$

then $f_{\subseteq}(S) \overset{(*)}{=} {\sum_{T \subseteq S}}' f_{=}(T) \quad \forall S \subset [n]$

$$\Leftrightarrow \quad f_{=}(S) = {\sum_{T \subseteq S}}' (-1)^{|S \setminus T|} f_{\subseteq}(T)$$



e.g. $f_{=}(\emptyset) = f_{\subseteq}(\emptyset)$

$$f_{=}(\{i\}) = f_{\subseteq}(\{i\}) - f_{\subseteq}(\emptyset)$$
$$f_{=}(\{i,j\}) = f_{\subseteq}(\{i,j\}) - f_{\subseteq}(\{i\}) - f_{\subseteq}(\{j\}) + f_{\subseteq}(\emptyset)$$

COR: Let $f_{\subseteq}(S) := \alpha(S,q) = {\sum_{\substack{w \in \mathfrak{S}_n: \\ D(w^{-1}) \subseteq S}}}' q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ k_1, \ldots, k_\ell \end{bmatrix}_q$

Then $f_{=}(S) = \beta(S,q) = \sum_{\substack{w \in \mathfrak{S}_n \\ D(w^{-1}) = S}} q^{\mathrm{inv}(w)} = \sum_{T \subseteq S} \alpha(T,q)(-1)^{|S \setminus T|}$

$$= {\sum_{\substack{\underline{k}' \models n \\ \text{coarsening } \underline{k}}}}' (-1)^{\ell(\underline{k}) - \ell(\underline{k}')} \begin{bmatrix} n \\ \underline{k}' \end{bmatrix}_q$$

e.g. $n=4$
$S=\{2\}$
$\updownarrow$
$(2,2)$

$$\beta(\{2\},q) = \alpha(\{2\},q) - \alpha(\emptyset,q)$$
$$= \begin{bmatrix} 4 \\ 2,2 \end{bmatrix}_q - \begin{bmatrix} 4 \\ 4 \end{bmatrix}_q$$
$$= \frac{[4]_q [3]_q}{[2]_q} - 1$$
$$= (1+q^2)(1+q+q) - 1$$
$$= \cancel{1} + q + 2q^2 + q^3 + q^4 - \cancel{1} = q + 2q^2 + q^3 + q^4$$

proof of PIE: Note $\{f_=(S)\}_{S\subseteq[n]}$ determines $\{f_\subseteq(S)\}_{S\subseteq[n]}$ uniquely via (*),

and conversely by induction on $|S|$, since (*) says

$$f_=(S) = f_\subseteq(S) - \sum_{T\subsetneq S}{}' \underbrace{f_=(T)}_{\text{already determined}}.$$

If we let $g(S) := \sum_{T\subseteq S}{}' (-1)^{|S\setminus T|} f_\subseteq(T)$

then fixing S,

$$\sum_{R\subseteq S} g(R) = \sum_{R\subseteq S} \sum_{T\subseteq R}{}' (-1)^{|R\setminus T|} f_\subseteq(T)$$

$$= \sum_{T\subseteq S} f_\subseteq(T) \underbrace{\sum_{\substack{R:\\ T\subseteq R\subseteq S}}{}' (-1)^{|R\setminus T|}}$$

$$= \sum_{\hat{R}:=R\setminus T\subseteq S\setminus T}{}' (-1)^{\hat{R}} = \begin{cases} 1 & \text{if } S=T \\ \sum_{k=0}^{|S\setminus T|=m} (-1)^k \binom{m}{k} \\ = (1+(-1))^m & \text{if } T\subsetneq S \\ = 0 \end{cases}$$

■

EXAMPLES of PIE

① ~~A determinantal~~ reformulation (Stanley §2.2)

PROP: If it happens that $f_\subseteq(S) = h(n)e(k_1)\cdots e(k_\ell)$ for some ~~functions~~

$h, e : \mathbb{Z} \longrightarrow \underset{\text{a commutative ring}}{R}$ when S = partial sums of $\underline{k} = (k_1, \ldots, k_\ell) \models n$

with $h(0)=e(0)=1$

$h(n)=0$ for $n<0$

then $f_=(S) = h(n)\cdot\det \begin{bmatrix} e(k_1) & e(k_1+k_2) & e(k_1+k_2+k_3) & \cdots \\ 1 & e(k_2) & e(k_2+k_3) & \\ 0 & 1 & \ddots & \ddots \\ & 0 & \ddots & \ddots \\ \bigcirc & & 0 & 1 \quad e(k_\ell) \end{bmatrix}$

e.g. $\overset{n=9}{f_=}(\underset{\underset{(3,2,4)}{\Uparrow}}{\{3,5\}}) = h(8)\det \begin{bmatrix} e(3) & e(5) & e(9) \\ 1 & e(2) & e(6) \\ 0 & 1 & e(4) \end{bmatrix} = h(8)\left(e(3)e(2)e(4) - e(5)e(4) + e(9) - e(3)e(6)\right)$

Sign-reversing involutions and identities involving signs (Stanley §2.6)

Some identities with +/− signs can be proven like this:

PROP: Given a set X with a sign function $\mathrm{sgn}: X \to \{\pm 1\}$

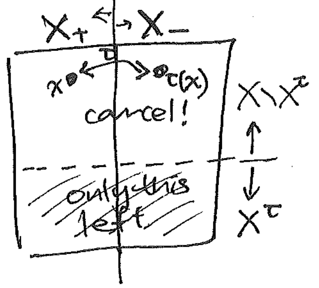
a weight function $\mathrm{wt}: X \to \underset{\text{an abelian group}}{R}$

and a sign-reversing, weight-preserving, involution
$\big(\mathrm{sgn}(\tau(x)) = -\mathrm{sgn}(x)\big)$ $\big(\mathrm{wt}(\tau(x)) = \mathrm{wt}(x)\big)$ $(\tau^2 = 1)$

$\tau: X \to X$

then
$$\sum_{x \in X} \mathrm{sgn}(x) \cdot \mathrm{wt}(x) = \sum_{x \in X^\tau := \{x \in X : \tau(x) = x\}} \mathrm{sgn}(x) \cdot \mathrm{wt}(x)$$

proof: $X =$



$$\mathrm{sgn}(x)\,\mathrm{wt}(x) + \underbrace{\mathrm{sgn}(\tau(x))}_{-\mathrm{sgn}(x)} \cdot \underbrace{\mathrm{wt}(\tau(x))}_{\mathrm{wt}(x)} = 0$$

∎

EXAMPLES:

① (Warm-up)
$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0$$
$$\Big\| $$
$$\sum_{\text{subsets } S \subseteq [n]} (-1)^{|S|}$$

$\mathrm{sgn}: X = 2^{[n]} \to \{\pm 1\}$
$S \mapsto (-1)^{|S|}$

$\mathrm{wt}: X = 2^{[n]} \to \mathbb{N}$
$S \mapsto 1$

$$\tau: \overset{2^{[n]}}{\overset{\|}{X}} \to X$$
$$S \mapsto \begin{cases} S - \{1\} & \text{if } 1 \in S \\ S \sqcup \{1\} & \text{if } 1 \notin S \end{cases}$$

is sign-reversing
weight-preserving
with $X^\tau = \varnothing$ (no fixed points).

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e.g. $$\alpha(S,q) = \begin{bmatrix} n \\ k_1,\dots,k_\ell \end{bmatrix}_q = \underbrace{[n]!_q}_{h(n)} \; \frac{1}{\underbrace{[k_1]!_q}_{e(k_1)}} \cdots \frac{1}{\underbrace{[k_\ell]!_q}_{e(k_\ell)}}$$

so $$\beta(S,q) = [n]!_q \det \begin{bmatrix} \frac{1}{[k_1]!_q} & \frac{1}{[k_1+k_2]!_q} & \cdots & \frac{1}{[n]!_q} \\ 1 & \frac{1}{[k_2]!_q} & & \vdots \\ & 1 & \ddots & \\ 0 & & 1 & \frac{1}{[k_\ell]!_q} \end{bmatrix}$$

$n=4$

$$\underset{\substack{\updownarrow \\ (2,2)}}{\beta(\{2\},q)} = [4]!_q \begin{bmatrix} \frac{1}{[2]!_q} & \frac{1}{[4]!_q} \\ 1 & \frac{1}{[2]!_q} \end{bmatrix} = [4]_q[3]_q[2]_q\left(\frac{1}{[2]_q[2]_q} - \frac{1}{[4]_q[3]_q[2]_q}\right)$$
$$= (1+q^2)(1+q+q^2) - 1 \quad ✓$$

② Similarly, if $f_{\geq}(S) = \sum_{T \supseteq S} f_{=}(T)$

then $$f_{=}(S) = \sum_{T \supseteq S} (-1)^{|T \setminus S|} f_{\geq}(T)$$

and in particular $$f_{=}(\emptyset) = \sum_{T} (-1)^{|T|} f_{\geq}(T)$$

e.g. if $A_1, \dots, A_n$ are subsets of some universe $\mathcal{U}$

then letting $$f_{\geq}(S) = \#\left(\bigcap_{i \in S} A_i\right) = \#\{u \in \mathcal{U} : \{i=1,\dots,n : u \in A_i\} \supseteq S\}$$

$$f_{=}(S) = \#\{u \in \mathcal{U} : \{i=1,\dots,n : u \in A_i\} = S\}$$
$$= \sum_{T \supseteq S} (-1)^{|T \setminus S|} \#\left(\bigcap_{i \in T} A_i\right)$$

$$\underbrace{\boxed{\begin{aligned} \text{and } \#\left(\mathcal{U} \setminus \left(\bigcup_{i=1}^{n} A_i\right)\right) = f_{=}(\emptyset) &= \sum_{T} (-1)^{|T|} \#\left(\bigcap_{i \in T} A_i\right) \\ &= |\mathcal{U}| - \sum_{i=1}^{n} \# A_i + \sum_{1 \leq i < j \leq n} \# A_i \cap A_j - \cdots \end{aligned}}}_{\text{A common form of PIE}}$$

e.g. $d_n = \#\{\text{derangements } \sigma \text{ in } \underset{\overset{\shortparallel}{\mathcal{U}}}{\mathfrak{S}_n}\} = \#\left(\mathcal{U} \setminus \bigcup_{i=1}^{n} A_i\right)$ where $A_i := \{\sigma\in\mathfrak{S}_n : \sigma(i)=i\}$

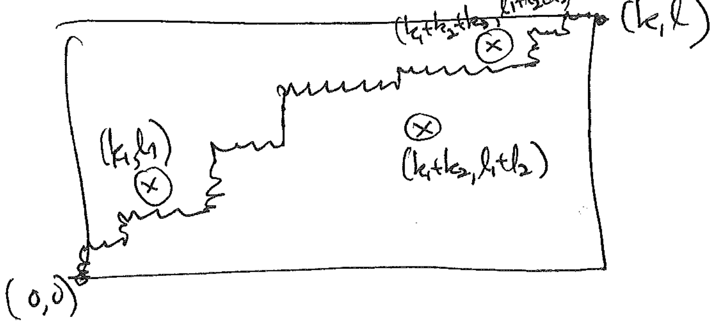
$$= \sum_{T\subseteq[n]} (-1)^{|T|} \# \underbrace{\left(\bigcap_{i\in T} A_i\right)}_{\#\{\sigma\in\mathfrak{S}_n:\ \sigma(i)=i\ \forall i\in T\} = (n-|T|)!}$$

$$= \sum_{T\subseteq[n]} (-1)^{|T|}(n-|T|)!$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot(n-k)! = n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$$

$$= n!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\dots+\frac{(-1)^n}{n!}\right) \quad ✓$$

③ How many lattice paths $(0,0) \longrightarrow (\underset{\overset{\shortparallel}{k_1+k_2+k_3+k_m}}{k}, \underset{\overset{\shortparallel}{l_1+l_2+l_3+l_m}}{l})$ avoid the points $(k_1,l_1), (k_1+k_2, l_1+l_2), \dots, (k_1+k_2+k_m, l_1+l_2+l_m)$?



If $A_i := \{\text{paths that hit } (k_1+\dots+k_i, l_1+\dots+l_i)\}$

then $\# A_i = \binom{k_1+\dots+k_i+l_1+\dots+l_i}{k_1+\dots+k_i}\binom{k_{i+1}+\dots+k_m+l_{i+1}+\dots+l_m}{k_{i+1}+\dots+k_m}$

$\# A_i\cap A_j = \dots$

and $\#\left(\mathcal{U}\setminus\left(\bigcup_{i=1}^{m} A_i\right)\right) = \sum_{T}(-1)^{|T|}\#\bigcap_{i\in T}A_i$

$$= \det\begin{vmatrix} \binom{k_1+l_1}{k_1} & \binom{k_1+k_2+l_1+l_2}{k_1+k_2} & \cdots & \binom{k+l}{k} \\ 1 & \binom{k_2+l_2}{k_2} & \binom{k_2+k_3+l_2+l_3}{k_2+k_3} & \vdots \\ 0 & 1 & \binom{k_3+l_3}{k_3} & \binom{k_3+k_4+l_3+l_4}{k_3+k_4} \\ 0 & 0 & 1 & \binom{k_4+l_4}{k_4} \end{vmatrix}$$

10/23/2015 ② THM (Euler's "Pentagonal Number thm")
(Stanley §1.8)
~~PROP~~ PROP 1.8.7

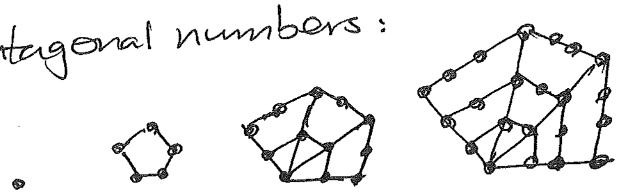
$$\underbrace{\prod_{j\geq 1}(1-q^j)}_{\substack{\text{denominator}\\ \text{for the}\\ \text{partition function } p(n)\\ \text{gen. fn.}}} = (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)\cdots$$

$$= 1-q-q^2+\cancel{q^3(1-1)}+\cancel{q^4(1-1)}+q^5(1+1-1)+\ldots$$

$$= 1\overbrace{-q-q^2}^{n=1}+\overbrace{q^5+q^7}^{n=2}\overbrace{-q^{12}-q^{15}}^{n=3}+\overbrace{q^{22}+q^{26}}^{n=4}-\cdots$$

$$= 1+\sum_{n\geq 1}(-1)^n\left(q^{\frac{n(3n-1)}{2}}+q^{\frac{n(3n+1)}{2}}\right) = \sum_{n=-\infty}^{+\infty}(-1)^n q^{\frac{n(3n-1)}{2}}$$

pentagonal numbers:



Before proving it, let's note a useful corollary for tabulating
$p(n) = \#\{\lambda : \lambda \vdash n\}$

COR: For $n\geq 1$
$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+\ldots,$

proof:
$$\sum_{n\geq 0} p(n)q^n = \frac{1}{\prod_{j\geq 1}(1-q^j)}$$

so
$$\left(\sum_{n\geq 0} p(n)q^n\right)\Big(\underbrace{\prod_{j\geq 1}(1-q^j)}_{(1-q^1-q^2+q^5+q^7-\ldots)}\Big) = 1$$

$$\Big\downarrow [q^n]$$

$$p(n)-p(n-1)-p(n-2)+p(n-5)+p(n-7)-\ldots = 0 \quad \blacksquare$$

It's quite efficient!

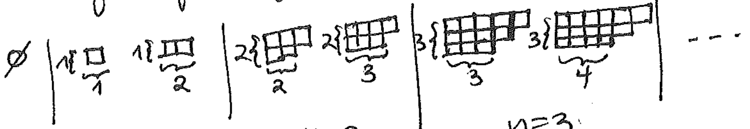
RMK: THM (Hardy & Ramanujan 1918) $p(n) \sim \dfrac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3}\,n}$

(63) F. Franklin's (1881) proof of Euler's P.N.T.

$$\mathrm{LHS} = \prod_{j\geq 0}(1-q^j) = \sum_{\substack{\lambda:\\ \lambda \text{ has distinct parts} \\ \text{i.e. } \lambda_1>\lambda_2>\dots>\lambda_\ell}} \overbrace{(-1)^{\ell(\lambda)}}^{\mathrm{sgn}(\lambda)} \overbrace{q^{|\lambda|}}^{wt(\lambda)}$$

$\{\lambda \text{ has distinct parts, i.e. } \lambda_1>\lambda_2>\dots>\lambda_\ell\} =: X$

$$\mathrm{RHS} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + \dots$$



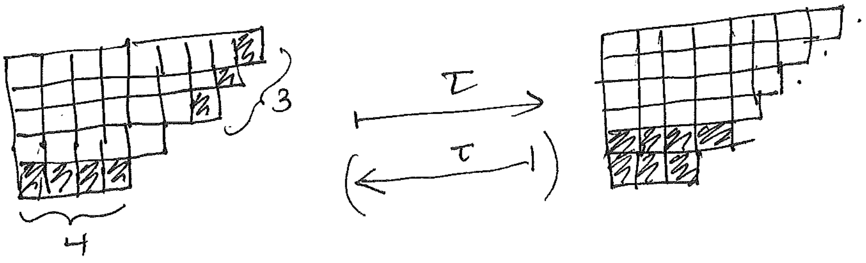
n=1 n=2 n=3

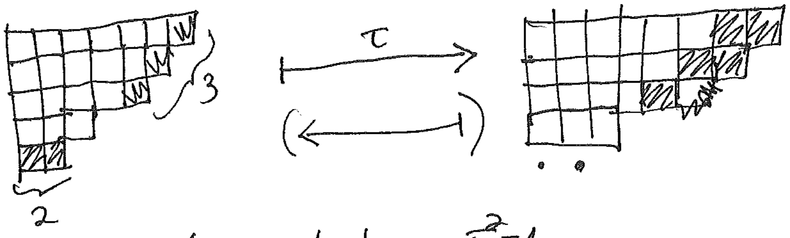
$(-1)^n\left(q^{\frac{3n(n-1)}{2}} + q^{\frac{3n(n+1)}{2}}\right)$

Franklin defined $\tau : X \to X$ by comparing, $X = \{\lambda \text{ with distinct parts}\}$

- smallest part
- longest initial run $\lambda_1, \lambda_1-1, \lambda_1-2, \dots$

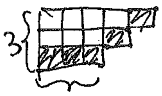
and moving the smaller one onto the bigger:





When one can do this, check
- $\tau^2 = 1$
- $\ell(\tau(\lambda)) = \ell(\lambda) \pm 1$ so $\mathrm{sgn}(\tau(\lambda)) = -\mathrm{sgn}(\lambda)$
- $|\tau(\lambda)| = |\lambda|$ so $wt(\tau(\lambda)) = wt(\lambda)$

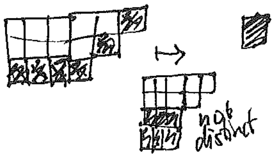
One can't do this if they have same size



n=3

$|\lambda| = \frac{3n(n-1)}{2}$

or the run is 1 smaller but they overlap:



$|\lambda| = \frac{3n(n+1)}{2}$

(64)

③ THM: (Kirchhoff's Matrix-Tree) Theorem

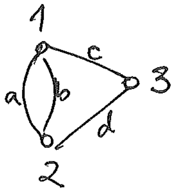
The number of spanning trees in a multi-graph $G = (V, E)$, $V = [n] = \{1, 2, \ldots, n\}$
(multiple/parallel edges allowed)

~~[illegible]~~ is $\det\left(\overline{L(G)}^{i,i}\right)$

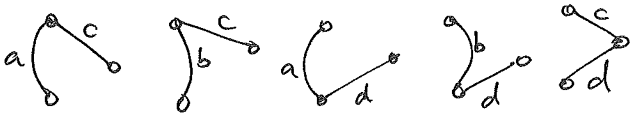
where $\overline{L(G)}^{ii} = \underbrace{L(G)}_{n\times n \text{ Laplacian matrix}}$ with row i, column i removed for any $i = 1, 2, \ldots, n$

having $L(G)_{v,w} = \begin{cases} \deg_G(v) & \text{if } v = w \\ -\#(\text{edges from } v \text{ to } w) & \text{if } v \neq w \end{cases}$

EXAMPLE: $G =$



has 5 spanning trees



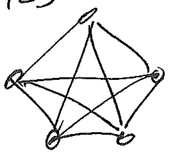
and
$$L(G) = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 3 & -2 & -1 \\ 2 & -2 & 3 & -1 \\ 3 & -1 & -1 & 2 \end{array}$$

has $\det\left(\overline{L(G)}^{1,1}\right) = \det\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} = 6 - 1 = 5$ ✓

$\det\left(\overline{L(G)}^{3,3}\right) = \det\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} = 9 - 4 = 5$ ✓

EXAMPLE: Let's prove Cayley's formula n^{n-2} for spanning trees in complete graph K_n on $[n]$ this way...

e.g. $n=5$



$$\overline{L(K_n)}^{n,n} = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & n-1 & -1 & \cdots & -1 \\ 2 & -1 & n-1 & & \vdots \\ \vdots & \vdots & & \ddots & -1 \\ n & -1 & -1 & \cdots & n-1 \end{array} = n\underbrace{I_{n-1}}_{\text{identity matrix}} - \underbrace{\mathbb{1}_{n-1}}_{\text{all 1's matrix}}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{bmatrix}$$

Who are eigenvalues of $\mathbb{1}_{n-1}$? It has rank 1, so $\underline{n-2 \text{ eigenvalues are } 0}$.

Also $\mathbb{1}_{n-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = (n-1)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, so $\underline{\text{one eigenvalue is } n-1}$

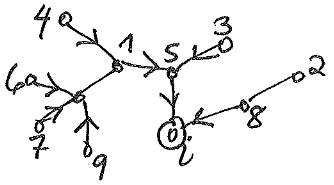
Thus $\mathbb{1}_{n-1}$ has eigenvalues $(\underbrace{0, 0, \ldots, 0}_{n-2}, n-1)$, so $\overline{L(K_n)}^{n,n}$ has eigenvalues $(\underbrace{n, n, \ldots, n}_{n-2}, 1)$ and $\det = n^{n-2}$

(65)

Instead of proving Kirchhoff's Thm, let's prove a weighted, directed version:

THM: If
$$L = \begin{array}{c|cccc} & 1 & 2 & \cdots & n \\ \hline 1 & a_{12}+a_{13}+\dots+a_{1n} & -a_{12} & -a_{13} \cdots & -a_{1n} \\ 2 & -a_{21} & a_{21}+a_{23}+\dots+a_{2n} & \ddots & \vdots \\ \vdots & -a_{31} & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & \\ n & -a_{3n} & \cdots & & a_{n1}+a_{n2}+\cdots a_{n,n-1} \end{array}$$
has $L_{ij} = \begin{cases} a_{i1}+a_{i2}+\dots+a_{in} & \text{if } i=j \\ -a_{ij} & \text{if } i \neq j \end{cases}$

then
$$\det\left(\overline{L}^{ii}\right) = \sum_{\substack{\text{arborescences } A \\ \text{directed toward } i}} \prod_{\text{arcs } i \to j \text{ in } A} a_{ij} \in \mathbb{Z}[a_{12}, a_{21}, \dots]$$



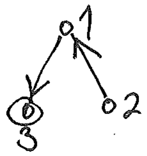
---

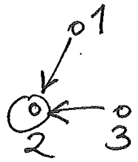
e.g. $n=3$

$$L = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & a_{12}+a_{13} & -a_{12} & -a_{13} \\ 2 & -a_{21} & a_{21}+a_{23} & -a_{23} \\ 3 & -a_{31} & -a_{32} & a_{31}+a_{32} \end{array}$$

$$\det\left(\overline{L}^{3,3}\right) = \det\begin{bmatrix} a_{12}+a_{13} & -a_{12} \\ -a_{21} & a_{21}+a_{23} \end{bmatrix} = (a_{12}+a_{13})(a_{21}+a_{23}) - (-a_{12})(-a_{21})$$
$$= \cancel{a_{12}a_{21}} + a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23} - \cancel{a_{12}a_{21}}$$

$$= a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{23}$$





Note above THM ⇒ Kirchhoff

by setting $a_{ij} = \#\begin{pmatrix}\text{edges,} \\ i \text{ to } j \\ \text{in } G\end{pmatrix} = a_{ji}$

<u>proof of</u> THM:

$$L = \begin{bmatrix} R_1 - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & R_2 - a_{22} & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \cdots & & R_n - a_{nn} \end{bmatrix} \quad \text{where } R_i = a_{i1} + a_{i2} + \dots + a_{ii} + \dots + a_{in} = \sum_{j=1}^{n} a_{ij}$$

$$= \left(R_i \delta_{ij} - a_{ij}\right)_{\substack{i=1,\to n \\ j=1,\to n}}$$

$$\Rightarrow \det\left(\overline{L}^{n,n}\right) = \sum_{\omega \in \mathfrak{S}_{n-1}} \operatorname{sgn}(\omega) \prod_{i=1}^{n-1} L_{i,\omega(i)}$$

$$= \sum_{\substack{S \subseteq [n-1] \\ \text{(to be fixed by } \omega)}} \prod_{i \in S} (R_i - a_{ii}) \sum_{\substack{\omega \in \mathfrak{S}_{[n-1] \setminus S} \\ \text{a derangement}}} \operatorname{sgn}(\omega) \prod_{i \in [n-1] \setminus S} (-a_{i,\omega(i)})$$

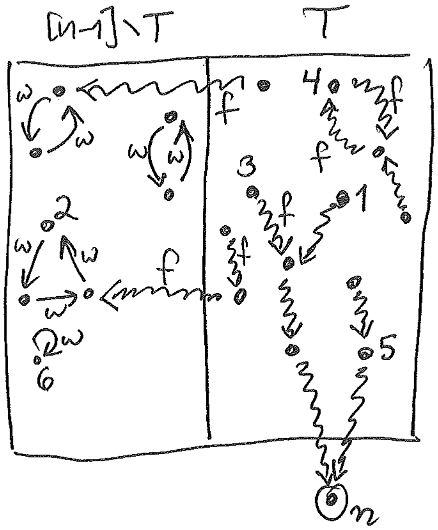
$$= \sum_{S \subseteq [n-1]} \sum_{T \subseteq S} \prod_{i \in T} R_i \prod_{S \setminus T} (-a_{ii}) \sum_{\substack{\text{derangements} \\ \omega \in \mathfrak{S}_{[n-1] \setminus S}}} \operatorname{sgn}(\omega) \prod_{i \in [n-1] \setminus S} (-a_{i,\omega(i)})$$

$$= \sum_{T \subseteq [n-1]} \underbrace{\prod_{i \in T} (a_{i1} + a_{i2} + \dots + a_{in})}_{\sum_{f: T \to [n]} \prod_{i \in T} a_{i,f(i)}} \cdot \sum_{\omega \in \mathfrak{S}_{[n-1] \setminus T}} \operatorname{sgn}(\omega) \prod_{i \in [n-1] \setminus T} (-a_{i,\omega(i)})$$

$$= \sum_{(T,f,\omega)} \underbrace{(-1)^{|[n-1] \setminus T|} \operatorname{sgn}(\omega)}_{\operatorname{sgn}(x)} \underbrace{\prod_{i \in T} a_{i,f(i)} \prod_{i \in [n-1] \setminus T} a_{i,\omega(i)}}_{wt(x)}$$

$$X := \left\{ \begin{array}{l} (T, f, \omega) \\ T \subseteq [n-1] \\ f: T \to [n] \\ \omega \in \mathfrak{S}_{[n-1] \setminus T} \end{array} \right.$$

Picture of (T, f, ω):



We can define an involution

$$\tau : X \longrightarrow X$$

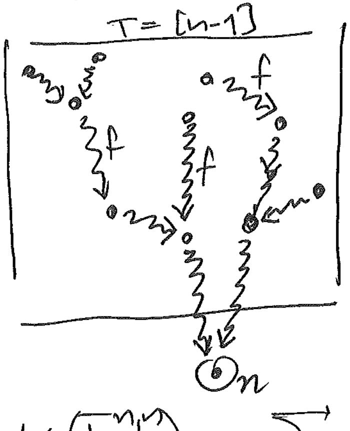
that eliminates all cycles in ω or f by switching them from ω to f or back from f to ω whichever cycle contains the smallest index $i \in [n-1]$

Check that it is an
- involution
- wt-presening
- sign-reversing

Who are its fixed points X^{τ}?

No cycles $\Rightarrow$ $[n-1] \setminus T$ is empty, i.e. $T = [n-1]$

and $f : [n-1] \longrightarrow [n]$ has no cycles



(easy)
LEMMA:
This forces f to be an arborescence directed toward n

Hence $\det\left(L^{n,n}\right) = \sum_{\substack{\text{arborescences } f \text{ on } [n] \\ \text{toward } n}} \prod_{i \in [n-1]} a_{i,f(i)}$ ∎

RMK/Digression on Euler tours and the BEST Thm (Ardila §3.1.4) (Stanley Vol II)

Kirchhoff's Thm. in its directed version lets us solve another, seemingly unrelated problem:

Given a directed graph (digraph) $D = (\underset{\text{vertices}}{V}, \underset{\text{arcs } (x,y)}{A})$

$\underset{x}{\bullet} \longrightarrow \underset{y}{\bullet}$

how many directed Euler tours does it have?

(= circularly ordered walks along directed arcs in A visiting each exactly once, returning to starting vertex)

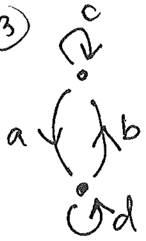
EXAMPLES:

①

has ~~n~~one:

(abc)

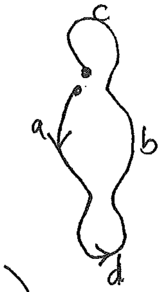
③



has ~~one~~ one:

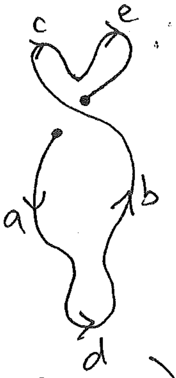
(adbc)

= (dbca)

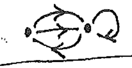


④

has two:

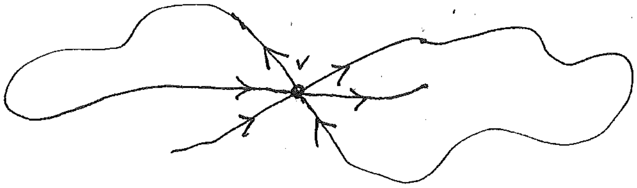


(adbec) ≠ (adbce)

② but  has none.

PROP: D has an Euler tour $\Leftrightarrow$ • its underlying undirected graph is connected, and
• $\text{outdeg}_D(v) = \text{indeg}_D(v)$ $\forall v \in V$.

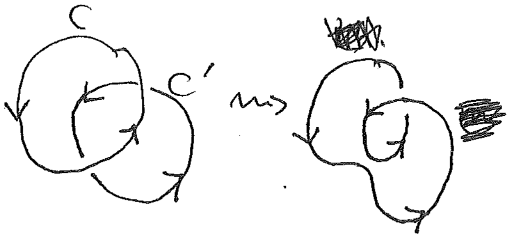
proof: ($\Longrightarrow$) is pretty clear, since the tour connects V and matches outgoing with incoming arcs at each v



If outdeg = indeg everywhere, pick v_0 to start and leave along any arc (then erase it), entering v_1 and leaving along some arc (then erase it). Repeat until you get stuck, which can only be at v_0, since outdeg = indeg is preserved elswhere.

This creates a directed cycle C, and D being connected means either C exhausts all of D, or some vertex on C has an arc not in C. Start there (with C erased) to produce a cycle C'.

Then "suture" C and C' like this:



Repeat until D is exhausted ∎

<u>THM</u> (B.E.S.T.)
(deBruijn, van Ardenne-Ehrenfest, Smith, Tutte) Fix some $v_0 \in V$.

If D has an Euler tour, then it has

$$\underbrace{\#\left(\begin{array}{c}\text{arborescences in } D\\ \text{directed toward } v_0\end{array}\right)}_{\text{easy to compute (Kirchhoff)}} \cdot \underbrace{\prod_{v \in V}\left(\text{outdeg}_D(v)-1\right)!}_{\text{even easier!}}$$

of them

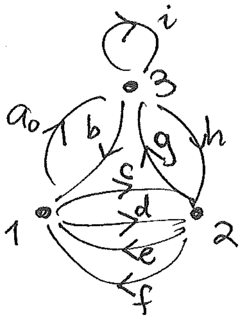
<u>proof</u>: <u>Start</u> all tours at some fixed arc a_0 emanating from v_0, by convention.

Given an Euler tour t in D create

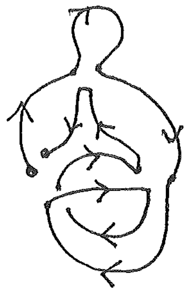
- $\alpha(t) := \{$the set of one arc for each $v \neq v_0$ which is the last arc out of v $\underset{v}{\bullet}\!\longrightarrow\!\bullet$ visited by $t\}$

- $(\omega_v^{(t)})_{v \in V} := \{$the linear order on the non-$\alpha(t)$ non-a_0 arcs out of t in which ~~t~~ visits them$\}$

e.g. $D=$

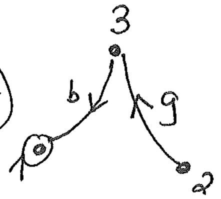


with $t=$



$= (a_0 i h f d e c g b)$

has $\alpha(t)$



$\omega_{v_0}^{(t)} = (d, c)$ (since a_0 is omitted)

$\omega_2^{(t)} = (f, e)$ (since $g \in \alpha(t)$)

$\omega_3^{(t)} = (i, h)$ (since $b \in \alpha(t)$)

10/30/2015 > CLAIM: $\alpha(t)$ is always an arborescence in D directed toward v_0, since it has exact $|V|-1$ arcs (one for each $v \in V - \{v_0\}$) and has a directed path from $v \to \cdots \to v_0$ for every $v \in V$ using backward induction on how late v is last visited by t.

Thus we get a map

$$\{\text{Euler tours } t \text{ in } D\} \xrightarrow{f} \{(\alpha, (\omega_v)_{v\in V}) : \alpha \text{ an arborescence toward } v_0 \text{ in } D \text{ and } (\omega_v)_{v\in V} \text{ linear orders on the } \begin{matrix}\text{non-}\alpha\\ \text{non-}a_0\end{matrix} \text{ arcs emanating from each } v \in V\}$$

CLAIM: f is invertible, that is every $(\alpha, (\omega_v))$ determines a unique tour t.

(Do an example !). let the "audience" pick $(\alpha, (\omega_v)_{v\in V})$, and calculate $t = f^{-1}(\alpha, (\omega_v))$.

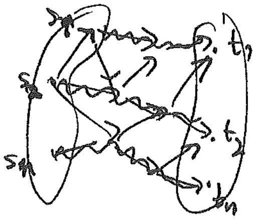
This finishes it, since target of f has the desired cardinality ∎

(71)

④ Lindström (1973) - Gessel (1985) - Viennot LEMMA: (Ardila §3.1.6)

Let D be an acyclic digraph with distinguished vertices $s_1, \dots, s_n$, $t_1, \dots, t_n$

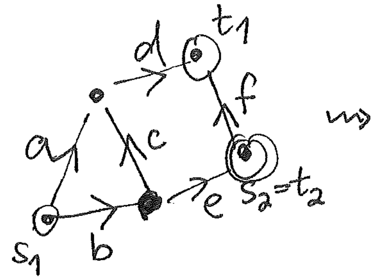
If $M = (m_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,n}}$ has $m_{ij} := \sum_{\substack{\text{paths } P \text{ in } D \\ \text{from } s_i \text{ to } t_j}} \underbrace{\omega(P)}_{:= \prod_{\text{arcs } a \text{ in } P} a}$



then $\det M = \sum_{\substack{\text{vertex-disjoint} \\ \text{paths } (P_1, \dots, P_n) \\ P_i : s_i \to t_{\omega(i)}}} \text{sgn}(\omega) \prod_{i=1}^{n} \omega(P_i)$

---

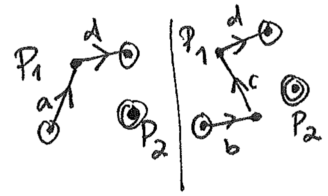
e.g. D =



$$\rightsquigarrow M = \begin{array}{c|cc} & t_1 & t_2 \\ \hline s_1 & ad+bcd+bef & be \\ s_2 & f & 1 \end{array}$$

has $\det M = (ad + bcd + \cancel{bef}) \cdot 1 - \cancel{be} f$

$= ad + bcd$



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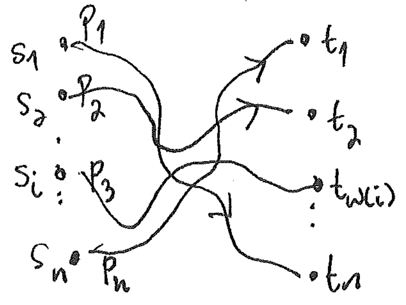
proof: $\det M = \sum_{\omega \in \mathfrak{S}_n} \text{sgn}(\omega) \prod_{i=1}^{n} \underbrace{m_{i,\omega(i)}}_{\sum_{P : s_i \to t_{\omega(i)}} \omega(P)} = \sum_{X := \left\{ \substack{\text{paths } (P_1, \dots, P_n) \\ P_i : s_i \to t_{\omega(i)}} \right.} \text{sgn}(\omega) \prod_{i=1}^{n} \omega(P_i)$



Want to define an involution $\tau : X \to X$ canceling down to $X^\tau = \{\text{vertex-disjoint } (P_1, \dots, P_n)\}$.

If $(P_1, \dots, P_n)$ are not vertex-disjoint,

- find P_{i_0} with smallest i_0 intersecting some path
- find earliest intersection vertex v along P_{i_0}
- find P_{j_0} with smallest $j_0 \neq i_0$ having $v \in P_{j_0}$

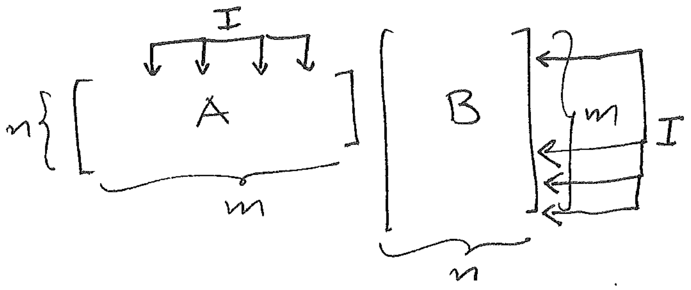
Then keep all other paths the same, and let P_{i_0}, P_{j_0} exchange the tails of their paths after v.

11/2/2015 >

COR 1: (Binet-Cauchy THM)

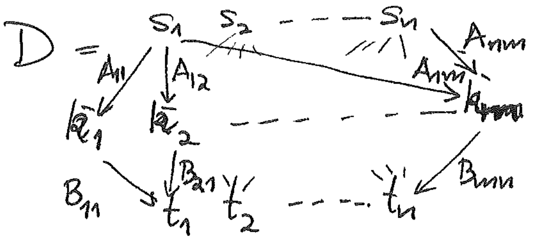
If A is $n\times m$, B is $m\times n$ then

$$\det(\underbrace{AB}_{n\times n}) = \sum_{\substack{K\subseteq[m]:\\ |K|=n}}' \det(A|_{\text{cols } K})\det(B|_{\text{rows } K})$$

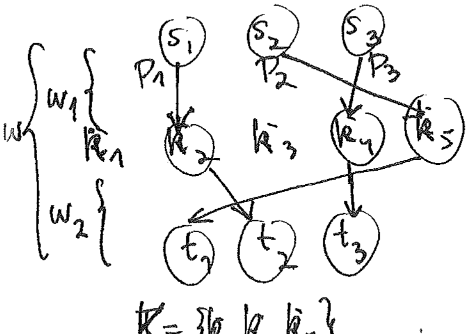


proof: $$(AB)_{ij} = \sum_{k=1}^{m}{}' A_{ik}B_{jk} = \sum_{\substack{\text{paths } P: s_i\to t_j\\ \text{in this}}} \omega(P)$$

D =



and hence $$\det(AB) = \sum_{\substack{\text{vertex-disjoint}\\ \text{paths } (P_1,\dots,P_n)\\ P_i: s_i \to t_{\omega(i)}}}' \operatorname{sgn}(\omega) \prod_{i=1}^{n} \omega(P_i)$$



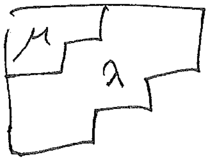
$K = \{k_2, k_4, k_5\}$

$$= \sum_{\substack{K\subseteq[m]\\ \parallel \\ \{k_1<\dots<k_n\}}}' \underbrace{\sum_{\substack{\omega_1\in\mathfrak{S}_n\\ \parallel\\ \text{bijections}\\ [n]\to K\\ \parallel\\ \{s_1,\dots,s_n\}}} \operatorname{sgn}(\omega_1)\prod_{i=1}^{n} A_{i,\omega_1(i)}}_{\det(A|_{\text{cols } K})} \cdot \underbrace{\sum_{\substack{\omega_2\in\mathfrak{S}_n\\ \parallel\\ \text{bijections}\\ K\to[n]\\ \parallel\\ \{t_1,\dots,t_n\}}} \operatorname{sgn}(\omega_2)\prod_{i=1}^{n} B_{i,\omega_2(i)}}_{\det(B|_{\text{rows } K})}$$

∎

COR 2 (Jacobi-Trudi identity)

Given partitions $\lambda = (\lambda_1 \geq \dots \geq \lambda_\ell)$ with $\mu_i \leq \lambda_i \; \forall i$
$\mu = (\mu_1 \geq \dots \geq \mu_\ell)$



then defining $h_r(x_1, \dots, x_n)$:= complete homogeneous symmetric polynomial of degree r
$r \geq 1$

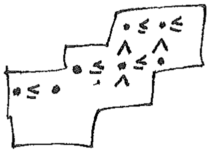
$$= \sum_{1 \leq i_1 \leq \dots \leq i_r \leq n} x_{i_1} x_{i_2} \cdots x_{i_r}$$

$$= x_1^r + x_1^{r-1} x_2 + \dots + x_1 x_2 \cdots x_r + \dots + x_n^r$$

and $h_0(x_1, \dots, x_n) := 1$

$h_{-r}(x_1, \dots, x_n) := 0$

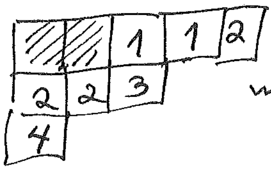
then $\det \left(h_{\lambda_i - i - (\mu_j - j)}(x_1, \dots, x_n) \right)_{\substack{i=1,\dots,\ell \\ j=1,\dots,\ell}} = \sum'_{\substack{\text{column-strict tableaux } T \\ \text{of shape } \lambda/\mu \\ \text{with entries in } [n]}} \prod_{i \in T} x_i$ =



=: skew Schur function $s_{\lambda/\mu}(x_1, \dots, x_n)$

e.g. $\lambda = (5, 3, 1)$

$\mu = (2, 0, 0)$ $n = 4$

T = 

$\rightsquigarrow \prod_{i \in T} x_i = x_1^2 x_2^3 x_3 x_4$

$$\det \begin{bmatrix} h_{5-2}^{(x_1,\dots,x_4)} & h_{5-0+1} & h_{5-0+2} \\ h_{3-2-1} & h_{3-0}^{(x_1,\dots,x_4)} & h_{3-0+1} \\ h_{1-2-2} & h_{1-0+1} & h_{1-0}^{(x_1,\dots,x_4)} \end{bmatrix} = s$$

(x_1, x_2, x_3, x_4)

$=$

$$\det \begin{bmatrix} h_3 & h_6 & h_7 \\ 1 & h_3 & h_4 \\ 0 & 1 & h_1 \end{bmatrix}$$

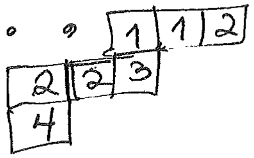
(74)

<u>proof</u>: Let D be a rectangular grid with arrows ↑ and →
having variables $x_1, x_2, \dots, x_n$ on the ↑ arrows
and 1 on the → arrows
with $(s_1, \dots, s_\ell)$ on the x_1-vertical at heights
$\mu - (1, 2, \dots, \ell)$
and $(t_1, \dots, t_\ell)$ on the x_n-vertical at heights
$\lambda - (1, 2, \dots, \ell)$:

$\mu = (2,0,0) \rightsquigarrow (+1,-2,-3)$

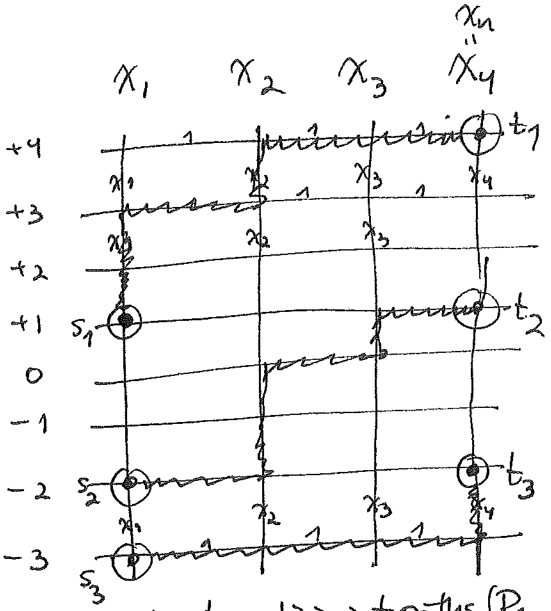
$\lambda = (5,3,1) \rightsquigarrow (+4,+1,-2)$

$n=4$



col-strict tableau T

⟷ EXERCISE!



vertex-disjoint paths $(P_1, \dots, P_\ell)$
where P_i takes <u>vertical</u> steps
dictated by entries in row i of T

Meanwhile $h_{\underbrace{(\lambda_i - i)}_{\text{height of } t_i} - \underbrace{(\mu_j - j)}_{\text{height of } s_j}}(x_1, \dots, x_n) = \sum_{\text{paths } P: s_j \to t_i} wt(P)$ ∎