12/25/2015 >

(90) Posets (Stanley Ch.3, Ardila §4)

~~DEF'N:~~ Recall a poset $(P,\leq)$ is a binary relation $x \leq y$ on a set P

which is reflexive $\quad x \leq x$

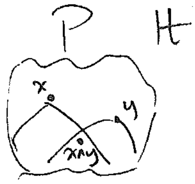
antisymmetric $\quad x \leq y,\ y \leq x \Rightarrow x = y$

transitive $\quad x \leq y,\ y \leq z \Rightarrow x \leq z$

It is graded if every chain $x_0 < x_1 < \ldots < x_\ell$ is finite

and all maximal chains have same length ℓ;

it is ranked if it has a bottom (minimum) element $\hat{0}$ and every $x \in P$ has all maximal chains in $[\hat{0}, x] = \{y \in P : \hat{0} \leq y \leq x\}$ of same length $\ell =: \mathrm{rank}_P(x)$. Its rank gen. fn is $F(P,x) := \sum_{p \in P} x^{\mathrm{rank}(p)}$



It is a meet semilattice if every $x, y \in P$ have some element $x \wedge y$ in P, called their meet, which is a greatest lower bound for x, y: any $z \leq x, y$ has $z \leq x \wedge y \leq x, y$.

Note
- $(x \wedge y) \wedge z = (x \wedge y) \wedge z$
- $x \wedge y = y \wedge x$
- $x \wedge x = x$
- $x \wedge y = x \Leftrightarrow x \leq y$

It's a join semilattice if $\forall x, y \in P\ \exists$ a join $x \vee y$ in P, a least upper bound: any $z \geq x, y$ has $z \geq x \wedge y \geq x, y$.

It's a lattice if it is both a meet- and join semi-lattice.

Note: $x \wedge (x \vee y) = x$
$= x \vee (x \wedge y)$

EXAMPLES:

① Finite chains $\underline{n} :=$

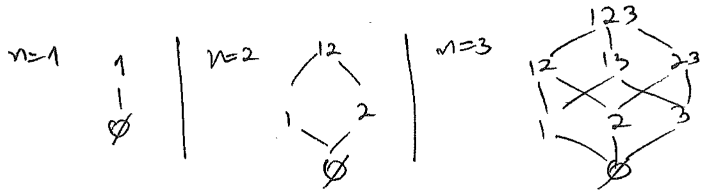
are graded lattices

$F(\underline{n}, x) = [n]_x = 1 + x + x^2 + \ldots + x^{n-1}$

② (Finite) Boolean algebras $B_n = 2^{[n]}$ are graded lattices, with $S \wedge T = S \cap T$

$S \vee T = S \cup T$

$\mathrm{rank}(S) = |S|$



$F(B_n, x) = \sum_{k=0}^{n} \binom{n}{k} x^k$

③ PROP: A finite meet semilattice $(P, \leq)$ always has a $\hat{0}$ (=minimum elt.) and if it also has a $\hat{1}$ (=maximum elt), then it is a lattice.

proof: Check that $(((x_1 \wedge x_2) \wedge x_3) \cdots) \wedge x_\ell)$ is a greatest lower bound for any finite subset $\{x_1, \ldots, x_\ell\}$ in a meet semilattice.

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Hence if $P = \{p_1, \ldots, p_\ell\}$ is a finite ~~meet~~ semilattice, then $\hat{0} = p_1 \wedge \cdots \wedge p_\ell$ exists in P.

Also, if P has a $\hat{1}$, then given $x, y \in P$ the set $\{x_1, \ldots, x_\ell\}$ of all upper bounds for x, y (i.e. $x_i \geq x, y$) is <u>non-empty</u> (as $\hat{1}$ is init), and one can check that $x_1 \wedge \cdots \wedge x_\ell = x \vee y$ ∎

④ $B_n(q) = L_n(q) = L(\mathbb{F}_q^n) := \{$all $\mathbb{F}_q$-linear subspaces $V \subseteq \mathbb{F}_q^n\}$
= (finite) <u>vector space lattices</u>

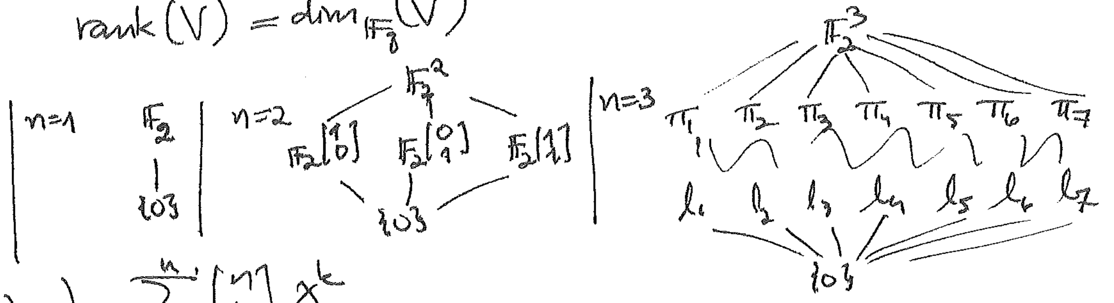
ordered by $\subseteq$ are <u>graded lattices</u>

with $V \wedge W := V \cap W$

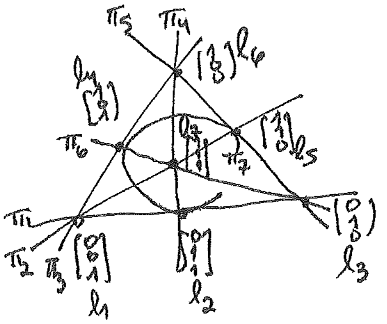
$V \vee W := V + W = \{v+w : v \in V, w \in W\}$

$\text{rank}(V) = \dim_{\mathbb{F}_q}(V)$

e.g. <u>$q=2$</u>



$$F(B_n(q), x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k$$



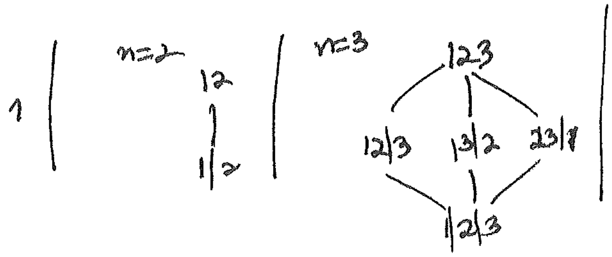
⑤ $\Pi_n = \{$set partitions of $[n]\}$
ordered by refinement
are <u>graded lattices</u>

with $\pi_1 \wedge \pi_2$ = common refinement of π_1, π_2

$\pi_1 \vee \pi_2$ = transitive closure of π_1, π_2's blocks

$\text{rank}(\pi) = n - \#\text{blocks}(\pi)$

e.g.



$$F(\Pi_n, x) = \sum_{k=1}^{n} S(n, n-k) x^{n-k}$$

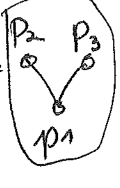
⑥ Given P, Q posets, $P \sqcup Q$ = disjoint union having $p \in P$, $q \in Q$ incomparable ($p \not\le q$, $q \not\le p$)

$P \times Q$ = Cartesian product with componentwise order:

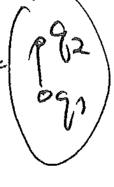
$$(p_1,q_1) \le (p_2,q_2) \iff \left\{ \begin{array}{c} p_1 \le_P p_2 \\ \text{and} \\ q_1 \le_Q q_2 \end{array} \right\}$$

$F(P,x) = 1+2x$ $\quad$ $F(Q,x) = 1+x$

e.g. if P =

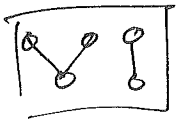


Q =

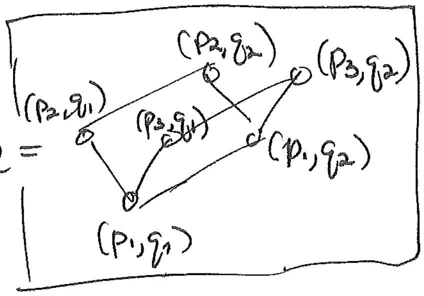


then

$P \sqcup Q$ =



$P \times Q$ =

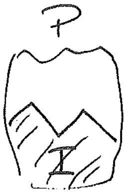


$F(P \times Q, x) = 1+3x+2x^2$
$= (1+2x)(1+x)$

P, Q lattices, ranked $\Rightarrow$ $P \times Q$ lattice, ranked $\quad$ $F(P \times Q, x) = F(P,x)\,F(Q,x)$

11/30/2015

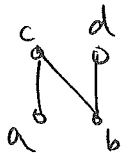
⑦ DEF'N: An (order) ideal $I \subset P$ a poset is a subset closed under going downward in P, i.e. $p \in I$ and $p' \le p \Rightarrow p' \in I$



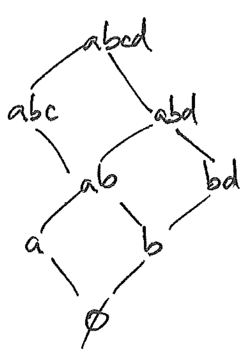
$J(P) :=$ {the lattice of all order ideals $I \subset P$ ordered via $\subseteq$} has $I_1 \wedge I_2 = I_1 \cap I_2$
$I_1 \vee I_2 = I_1 \cup I_2$

which makes it a distributive lattice, i.e. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(because $\cap, \cup$ satisfy these properties)

e.g. P =



has J(P) =



$$F(J(P),x) = \sum_{\substack{\text{ideals} \\ I \subseteq P}} x^{|I|}$$

When P is finite, J(P) is ranked with rank(I) = |I|.

EXERCISE: Show $J(P \sqcup Q) \cong J(P) \times J(Q)$

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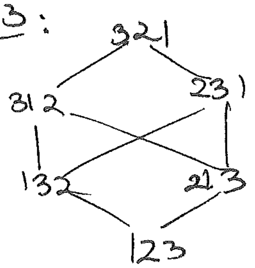
⑧ Two posets on $\mathfrak{S}_n$ that can be defined via transitive closure:

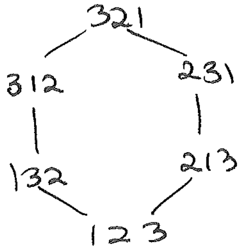
(strong) Bruhat order: trans closure of $x < y$ when $x(i,j) = y$ for some $1 \le i < j \le n$ and $inv(x) < inv(y)$)

(right) weak order: — " — $x(i,i+1) = y$ for some $1 \le i \le n-1$ and $inv(x) < inv(y)$)

e.g. $n=3$:



Bruhat order



weak order

- Both are ranked, with $rank(w) = inv(w)$, so have $F(P,q) = \sum_{w \in \mathfrak{S}_n} q^{inv(w)} = [n]!_q$
- Weak order is a lattice (not obvious); Bruhat order is not.

⑨ Dominance order on {partitions $\lambda \vdash n$}:

$\mu \trianglelefteq \lambda$ if $\mu_1 \le \lambda_1$

$\mu_1 + \mu_2 \le \lambda_1 + \lambda_2$

$\vdots$

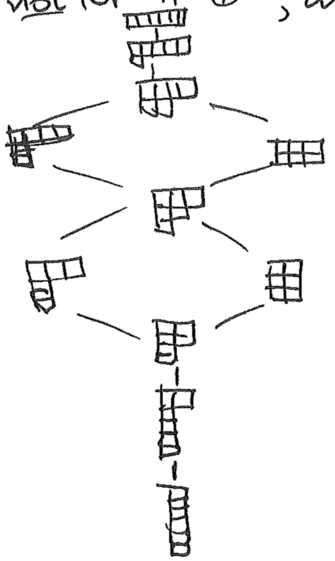
$\mu_1 + \dots + \mu_k \le \lambda_1 + \lambda_2 + \dots + \lambda_k$

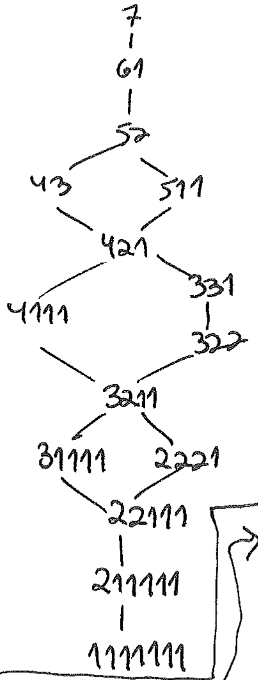
$\mu = (\mu_1, \mu_2, \dots)$, $\mu_1 \ge \mu_2 \ge \dots$

$\lambda = (\lambda_1, \lambda_2, \dots)$, $\lambda_1 \ge \lambda_2 \ge \dots$

It turns out to be a total or linear order for $n = 1,2,3,4,5$

but not for $n=6$, and not even ranked for $n=7$





EXERCISE: It is always self-dual, i.e. $P \cong P^{opp} = P^*$ via $\lambda \mapsto \lambda^t$

EXERCISE: It is a lattice, in which if $\lambda \wedge \mu = \rho$, $\lambda \vee \mu = \nu$, then $\rho_1 + \dots + \rho_k = \min(\lambda_1 + \dots + \lambda_k, \mu_1 + \dots + \mu_k)$

$\nu_1 + \dots + \nu_k = \max(\lambda_1 + \dots + \lambda_k, \mu_1 + \dots + \mu_k)$

Distributive lattices (Stanley §3.4)

DEF'N-PROP: In a lattice L,

(a) $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ $\forall x,y,z \in L$

(b) $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$ $\forall x,y,z \in L$

and equality in (a) holds $\forall x,y,z \in L$

$\iff$ equality in (b) holds $\forall x,y,z \in L$

in which case L is called *distributive*.

EXAMPLES:

① For a poset P, $J(P) = \{\text{order ideals } I \subseteq P\}$ is a distributive lattice

② L_1, L_2 dist. $\Rightarrow L_1 \times L_2$ distributive, and same for $L_1 \times \cdots \times L_k$

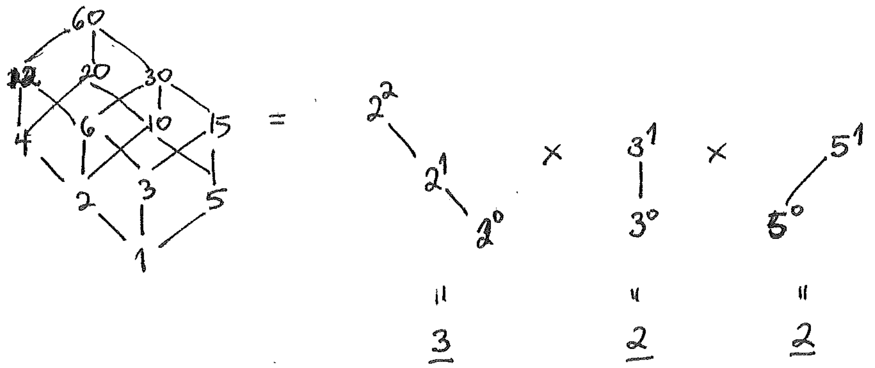
③ The divisor *poset* $D_n = \{\text{all divisors of } n\}$ for $n = 1, 2, \ldots$ is distributive, since if $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ for distinct primes p_i

then $D_n \cong \underline{a_1+1} \times \underline{a_2+1} \times \cdots \times \underline{a_k+1}$, and each chain $\underline{a_i+1}$ is distributive

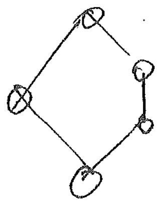
$d = p_1^{b_1} \cdots p_k^{b_k} \longmapsto (b_1+1, b_2+1, \ldots, b_k+1)$

e.g. $n = 60 = 2^2 \cdot 3^1 \cdot 5^1$

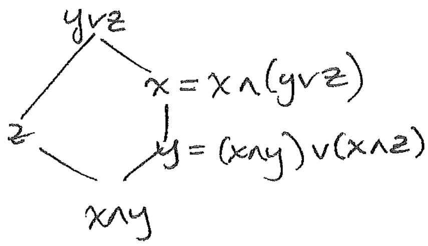
has $D_{60} \cong \underline{3} \times \underline{2} \times \underline{2}$

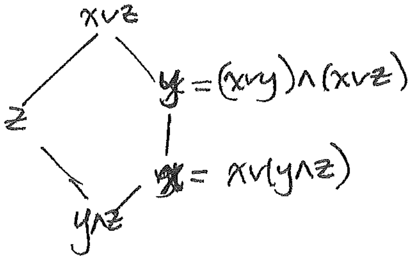


④

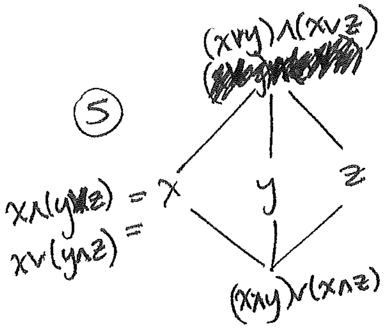


is not distributive:



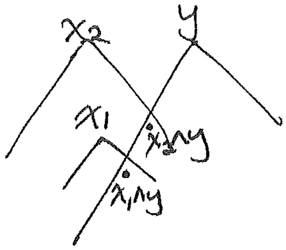


⑤



is not distributive:

12/2/2015 proof of DEF'N-PROP:



Note that $x_1 \le x_2$ in L $\Rightarrow$ $x_1 \wedge y \le x_2 \wedge y$

so $x\wedge(y\vee z) \ge x\wedge y,\ x\wedge z$ $\Big\} \Rightarrow$ $x\wedge(y\vee z) \ge (x\wedge y)\vee(x\vee z)$ proving (a).

One proves (b) dually, i.e. switching $\le$ for $\ge$ and $\vee$ for $\wedge$

Now assuming (a) holds with equality $\forall\, x,y,z \in L$

i.e. $x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z)$

then to prove (b) holds with equality: ~~it suffices to prove the opposite inequality, i.e. that $x\vee(y\wedge z) \overset{?}{\ge} (x\vee y)\wedge(x\vee z)$~~

$$(x\vee y)\wedge(x\vee z) \overset{(a)}{=} \big((x\vee y)\wedge x\big) \vee \big((x\vee y)\wedge z\big)$$

$$\overset{(a)}{=} x \vee \big((x\wedge z)\vee(y\wedge z)\big)$$

$$= x \vee (y\wedge z) \quad \text{since } x \ge x\wedge z \ \blacksquare$$

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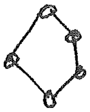
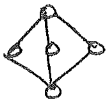
RMK: Garrett Birkhoff $\overset{1948}{}$ showed a lattice L is distributive

$\Longleftrightarrow$ L has no 5 element sublattice iso. to  or 

($\xrightarrow{\text{obvious}}$
$\xleftarrow{\text{not obvious; not hard}}$)

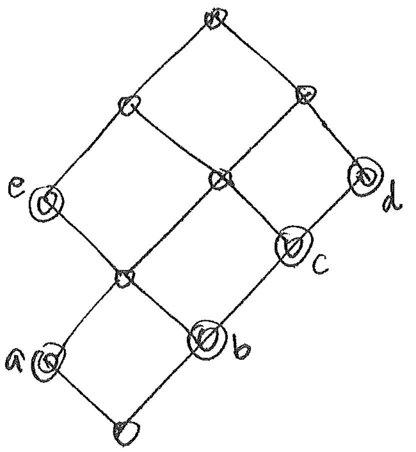
More importantly, he showed the following

THM (Birkhoff's Fund Thm of Finite Distributive Lattices)

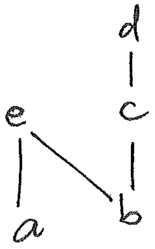
Every finite distributive lattice L is isomorphic to J(P) for a poset P defined uniquely up to isomorphism, namely $P \cong \mathrm{Irr}(L) := \{\text{the join-irreducible } p \in L\}$ with the induced partial order as a subposet of L

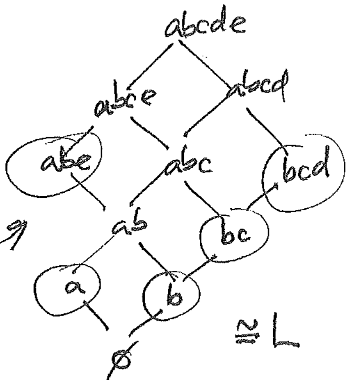
$p = x_1 \vee \cdots \vee x_\ell$ for some $\{x_1, \ldots, x_\ell\} \subset L$ $\Rightarrow p = x_i$ for some i.

EXAMPLE:

L = 

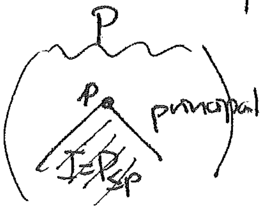
is distributive, with elements of P=Irr(L) labeled

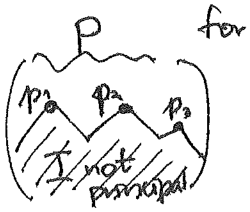
P=Irr(L) = 

J(P) = 

≅ L

Note that the join-irreducibles in J(P) = principal order ideals $I = P_{\leq p} = \{q \in P : q \leq p\}$ for some $p \in P$





proof of Birkhoff's Thm:

Given L finite and distributive, define maps

$$\begin{tikzcd} L \arrow[r, shift left, "f"] & J(P) \arrow[l, shift left, "g"] \end{tikzcd} \qquad \text{where } P = \mathrm{Irr}(L)$$

$$\begin{tikzcd} x \arrow[r, mapsto, "f"] & f(x) := \{p \in \mathrm{Irr}(L) : p \leq x\} \\ g(I) := p_1 \vee \cdots \vee p_\ell & I = \{p_1, \ldots, p_\ell\} \arrow[l, mapsto, "g"'] \end{tikzcd}$$

It's not hard to see both f, g are order-preserving i.e. $x \leq y \Rightarrow f(x) \leq f(y)$, $I \subseteq J' \Rightarrow g(I) \leq g(J')$.

We claim that in any finite lattice (not nec. distributive)

one has $g(f(x)) = \bigvee_{\substack{p \in \mathrm{Irr}(L): \\ p \leq x}} p = x$:

Certainly $\bigvee_{\substack{p \in \mathrm{Irr}(L): \\ p \leq x}} p \leq x$ since each $p \leq x$, but also one can

write $x = p_1 \vee p_2 \vee \cdots \vee p_\ell$ with each p_i join-irreducible (either $x \in \mathrm{Irr}(L)$ or write using downward induction on x in L $x = x_1 \vee x_2$ where $x_1 \lneq x$, $x_2 \lneq x$; repeat).

Hence $x = \bigvee_{\substack{p \in \mathrm{Irr}(L): \\ p \leq x}} p = g(f(x))$.

On the other hand, $f(g(\underset{\{p_1, \ldots, p_\ell\}}{\overset{}{I}})) = \{q \in \mathrm{Irr}(L) : q \leq p_1 \vee \cdots \vee p_\ell\} \supseteq I$

$$\begin{aligned} \text{but } q \leq p_1 \vee \cdots \vee p_\ell \Rightarrow q &= q \wedge (p_1 \vee \cdots \vee p_\ell) \\ \text{distributivity} \quad &= (q \wedge p_1) \vee \cdots \vee (q \wedge p_\ell) \\ q \in \mathrm{Irr}(L) \quad &\Rightarrow q = q \wedge p_i \text{ for some } i \\ &\Rightarrow q \leq p_i \in I \\ I \text{ is an order ideal} \quad &\Rightarrow q \in I \end{aligned}$$

Hence $f(g(I)) = I$. ∎

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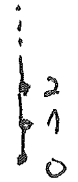
RMK: Certain ∞ distributive lattices are important...

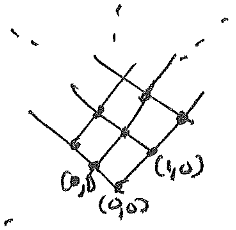
DEF'N: A *finitary* distributive lattice is a dist. lattice with a $\hat{0}$ which is *locally finite*

↑ all intervals $[x,y]:=\{z\in L : x\le z\le y\}$ are finite

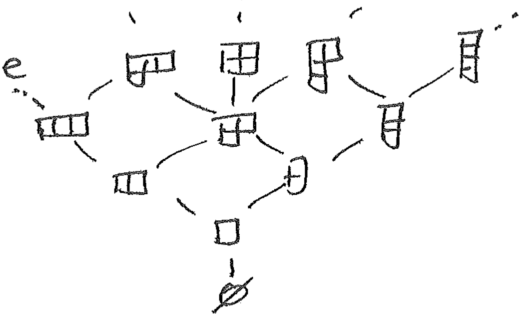
EXAMPLES: ① $\mathbb{N}$ =



② $\mathbb{N}^d$ e.g. d=2 $\mathbb{N}^2$ =



③ $\mathbb{Y}$ = Young's lattice on partitions



$\mu\wedge\lambda=\mu\cap\lambda$

$\mu\vee\lambda=\mu\cup\lambda$

One can easily show this gen'lization of Birkhoff's Thm:

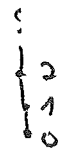
THM: Every *finitary* dist. lattice L is isomorphic to

$J_f(P):=\{$all *finite* order ideals $I\subseteq P\}$

for some poset P having all *principal* order ideals $P_{\le p}$ *finite*, defined uniquely up to iso., namely $P\cong \mathrm{Irr}(L)$.

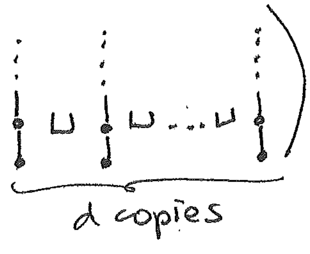
EXAMPLES:

① $\mathbb{N}$ =



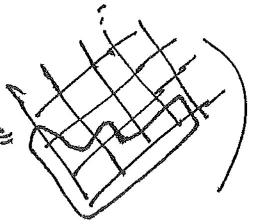
$= J_f\Big($

$\Big)$ ② $\mathbb{N}^d \cong J_f\Big($



$\Big)$

③ $\mathbb{Y} = J_f\Big(\mathbb{N}^2 \cong$



$\Big)$

(99) Möbius inversion (Stanley §3.6, 3.7)

Let~~'s~~ re-interpret inclusion-exclusion as being about the poset $P = B_n = 2^{[n]}$ and functions $f = f_= : P \to R$ some comm. ring

where we [illegible] were given a new function $g = f_{\subseteq} : P \to R$ such that $g(S) = \sum_{T \subseteq S} f(T)$

i.e. $g(y) = \sum_{x \in P} \zeta(x,y) f(x)$

where $\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \text{ in } P \\ 0 & \text{else} \end{cases}$

and we could invert to get f via

$$f_=(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_{\subseteq}(T)$$

i.e. $f(y) = \sum_{x \in P} \mu(x,y) g(x)$ where $\mu(\overset{T}{x}, \overset{S}{y}) = \begin{cases} (-1)^{|\overset{S}{y} \setminus \overset{T}{x}|} & \text{if } x \leq y \text{ in } P \\ 0 & \text{else} \end{cases}$

This works for other locally finite posets P, once we figure out where $\zeta(\cdot,\cdot)$, $\mu(\cdot,\cdot)$ should live...

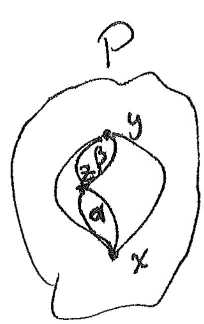
DEF'N: The incidence algebra $I(P,R)$ of a loc. fin. poset P (over a comm. ring R) is the ~~set~~ ring of all functions $\alpha: \mathrm{Int}(P) \to R$

$\{\text{intervals } [x,y] \text{ in } P\}$

with pointwise addition: $(\alpha+\beta)(x,y) = \alpha(x,y) + \beta(x,y)$

and convolution product: $(\alpha * \beta)(x,y) = \sum_{z \in [x,y]} \alpha(x,z)\beta(z,y)$

a finite sum!



and 2-sided identity element: $\delta(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$ = Kronecker delta

(100)

We'll want to know that the zeta function $\zeta(x,y)=\begin{cases}1 & \text{if } x\le y\\ 0 & \text{else}\end{cases}$

is always invertible in $I(P,R)$:

PROP: $\alpha\in I(P,R)$ has a (2-sided) inverse $\iff \alpha(x,x)\in R^{\times}\ \forall x\in P$

proof: $\alpha*\beta=\delta$

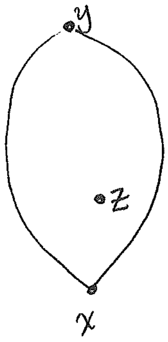
$$\iff (\alpha*\beta)(x,y)=\delta(x,y)=\begin{cases}1 & \text{if } x=y\\ 0 & \text{if } x\neq y\end{cases}\qquad\forall x,y\in P$$

$$\|$$

$$\sum_{z\in[x,y]}\alpha(x,z)\beta(z,y)$$

which forces $\alpha(x,x)\beta(x,x)=1$, so $\left\{\begin{matrix}\alpha(x,x)\in R^{\times}\\ \beta(x,x)=\alpha(x,x)^{-1}\end{matrix}\right\}\ \forall x\in P$

and then when $\alpha(x,x)\in R^{\times}$, the values for $\beta^{(x,y)}$ are uniquely determined by induction on $\#[x,y]$ via



$$\alpha(x,x)\beta(x,y)+\sum_{z\in(x,y]}{}'\,\alpha(x,z)\beta(z,y)=0$$

$$\Rightarrow\quad \beta(x,y)=-\alpha(x,x)^{-1}\sum_{z\in(x,y]}\alpha(x,z)\underbrace{\beta(z,y)}_{\#[z,y]<\#[x,y]}.$$

Note $\alpha(x,x)\in R^{\times}$ will also give a left inverse $\beta'(\cdot,\cdot)$

defined recursively by $\beta'(x,y)=-\alpha(y,y)^{-1}\sum_{z\in[x,y)}{}'\,\beta'(x,z)\alpha(z,y)$

but then associativity of $*$ forces $\beta'=\beta'*(\alpha*\beta)=(\beta'*\alpha)*\beta=\beta$ $\blacksquare$

COR: $\zeta(\cdot,\cdot)$ has an inverse, called the Möbius function $\mu=\zeta^{-1}$

defined recursively by $\boxed{\mu(x,x)=1\quad\forall x\in P}$

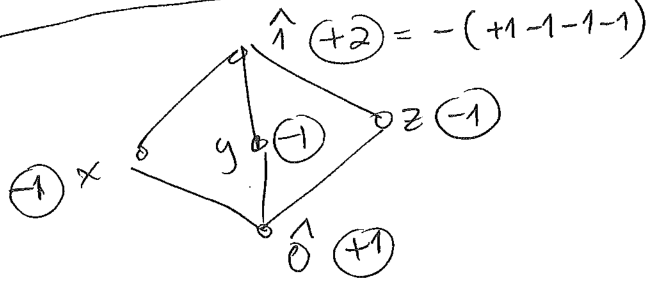
and either $\mu(x,y)=-\sum_{z\in(x,y]\text{ i.e. }x<z\le y}\mu(z,y)\quad\forall x<y$

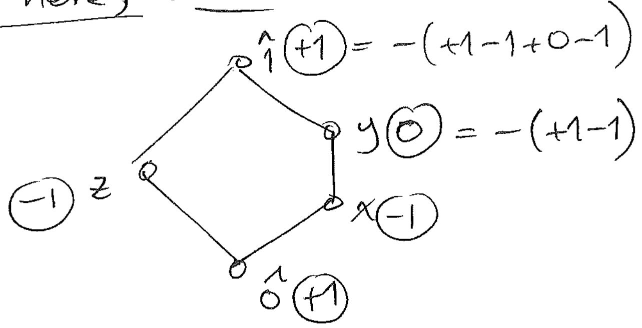
or $\boxed{\mu(x,y)=-\sum_{z\in[x,y)\text{ i.e. }x\le z<y}\mu(x,z)\quad\forall x<y}$

(101)

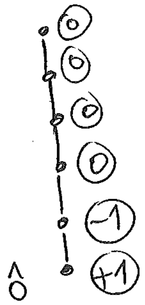
EXAMPLES of $\mu(\cdot,\cdot)$

① Let's compute $\mu(\hat{0},p)$ $\forall p$ here, values circled:



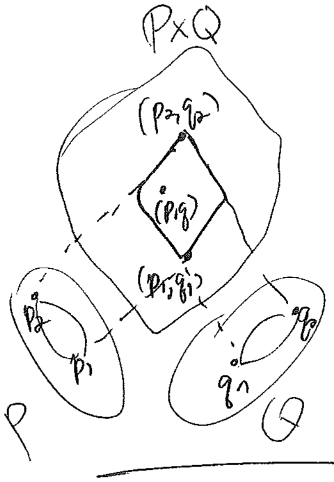


② In a finite chain, $\mu(x,y) = \begin{cases} +1 & \text{if } x=y \\ -1 & \text{if } x \lessdot y \\ 0 & \text{else} \end{cases}$



③ **PROP:** In a product $P\times Q$ $\quad \mu_{P\times Q}((p_1,q_1),(p_2,q_2)) = \mu_P(p_1,p_2)\,\mu_Q(q_1,q_2)$

proof: The function [illegible] $\alpha(\cdot,\cdot) \in I(P\times Q, \mathbb{Z})$ defined by the RHS satisfies the correct initial condition and recurrence: $\alpha((p,q),(p,q)) = \underbrace{\mu_P(p,p)}_{+1}\underbrace{\mu_Q(q,q)}_{+1} = +1$ ✓



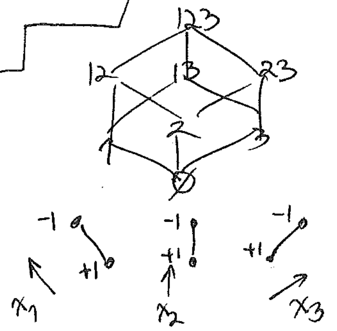
$$\sum_{\substack{(p,q) \\ \in[(p_1,q_1),(p_2,q_2)]}} \mu_P(p_1,p)\,\mu_Q(q_1,q) = \underbrace{\left(\sum_{p\in[p_1,p_2]}\mu_P(p_1,p)\right)}_{0}\cdot\underbrace{\left(\sum_{q\in[q_1,q_2]}\mu_Q(q_1,q)\right)}_{0}$$

$= 0$ ✓ ■

④ COR: In $B_n = 2^{[n]} \cong \underline{2}^n = \underline{2}\times\underline{2}\times\cdots\times\underline{2}$,

$$\mu(T,S) = (-1)^{|S\setminus T|}$$

for $T\subseteq S$.



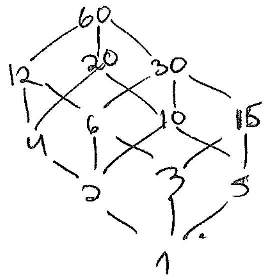
⑤ The number-theoretic Möbius function $\mu(m) = \begin{cases} (-1)^k & \text{if } m = p_1^1 \cdots p_k^1 \text{ is squarefree with } k \text{ prime factors} \\ 0 & \text{if } m \text{ is } \underline{\text{not}} \text{ squarefree} \end{cases}$

Is really computing $\mu_{D_n}(d_1, d_2) = \mu\left(\frac{d_2}{d_1}\right)$

for $d_1 \mid d_2$ in the divisor poset $D_n \cong \underline{(a_1+1)} \times \cdots \times \underline{(a_k+1)}$

when $n = p_1^{a_1} \cdots p_k^{a_k}$

e.g. $n = 60 = 2^2 \cdot 3^1 \cdot 5^1$



$\mu(3,12) = \mu\left(\frac{12}{3}\right) = \mu(4) = \mu(2^2) = 0$
$\uparrow$ not squarefree

similarly,
$\mu(3,60) = \mu\left(\frac{60}{3}\right) = \mu(20) = \mu(2^2 \cdot 5) = 0$

But $\mu(2,60) = \mu\left(\frac{60}{2}\right) = \mu(30) = \mu(2^1 \cdot 3^1 \cdot 5^1) = (-1)^3$

Now let's state and use...

<u>THM</u> (Möbius inversion formula)

If a poset P has all $P_{\le p}$ finite, and $f, g : P \to \underset{\text{a comm. ring}}{R}$ are related by $g(y) = \sum_{\substack{x \in P: \\ x \le y}} f(x)$

then $f(y) = \sum_{\substack{x \in P: \\ x \le y}} \mu(x,y)\, g(x)$

(and dually, if all $P_{\ge p}$ are finite, with $g(y) = \sum_{x : x \ge y} f(x)$

then $f(y) = \sum_{x : x \ge y} \mu(y,x)\, g(x)$)

(103)

proof: The free R-module $R^P := \{\text{functions } f : P \to R\}$

(with pointwise addition and scaling by elts of R)

is actually a (right) $I(P,R)$-module, meaning that $\alpha \in I(P,R)$ act on such f via $(f\cdot\alpha)(y) = \sum_{x\in P} f(x)\alpha(x,y)$

and $(f\cdot\alpha)\beta = f\cdot(\alpha*\beta)$ since

$$((f\cdot\alpha)\beta)(y) = \sum_{x\in P} (f\cdot\alpha)(x)\beta(x,y)$$

$$= \sum_{x\in P}\sum_{x'\in P} f(x')\alpha(x',x)\beta(x,y)$$

$$= \sum_{x'\in P} f(x') \underbrace{\sum_{x\in P}\alpha(x',x)\beta(x,y)}_{(\alpha*\beta)(x',y)}$$

$$= (f\cdot(\alpha*\beta))(y)$$

Then $g(y) = \sum_{\substack{x\in P:\\ x\le y}} f(x) = \sum_{x\in P} f(x)\zeta(x,y)$

[redacted]

i.e. $g = f\cdot\zeta$

$\big\downarrow$ act on right by $\zeta^{-1} = \mu$

$g\cdot\mu = f$

i.e. $\sum_{x\in P} g(x)\mu(x,y) = f(y)$

$$\parallel$$

$$\sum_{\substack{x\in P:\\ x\le y}} \mu(x,y)g(x) \qquad \blacksquare$$

COR 1: Inclusion-Exclusion, for $P = B_n$.

(104)

12/7/2015? COR 2. (Number-theoretic Möbius Inversion)

If $f, g : \underset{\{1,2,3,\ldots\}}{\overset{}{\mathbb{P}}} \to \mathbb{R}$ are related by $g(n) = \sum_{\substack{d: \\ d|n}} f(d)$

then $f(n) = \sum_{\substack{d: \\ d|n}} \underbrace{\mu\left(\frac{n}{d}\right)}_{= \mu(d,n) \text{ in divisor poset.}} g(d)$

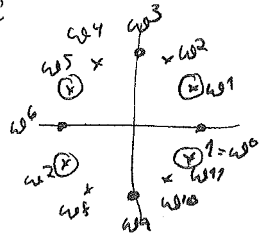
EXAMPLES

① Euler's phi-function $\varphi(n) = |(\mathbb{Z}/n\mathbb{Z})^\times|$

$= |\{\overline{m} \in \mathbb{Z}/n\mathbb{Z} : \gcd(m,n) = 1\}|$

$= \#$ prim. n^{th} roots-of-1 in $\mathbb{C}$

e.g. $\varphi(12) = 4 = |\{1, 5, 7, 11\}|$



It satisfies $f(n) = n = |\mathbb{Z}/n\mathbb{Z}| = \sum_{\substack{d: \\ d|n}} \underbrace{\varphi(d)}_{\#\{\text{prim. } d^{th} \text{ roots in } \mathbb{C}\}}$

$= \#\left\{\begin{matrix} n^{th} \text{ roots of 1} \\ \text{in } \mathbb{C} \\ \text{(not nec. prim.)} \end{matrix}\right\}$

e.g. $\{0, 1, \ldots, 12\} = \underset{d=1}{\{0\}} \sqcup \underset{d=2}{\{6\}} \sqcup \underset{d=3}{\{4, 8\}} \sqcup \underset{d=4}{\{3, 9\}} \sqcup \underset{d=6}{\{2, 10\}} \sqcup \underset{d=12}{\{1, 5, 7, 11\}}$

Hence by Möbius inversion, $\varphi(n) = \sum_{\substack{d: \\ d|n}} \mu\left(\frac{n}{d}\right) f(d)$

$= \sum_{d|n} \mu\left(\frac{n}{d}\right) d$

If $n = p_1^{a_1} \cdots p_k^{a_k}$

$$= \sum_{\substack{S \subset \{1,2,\ldots,k\} \\ \| \\ \{s_1,\ldots,s_\ell\}}} \mu\left(\frac{\text{[illegible]}}{p_{s_1} p_{s_2} \cdots p_{s_\ell}}\right) \cdot \frac{n}{p_{s_1} p_{s_2} \cdots p_{s_\ell}}$$

$$= \sum_{S \subset \{1,2,\ldots,k\}} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i}$$

$$= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} \left(p^{a_i} - p^{a_i - 1}\right)$$

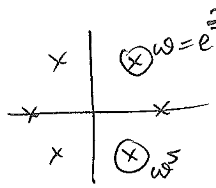
(105)

② EXERCISE: Show that $f(n) := \sum_{\substack{\text{primitive}\\ n^{\text{th}} \text{ roots of}\\ \text{unity } \gamma \text{ in } \mathbb{C}}}' \gamma = \mu(n)$ ← number-theoretic Möbius function

by checking that $\sum_{d|n}' f(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{if } n \geq 2 \end{cases}$

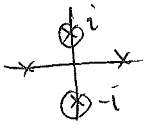
(and why does this suffice?)

e.g. $n=6$



$f(6) = \omega + \omega^5 = +1 = \mu(\underset{2^1 \cdot 3^1}{\overset{\shortparallel}{6}})$

$n=4$

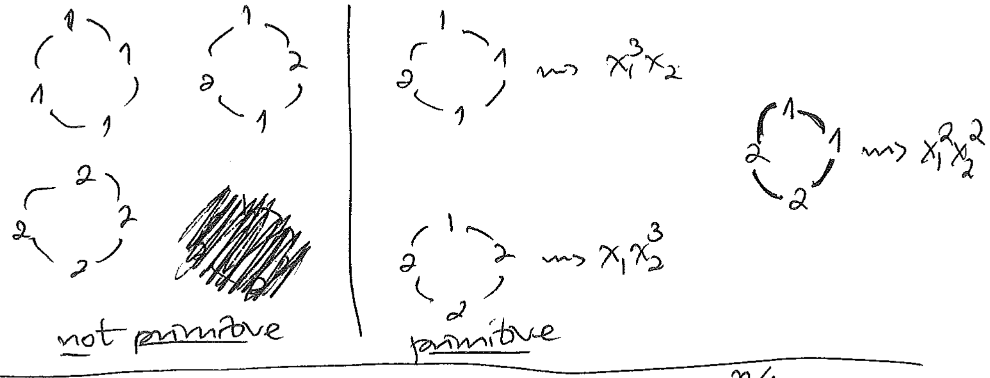


$f(4) = i + (-i) = 0 = \mu(\underset{2^2}{\overset{\shortparallel}{4}})$

---

③ DEF'N: A necklace of size n with q colors is a word $(w_1, \ldots, w_n) \in \{1, 2, \ldots, q\}^n$ considered up to cyclic rotation, and is primitive if its equiv. class $[w]$ has size n

e.g. $n=4$, $q=2$



not primitive | primitive

PROP: $\sum'_{\substack{\text{prim. necklaces } [w]\\ \text{of size } n\\ \text{with } q \text{ colors}}} \overbrace{x_{w_1} \cdots x_{w_n}}^{x_w} = \frac{1}{n} \sum_{d|n} \mu(d) \big(\underbrace{x_1^d + \ldots + x_q^d}_{\substack{\text{power sum symmetric function}\\ p_d(x_1, x_2, \ldots, x_q)}} \big)^{n/d}$

and $\#\{\text{of such necklaces}\} = \frac{1}{n} \sum_{d|n} \mu(d) q^{n/d}$

---

e.g. $n=4$, $q=2$

$$x_1^3 x_2 + x_1 x_2^3 + x_1^2 x_2^2 \overset{?}{=} \frac{1}{4} \sum_{d|4} \mu(d) (x_1^d + x_2^d)^{n/d} = \frac{1}{4}\Big(\underset{d=1}{+(x_1+x_2)^4} \underset{d=2}{-(x_1^2+x_2^2)^2} + \underset{d=4}{0 \cdot (x_1^4+x_2^4)^1} \Big)$$

$$= \frac{1}{4}\Big(x_1^4 + 4x_1^3 x_2 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + x_2^4 - (x_1^4 + x_2^4 + 2x_1^2 x_2^2)\Big)$$

$$= \frac{1}{4}(4x_1^3 x_2 + 4x_1^3 x_2 + 4x_1^2 x_2^2)$$

$$= x_1^3 x_2 + x_1^3 x_2 + x_1^2 x_2^2 \checkmark$$

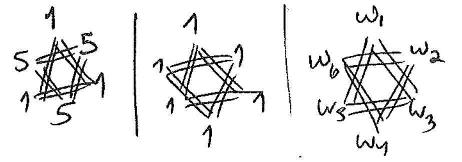
proof: Fixing n, if we ~~[illegible]~~ define for $d|n$

$$f(d) := \sum_{\substack{w \in \{1,\dots,q\}^n: \\ [w] \text{ has size } d}} x_w$$

$$g(d) := \sum_{\substack{w \in \{1,\dots,q\}^n: \\ [w] \text{ has size} \leq d}} x_w = \sum_{e|d} f(e)$$

then we want $\frac{1}{n} f(n)$, and we have $g(d) = \left(x_1^{n/d} + x_2^{n/d} + \dots + x_q^{n/d}\right)^d = p_{n/d}^d$

e.g. $n=6$
$d=2$



$\rightsquigarrow \left(x_1^3 + x_2^3 + \dots + x_q^3\right)^2 = g(2)$ (for $n=6$)

Hence $\frac{1}{n} f(n) \underset{\substack{\uparrow \\ \text{Möbius Inv.}}}{=} \frac{1}{n} \sum_{d|n} \mu(n/d) g(d) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) p_{n/d}^d = \frac{1}{n} \sum_{d|n} \mu(d) p_d^{n/d}$ ■

④ P. Hall's application (1936) — Given a finite group G, how to compute

$$f(G) := \#\{\text{subsets } A \subseteq G \text{ generating } G, \text{ i.e. } \langle A \rangle = G\} ?$$

For a subgroup $H \leq G$, easy to compute

$$g(H) := \#\{\text{subsets } A \subseteq G \text{ generating some } K \leq H\}$$
$$= \#\{\text{subsets } A \subseteq H\} = 2^{|H|}$$

But $g(H) = \sum'_{\substack{K: K \leq H \\ \uparrow}} f(K)$

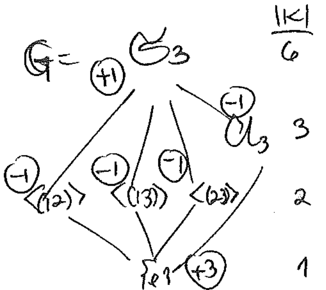
in the lattice of subgroups $L(G)$
$H_1 \wedge H_2 = H_1 \cap H_2$
$H_1 \vee H_2 = \langle H_1 \cup H_2 \rangle$

so $f(H) = \sum'_{K: K \leq H} \mu(K,H) \, g(\cancel{H}K)$

$= \sum_{K: K \leq H} \mu(K,H) \, 2^{|K|}$ i.e. $\boxed{f(G) = \sum'_{K: K \leq G} \mu(K,G) 2^{|K|}}$

$\mathfrak{S}_3$
$\mu(K,G)$
circled

e.g. $G =$



so $f(\mathfrak{S}_3) = \sum'_{K: K \leq \mathfrak{S}_3} \mu(K, \mathfrak{S}_3) 2^{|K|} = 2^6 - (2^2 + 2^2 + 2^2 + 2^3) + 3 \cdot 2^1$

$= 64 - 20 + 6$

$= 50$

(107)

More Möbius functions (§ 3.9, 3.8 Stanley)

Let's develop Weisner's formula & Cross-cut formula for $\mu(\cdot,\cdot)$ in a lattice, before computing μ in $\Pi_n, \mathcal{L}_n(q), J(P)$.

An algebraic tool is helpful:

DEF'N: For a lattice L, its Möbius algebra $A(L,\Bbbk)$ over a field $\Bbbk$ is $\Bbbk^L$ with a $\Bbbk$-basis $\{f_x\}_{x\in L}$ that multiplies by this rule: $f_x f_y = f_{x\wedge y}$

(= semigroup alg. $/\Bbbk$ for $\wedge$ on L)

PROP: For a finite lattice L, There is a ring isomorphism

$$A(L,\Bbbk) \xrightarrow{\varphi} \Bbbk^{|L|} := \{\underbrace{\Bbbk\times\cdots\times\Bbbk}_{|L| \text{ times}} \text{ with } \Bbbk\text{-basis } \{e_x\}_{x\in L} \text{ multiplying as orthogonal idempotents: } e_x^2 = e_x,\ e_x e_y = 0 \text{ if } x\neq y\}$$

$$f_y \longmapsto \sum_{x:x\leq y} e_x$$

which has $\varphi^{-1}(e_y) = \sum_{x:x\leq y} \mu(x,y) f_x := \delta_y$, so $f_y = \sum_{x:x\leq y} \delta_x$

Hence $\{\delta_y\}_{y\in L}$ are a $\Bbbk$-basis of orthog. idempotents in $A(L,\Bbbk)$

$$\delta_x^2 = \delta_x$$
$$\delta_x\delta_y = 0 \text{ if } x\neq y.$$

proof: φ is a $\Bbbk$-vector space iso. since its matrix is unitriangular

$$\varphi = e_x\left\{\overbrace{\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}^{f_y}\right.$$ for any linear ordering of L that extends $\leq$.

Also $\varphi(f_y f_z) = \varphi(f_{y\wedge z}) = \sum_{x:x\leq y\wedge z} e_x$

$$\varphi(f_y)\varphi(f_z) = \Big(\sum_{x:x\leq y} e_x\Big)\Big(\sum_{w:w\leq z} e_w\Big) = \sum_{\substack{(x,w):\\ x\leq y\\ w\leq z}} e_x e_w = \sum_{\substack{x:\\ x\leq y,z}} e_x = \sum_{x:x\leq y\wedge z} e_x$$

The fact that $\varphi^{-1}(e_y) = \sum_{x:x\leq y}\mu(x,y) f_x$ comes from

$\sum_{x:x\leq y} \varphi^{-1}(e_x) = f_y$ via Möbius inversion ∎

COR 1: (Weisner's Thm) If $a \lneq \hat{1}$ in a finite lattice L, then $\sum'_{x:\, a\wedge x=\hat{0}} \mu(x,\hat{1}) = 0$. Dually if $a \gneq \hat{0}$, then $\sum'_{x:\, a\vee x=\hat{1}} \mu(\hat{0},x) = 0$.

proof: Compute in 2 ways

$$f_a\,\delta_{\hat{1}} = \Big(\sum_{b:\, b\le a}\delta_b\Big)\delta_{\hat{1}} = 0$$

since $b\le a \Rightarrow b\ne\hat{1}$

$$f_a\,\delta_{\hat{1}} = f_a\cdot\Big(\sum_{x:\, x\le\hat{1}}\mu(x,\hat{1})f_x\Big) = \sum_{x\in L}\mu(x,\hat{1})f_{a\wedge x}$$

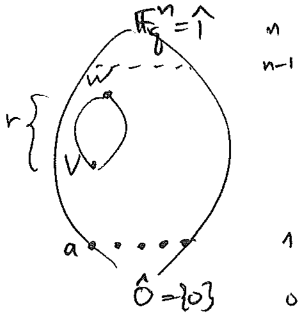
extract coeff. of $f_{\hat{0}}$

$$0 = \sum_{\substack{x\in L:\\ a\wedge x=\hat{0}}}\mu(x,\hat{1}) \quad \blacksquare$$

EXAMPLES:

① **PROP:** In $\mathcal{L}_n(q)$, $\mu(\hat{0},\hat{1}) = (-1)^n q^{\binom{n}{2}}$

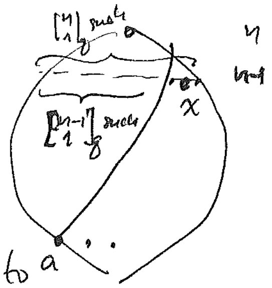
and hence $\mu(V,W) = (-1)^r q^{\binom{r}{2}}$ if $\dim(W/V)=r$



proof: Pick a line a, and then

$$0 = \sum_{x:\, a\vee x=\hat{1}}\mu(\hat{0},x)$$

$$\mu(\hat{0},\hat{1}) = -\sum_{\substack{x\lneq\hat{1}:\\ a\vee x=\hat{1}}}\mu(\hat{0},x)$$



forces x to have dim. $n-1$ since $\dim(\underbrace{x+a}_{x\vee a}) \le \dim(x)+\overbrace{\dim(a)}^{1} - \dim(\underbrace{x\cap a}_{x\wedge a}) \le \dim(x)+1$

$$= -\left(\begin{bmatrix}n\\1\end{bmatrix}_q - \begin{bmatrix}n-1\\1\end{bmatrix}_q\right)\cdot\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1})$$

$$= -\big((1+q+\dots+q^n)-(1+q+\dots+q^{n-1})\big)\cdot\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1})$$

$$= -q^{n-1}\mu_{\mathcal{L}_{n-1}(q)}(\hat{0},\hat{1}) \underset{\substack{\uparrow\\ \text{iterate}}}{=} (-1)^n q^{(n-1)+(n-2)+\dots+2+1+0}$$

$$= (-1)^n q^{\binom{n}{2}} \quad \blacksquare$$

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② This argument generalizes...

DEFIN: A ~~[illegible]~~ ranked lattice L is (upper-) semimodular if

$$r(x\vee y)+r(x\wedge y)\le r(x)+r(y)\quad \forall x,y\in L$$

e.g. (finitary) distributive lattices

$L_n(q)$

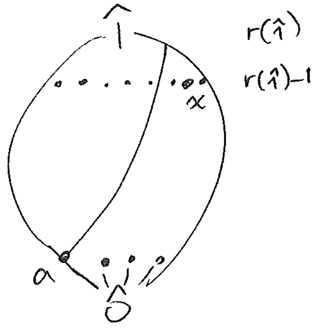
Π_n (EXERCISE)

PROP: L finite and upper-semimodular ⇒ $\mu(\cdot,\cdot)$ alternates in sign

i.e. $(-1)^{r(y)-r(x)}\mu(x,y)\ge 0$

proof: WLOG $x=\hat0$, $y=\hat1$ and pick any atom $a \gtrdot \hat0$

to apply Weisner, giving $0=\sum'_{\substack{x\in L:\\ x\vee a=\hat1}}\mu(\hat0,x)$



$$\mu(\hat0,\hat1)=-\sum'_{\substack{x\lneq\hat1:\\ x\vee a=\hat1}}\overbrace{\mu(\hat0,x)}^{\text{has sign } (-1)^{r(\hat1)-1} \text{ by induction}}$$

} ⇒ forces x to be of rank $r(\hat1)-1$

since $r(x\vee a)\le r(x)+r(a)-r(a\wedge x)$
$\le r(x)+1$.

$$\Rightarrow (-1)^{r(\hat1)} r(\hat0,\hat1)\ge 0 \quad ■$$

③ We could similarly use Weisner, but instead let's prove via Möbius inversion

PROP: In Π_n, $\sum_{\pi\in\Pi_n}\mu(\hat0,\pi)t^{\#\text{blocks}(\pi)} = t(t-1)(t-2)\cdots(t-(n-1))=\sum_{k=1}^{n}s(n,k)t^k$

Π_n ‖ set partition lattice

⇝ coeff. of t^{n-1}

$$\mu(\hat0,\hat1)=(-1)^{n-1}(n-1)!$$

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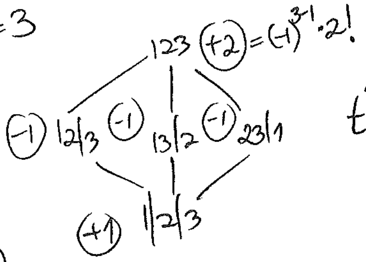
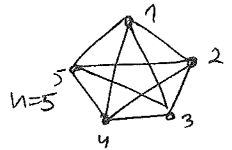
e.g. n=3



$$t^3-3t^2+2t^1 = t(t-1)(t-2)$$

$\mu(\hat0,\pi)$ circled

<u>proof</u>: It suffices to prove it for $t \in \{1,2,3,\ldots\}$ by computing two ways $\chi(K_n,t) = \#\{\underline{\text{proper}} \text{ vertex } t\text{-colorings of } K_n\}$



$$= \underset{\substack{\uparrow\\ \text{color 1}}}{t}\underset{\substack{\uparrow\\ \text{then}\\ \text{color 2}}}{(t-1)}(t-2)\cdots\underset{\substack{\uparrow\\ \text{etc.}}}{(t-(n-1))}$$

$$= \#\left\{\begin{array}{l}\text{vertex } t\text{-colorings } c \text{ of } K_n\\ \text{whose associated } \underline{\text{color}}\ \underline{\text{partition}}\ \pi(c) = \hat{0}\end{array}\right\}$$

If $f(\pi) = \#\left\{\begin{array}{c}\text{vertex } t\text{-colorings } c \text{ of } K_n\\ \text{having } \pi(c) = \pi\end{array}\right\}$.

~~then~~ $g(\pi) = \#\left\{\begin{array}{c}\text{———— '' ————}\\ \pi(c) \geq \pi\end{array}\right\} = \sum'_{\tau:\ \tau\geq\pi} f(\pi)$.

$= t^{\#\text{blocks}(\pi)}$

Hence $f(\pi) = \sum_{\sigma:\ \sigma\geq\pi} \mu(\pi,\sigma)\, g(\sigma)$

so $\chi(K_n,t) = f(\hat{0}) = \sum_{\sigma:\ \sigma\geq\hat{0}} \mu(\hat{0},\sigma)\, t^{\#\text{blocks}(\sigma)}$ ∎

<u>RMK</u>: This determines $\mu(\pi,\sigma)$ for all $\pi,\sigma\in\Pi_n$ as follows:

If σ has blocks $S_1,\ldots,S_\ell$, and π refines these into $n_1,\ldots,n_\ell$ blocks respectively,

then $[\pi,\sigma]_{\Pi_n} \cong \Pi_{n_1}\times\Pi_{n_1}\times\cdots\times\Pi_{n_\ell}$

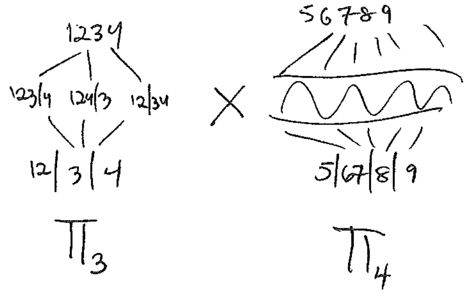
so $\mu_{\Pi_n}(\pi,\sigma) = (-1)^{n_1-1}(n_1-1)!\ \cdots\ (-1)^{n_\ell-1}(n_\ell-1)!$

e.g. $\sigma = 1234\|56789$

$\pi = 12|3|4\|5|67|8|9$

$n_1=3 \quad n_2=4$

$\Rightarrow [\pi,\sigma] \cong$



(111)

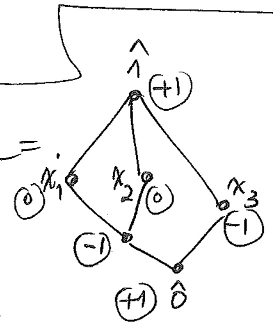
To deal with μ in $J(P)$ let's use ...

Rota's Crosscut Thm:
In a finite lattice L, with coatoms $\{x_1, --, x_\ell\}$
(= elts. $x \lessdot \hat{1}$)

$$\mu(\hat{0},\hat{1}) = \sum_{\substack{S \subset \{x_1,--,x_\ell\}: \\ \wedge S = \hat{0}}} (-1)^{|S|}$$

In particular, $\mu(\hat{0},\hat{1})$ if $\hat{0}$ is not a meet of coatoms or if $\hat{1}$ is not a join of atoms

e.g. $L =$



$\mu(\hat{0},y)$ circled

S	$(-1)^{\lvert S \rvert}$
$\{1,3\}$	$+1$
$\{2,3\}$	$+1$
$\{1,2,3\}$	-1
	$+1$

$+1 \overset{\shortparallel}{} \mu(\hat{0},\hat{1})$ ✓

proof: In $A(L,\mathbb{k})$, compute 2 ways

$$\prod_{i=1}^{\ell}(f_{\hat{1}} - f_{x_i})$$

$$= \sum_{S \subset \{x_1,--,x_\ell\}} (-1)^{|S|} \prod_{x_i \in S} f_{x_i} = \sum_{S \subset \{x_1,--,x_\ell\}} (-1)^{|S|} f_{\wedge S}$$

$$= \prod_{i=1}^{\ell}\Big(\sum_{y \not\leq x_i} \delta_y\Big) = \sum_{\substack{(y_1,--,y_\ell): \\ y_i \not\leq x_i}} \delta_{y_1}\delta_{y_2}\cdots\delta_{y_\ell} = \sum_{\substack{y \not\leq x_i \\ \forall i=1,--,\ell}} \delta_y = \delta_{\hat{1}}$$

$\uparrow$ every $y \lneq \hat{1}$ lies below some coatom x_i

$\Big\}$ extract coeff. of $f_{\hat{0}}$

THM. ▣

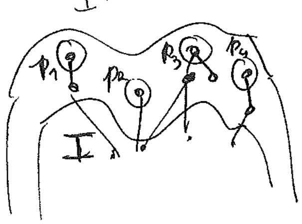
COR: In a finite distributive lattice $L = J(P)$,

$$\mu(I,I') = \begin{cases} (-1)^{|I' \setminus I|} & \text{if } I' \setminus I \text{ is an antichain in } P \\ 0 & \text{else} \end{cases}$$

proof: Check that the coatoms $x_1,--,x_\ell$ are $I' - \{p_i\}$ for maximal elts of $I' \setminus I$

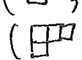
so their meet $x_1 \wedge \ldots \wedge x_\ell = I' \setminus \{p_1,--,p_\ell\} = I \iff$ every elt. of $I' \setminus I$ is maximal

i.e. $I' \setminus I$ is an antichain ▣



EXAMPLE: In Young's lattice $\mathbb{Y}$, $\mu(\lambda,\rho) = \begin{cases} (-1)^{|\rho/\lambda|} & \text{if } \rho/\lambda \text{ has no 2 cells in a row or column} \\ 0 \text{ else} \end{cases}$

e.g. $\mu($  ,  $) = 0$

$\mu($  ,  $) = (-1)^3$

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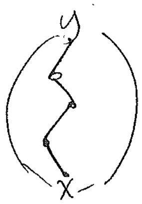
Möbius functions have a connection to topology via..

PROP (P. Hall's Thm)



$$\mu(x,y) = \sum_{\substack{\text{chains} \\ x=x_0<x_1<\cdots<x_\ell=y}} (-1)^\ell$$

proof: Call the RHS $\mu'(x,y)$, and let's check

$$\sum_{z:\, x\leq z\leq y} \mu'(x,y) \overset{?}{=} \begin{cases} 1 & \text{if } x=y \quad (\text{EASY} \checkmark) \\ 0 & \text{if } x<y \end{cases}$$

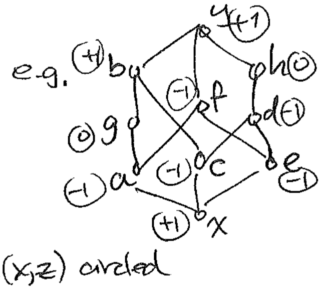
$$\Big\| $$

$$\sum_{(z,\ \underset{x}{\overset{\|}{x_0}}<x_1<\cdots<\underset{z}{\overset{\|}{x_\ell}})} (-1)^\ell = 0$$ via a sign-reversing involution that adds/removes y from the end of the chain as x_ℓ, depending on whether $x_\ell = z$ or not. ▨

Re-interpreting this, $x_1<x_2<\cdots<x_{\ell-1}$ is a chain in the open interval $(x,y) := \{z\in P : x<z<y\}$, which has its order complex $\Delta(x,y)$ ...

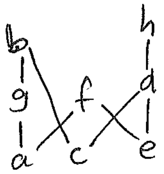
DEF'N: The order complex of a poset Q is the (abstract) simplicial complex $\Delta Q \subset 2^Q$ on vertex set Q with faces/simplices F = chains in Q (and $\dim F := |F|-1$)

e.g.

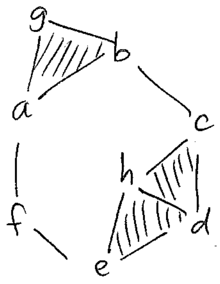


$\mu(x,z)$ circled

$\rightsquigarrow (x,y) =$



$\rightsquigarrow \Delta(x,y) =$



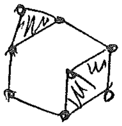
$= \{\emptyset, a,b,c,d,e,f,g, ab,bc,cd,\ldots,ag,bg,eh,dh,cd, abg, deh, cdh\}$

Then P. Hall's Thm says this:

PROP: $$\mu(x,y) = \sum_{\substack{\text{faces F of} \\ \Delta(x,y)}} (-1)^{\dim F} =: \tilde{\chi}(\Delta(x,y)) \overset{\substack{\text{Euler-Poincaré Thm} \\ \downarrow}}{=\!=} \sum_{i=-1}^{\infty} \dim_k \tilde{H}_i(\Delta(x,y), k)$$

$\tilde{\chi}$ ↑ (reduced) Euler characteristic

$\tilde{H}_i$ ↑ reduced homology groups

e.g. above $\mu(x,y) = \tilde{\chi}\Big($  $\Big) \underset{\uparrow}{=} \tilde{\chi}\Big($  $\underset{=\underline{\text{circle}}\, S^1}{}\Big) = (-1)^1$ since $\tilde{H}_i(S^1,k) = \begin{cases} 0 & i=-1 \\ 0 & i=0 \\ \Bbbk & i=1 \\ 0 & i\geq 2 \end{cases}$

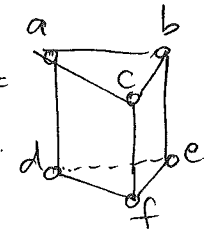
homotopy invariance of $\tilde{H}_i$ & $\tilde{\chi}$

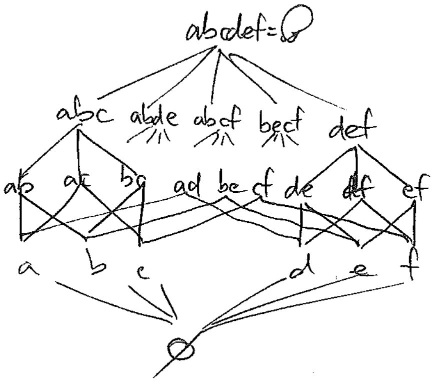
This helps compute $\mu(x,y)$ sometimes when the topology of $\Delta(x,y)$ is known...

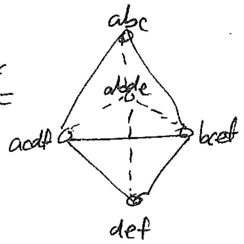
convex hull of finitely many pts in $\mathbb{R}^n$

<u>THM</u>: For a <u>convex</u> polytope $\mathcal{P}$ with poset of faces $\mathcal{F}(\mathcal{P})$

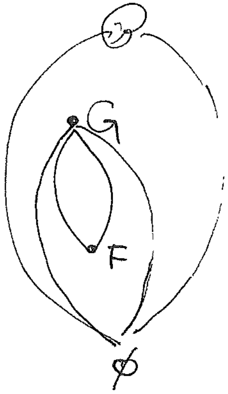
$$\mu(F,G) = (-1)^{\dim G - \dim F}$$

e.g. $\mathcal{P}=$ 

$\rightsquigarrow \mathcal{F}(\mathcal{P})=$ 

$\mathcal{P}^*=$ 

<u>proof sketch</u>: WLOG $G=\mathcal{P}$ since each face G is itself a convex polytope.



WLOG $F=\emptyset$, since $\mathcal{F}(\mathcal{P})^{\text{opp}} = \mathcal{F}(\underbrace{\mathcal{P}^*}_{})$

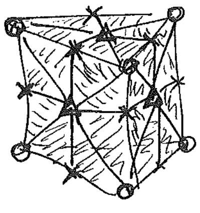
(polar) dual polytope in $(\mathbb{R}^n)^* \cong \mathbb{R}^n$
$= \{v\in\mathbb{R}^n : \langle v,p\rangle \leq 1 \ \forall p\in\mathcal{P}\}$

so $\mu(F,G) = \mu(G^*,F^*)$

$\mu(F,\mathcal{P}) = \mu(\emptyset,F^*)$

Then for $\mu(\emptyset,\mathcal{P}) = \tilde{\chi}(\underbrace{\Delta(\emptyset,\mathcal{P})}_{})$

isomorphic to the barycentric subdivision of the boundary of $\mathcal{P}$



$= \tilde{\chi}(S^{\dim(\mathcal{P})-1})$

$= (-1)^{\dim\mathcal{P}-1}$

$= (-1)^{\dim\mathcal{P}-\dim\emptyset}$ since $\dim\emptyset=-1$ $\blacksquare$

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DEF'N: A poset P which is ranked and has $\mu(x,y) = (-1)^{r(y)-r(x)}$ $\forall x,y \in P$ is called Eulerian

EXAMPLES

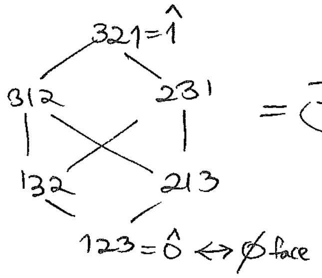
① Face lattices $\mathcal{F}(P)$ for convex polytopes are Eulerian

② THM (Verma) (Strong) Bruhat order on $\mathfrak{S}_n$ is Eulerian

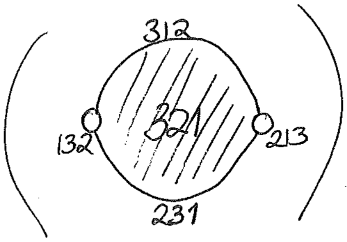
Not obvious!

In fact, each interval $[x,y]$ in Bruhat order on $\mathfrak{S}_n$ turns out to be the face poset $\mathcal{F}(X)$ for a certain regular cellular ~~sphere~~ ball $\mathbb{B}^{r(x)-r(y)-1}$ (Björner-Wachs)

e.g. $\mathfrak{S}_3 =$

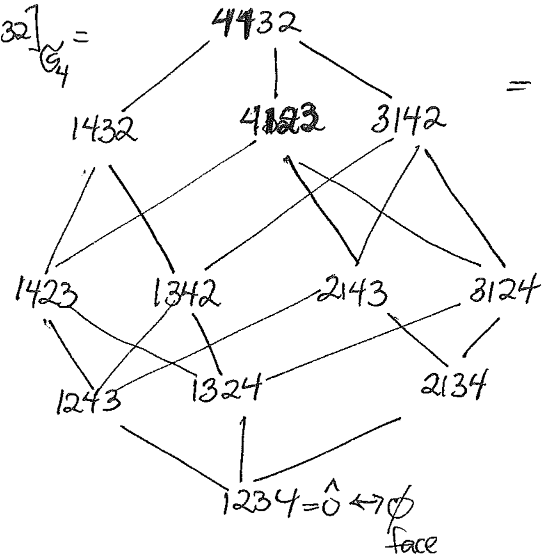


$= \mathcal{F}\Big($

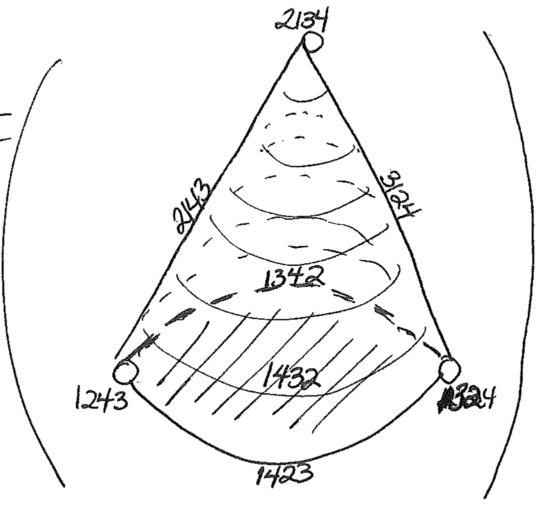


$\Big)$

$[1234, 4432]_{\mathfrak{S}_4} =$



$= \mathcal{F}\Big($



$\Big)$

front face ↔ 4123
back face ↔ 3142
interior ↔ 4132

Wish list of things I'd have liked to do this semester:

- Hypergeometric notation & identities

    + some q-series

- Rook theory

- Sperner theory of posets

    e.g. • Dilworth's Thm.
    Mirsky's ("dual Dilworth") Thm.
    + Greene-Kleitman partition of a poset

    - Sperner's Thm.
    - LYM inequality
    - Peek posets & symmetric chain decompositions
    - Unimodality

- Finite (Atomic distributive lattices) = (Boolean algebras $\mathcal{B}_n$)
  (—"— modular —"—) = (Vector space lattices $\mathcal{L}_n(q)$
  + projective planes in rank 3)
  (—"— upper semimodular —"—) = (lattices of flats of matroids
  or geometric lattices)