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Reflection decompositions in the classical Weyl groups

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Abstract

In this paper, we present formulas for the number of decompositions of elements of the Weyl groups of type A_n, D_n and B_n as products of a number of reflections that is not necessarily minimal. For this purpose, we consider the poset of conjugacy classes of W introduced in Bédard and Goupil (1992) for the symmetric group. This poset describes the action of the set of reflections of a reflection group on its conjugacy classes. In particular, we show how the reflection decompositions in the symmetric group $\mathfrak{S}_n$ are related to the reflection decompositions in D_n.

1. Introduction

For each element x of a Coxeter group W, there holds a presentation

$$x=\rho_1 \cdots \rho_{r(x,W)} \tag{1.1}$$

of x as a product of reflections $\rho_1, \ldots, \rho_{r(x,W)}$ belonging to W where $r(x, W)$ is minimal for given $x \in W$. The elements in a given conjugacy class C_i of W always have the same number of reflections $r(C_i)$ in their minimal decomposition (1) and the number of these minimal decompositions, denoted $p(C_i)$, also depends only on the conjugacy class C_i.

Let R be the formal sum of all the reflections of W which lies in the center $\mathscr{C}[W]$ of the group algebra of W over the real number field $\mathbb{R}$. The number $p(C_i)$ is the coefficient of the conjugacy class C_i in the $r(C_i)$th power of R in $\mathscr{C}[W]$:

$$p(C_i)=R^{r(C_i)}|_{C_i}, \tag{1.2}$$

In the symmetric group $\mathfrak{S}_n$, the numbers $p(C_i)$ are essentially obtained from the classical result of Dénes [3] stating that the number of minimal decompositions of

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a k-cycle is equal to the number k^{k-2} of trees on k vertices. The purpose of this paper is first to give a description of the coefficients $p(C_i)$ for the Weyl groups of type B_n and D_n and then to relate these coefficients to the enumeration of the decompositions of elements of $\mathfrak{S}_n$ into more than the minimum number of transpositions. In Section 2, we describe the poset $\mathfrak{T}(W)$ of conjugacy classes of W and recall the formulas for the numbers $p(C_i)$ in the symmetric group. In Sections 3 and 4, we give the formulas for the numbers $p(C_i)$ in the groups D_n and B_n respectively and in Section 5, we show how we can use our results for the group D_n to enumerate the number of decompositions of elements σ of $\mathfrak{S}_n$ with cycle type $\lambda=(k_1,k_2)$ into more than the minimum number $r(\sigma,\mathfrak{S}_n)$ of transpositions.

2. Definitions

Let us first recall some facts about the Weyl groups of type A_n, B_n and D_n which will be used throughout this text. The conjugacy classes $C(\mu,\mathfrak{S}_n)$ of the symmetric group S_n are indexed by partitions μ of n written $\mu\vdash n$. The number n is called the *weight* of the partition μ and we write $|\mu|=n$. We write $\mu=1^{\alpha_1}2^{\alpha_2}\cdots n^{\alpha_n}$ to mean that a permutation $x\in C(\mu,\mathfrak{S}_n)$ contains α_i cycles of length i. If $\lambda=1^{\beta_1}2^{\beta_2}\cdots n^{\beta_n}$ is another partition of n, then the sum $\lambda+\beta$ is defined as $\lambda+\beta=1^{\alpha_1+\beta_1}2^{\alpha_2+\beta_2}\cdots n^{\alpha_2+\beta_n}$.

The hyperoctahedral group B_n can be seen as the group of signed permutation matrices of order n. It contains the group D_n as a subgroup of index 2. The group D_n consists of the signed permutations matrices of order n with an even number of entries equal to -1.

In similarity with the symmetric group, it is convenient to represent the elements σ of B_n by a sequence of n integers $\sigma_1\sigma_2\cdots\sigma_n$ with $-n\leqslant\sigma_i\leqslant n$ for all $i=1,\ldots,n$. The integer σ_i can be seen as the image of i under σ. Two k-cycles in B_n are conjugate iff the number of -1 in their matrix or row representation have the same parity. For example the two 5-cycles $(3,4,2,-5,1)$ and $(-5,3,-4,1,-2)$ are in the same conjugacy class of B_n. We may therefore call the cycles and the conjugacy classes of B_n *positive* or *negative* according to their even or odd number of negative elements in their row or matrix representation and the conjugacy classes of B_n are indexed by pairs of partitions (λ,μ) such that $|\lambda|+|\mu|=n$ where the partition λ describes the distribution of positive cycles and μ gives the distribution of the negative cycles. The set of reflections of B_n contains two conjugacy classes noted R_1 and R_2. The class R_1 contains the positive reflections $\rho_{i,j}$ and $\rho_{-i,-j}$ and the class R_2 contains the negative reflections ρ_{-i}. These reflections are defined by the following relations:

$$\rho_{i,j}(b_i-b_j)=-b_i+b_j,\qquad \rho_{i,j}(b_k)=b_k\qquad k\neq i,j \tag{2.1a}$$

$$\rho_{-i,-j}(b_i+b_j)=-b_i-b_j,\qquad \rho_{-i,-j}(b_k)=b_k,\qquad k\neq i,j, \tag{2.1b}$$

$$\rho_{-i}(b_i)=-b_i,\qquad \rho_{-i}(b_k)=b_k\qquad k\neq i. \tag{2.1c}$$

where $\{b_1, b_2, \cdots, b_n\}$ is the standard basis of $\mathbb{R}^n$. The cardinality of a conjugacy class $C(\lambda, \mu; B_n)$ of B_n is given by the formula

$$|C(\lambda, \mu; B_n)| = |C(\lambda+\mu; \mathfrak{S}_n)| \prod_{i=1}^{n} \binom{\alpha_i+\beta_i}{\alpha_i} 2^{r(\lambda+\mu;\, \mathfrak{S}_n)}. \tag{2.2}$$

The conjugacy classes of D_n are precisely the positive conjugacy classes of B_n except when the cycles are all of even length (odd permutations) and positive ($\mu = \emptyset$); in which case they split into two conjugacy classes of equal size. Since it is irrelevant to our discussion to distinguish these two conjugacy classes of D_n, we will write $C(\lambda, \emptyset; D_n)$ for the disjoint union of these two conjugacy classes. The product $R * C_i$ in the center $\mathscr{C}[W]$ of the set of reflections R with any conjugacy class of W decomposes as a linear combination of conjugacy classes. We describe in the next proposition this decomposition for the groups $\mathfrak{S}_n$, B_n and D_n.

Proposition 2.1. *Let $\mu = 1^{\alpha_1} \cdots n^{\alpha_n}$ and $\lambda = 1^{\beta_1} \cdots n^{\beta_n}$ be partitions such that $|\mu| + |\lambda| = n$. The coefficients in the decomposition of the product $R * C(\lambda, \mu; W)$ of the set of reflections with any other conjugacy class of W in the basis of conjugacy classes are given in Fig. 1 for the symmetric group $\mathfrak{S}_n$ and in Fig. 2 for the groups B_n and D_n. In both figures, the weight of an oriented edge going from a class $C(\lambda, \mu; W)$ to the class $C(\lambda^*, \mu^*; W)$ represents the coefficient of the class $C(\lambda^*. \mu^*; W)$ in the product $R * C(\lambda, \mu; W)$. In Fig. 1, since we need only one partition for the symmetric group, we set $\mu = \mu^* = \emptyset$.*

Proof. These coefficients are obtained by using simple combinatorial arguments that are developed in a case-by-case study. We omit the details. $\square$

In Fig. 2, the superscripts in the β's and α's give the multiplicities of positive and negative cycles respectively. Thus in both figures, cases (a) and (b) involve conjugacy classes with only positive cycles. Case (c) describes the action of joining a positive cycle and a negative cycle to form a bigger negative cycle by using a positive reflection. Case (d) is the only case involving the action of a negative reflection, it transforms a positive cycle into a negative cycle of the same length. Thus if we want to describe the action of reflections in the group D_n, we only have to discard case (d) in Fig. 2. The action of the set of reflections in the symmetric group given by Fig. 1 first appeared in [6]. It is similar to cases (a) and (b) of Fig. 2 for only upward weights differ by a factor 2.

Observe that for any reflection group W we can construct a ranked poset $\mathfrak{T}(W)$ of conjugacy classes using the action by multiplication of the set of reflections in the following way. The *rank* of a conjugacy class C_i is the minimal number $r(C_i)$ of reflections in the decomposition of an element $\sigma \in C_i$ as a product of reflections. The partial order on the conjugacy classes is given by the condition

$$C_j \textit{ covers } C_i \Leftrightarrow r(C_j) = r(C_i) + 1 \quad \textit{and} \quad \forall \sigma \in C_j, \quad \exists \beta \in C_i, \quad \rho \in R$$

$$\text{such that } \sigma = \beta\rho. \tag{2.3}$$

$$\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots j^{\beta_j}\dots (i+j)^{\beta_{i+j}}\dots n^{\beta_n}\\
(i\neq j)\quad ij\beta_i\beta_j \qquad \uparrow\downarrow \qquad (i+j)\beta_{i+j}+1\\
1^{\beta_1}\dots i^{\beta_i-1}\dots j^{\beta_i-1}\dots (i+j)^{\beta_{i+j}+1}\dots n^{\beta_n}\\
\text{(a)}
\end{array}
\qquad
\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots (2i)^{\beta_{2i}}\dots n^{\beta_n}\\
i^2\binom{\beta_i}{2} \qquad \uparrow\downarrow \qquad i(\beta_{2i}+1)\\
1^{\beta_1}\dots i^{\beta_i-2}\dots (2i)^{\beta_{2i}+1}\dots n^{\beta_n}\\
\text{(b)}
\end{array}$$

Fig. 1. Action by multiplication of R on conjugacy classes of $\mathfrak{S}_n$.

$$\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots j^{\beta_j}\dots (i+j)^{\beta_{i+j}}\dots n^{\beta_n}\\
(i\neq j)\quad 2ij\beta_i\beta_j \qquad \uparrow\downarrow \qquad (i+j)\beta_{i+j}+1\\
1^{\beta_1}\dots i^{\beta_i-1}\dots j^{\beta_j-1}\dots (i+j)^{\beta_{i+j}+1}\dots n^{\beta_n}\\
\text{(a)}
\end{array}
\qquad
\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots (2i)^{\beta_{2i}}\dots n^{\beta_n}\\
2i^2\binom{\beta_i}{2} \qquad \uparrow\downarrow \qquad i(\beta_{2i}+1)\\
1^{\beta_1}\dots i^{\beta_i-2}\dots (2i)^{\beta_{2i}+1}\dots n^{\beta_n}\\
\text{(b)}
\end{array}$$

$$\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots -j^{\alpha_j}\dots -(i+j)^{\alpha_{i+j}}\dots n^{\beta_n}\\
2ij\beta_i\beta_j \qquad \uparrow\downarrow \quad (i+j)(\alpha_{i+j}+1)\\
1^{\beta_1}\dots i^{\beta_i-1}\dots -j^{\alpha_j-1}\dots -(i+j)^{\alpha_{i+j}+1}\dots n^{\beta_n}\\
\text{(c)}
\end{array}
\qquad
\begin{array}{c}
1^{\beta_1}\dots i^{\beta_i}\dots -i^{\alpha_i}\dots n^{\beta_n}\\
i\beta_i \qquad \uparrow\downarrow \qquad i(\alpha_i+1)\\
1^{\beta_1}\dots i^{\beta_i-2}\dots -i^{\alpha_i+1}\dots n^{\beta_n}\\
\text{(d)}
\end{array}$$

$$\begin{array}{c}
1^{\beta_1}\dots -i^{\alpha_i}\dots -j^{\alpha_j}\dots (i+j)^{\beta_{i+j}}\dots n^{\beta_n}\\
(i\neq j)\quad (i+j)\beta_{i+j} \qquad \uparrow\downarrow \qquad 2ij(\alpha_i+1)(\alpha_j+1)\\
1^{\beta_1}\dots -i^{\alpha_i+1}\dots -j^{\alpha_j+1}\dots -(i+j)^{\beta_{i+j}-1}\dots n^{\beta_n}\\
\text{(e)}
\end{array}
\qquad
\begin{array}{c}
1^{\beta_1}\dots -i^{\alpha_i}\dots (2i)^{\beta_{2i}}\dots n^{\beta_n}\\
i\beta_{2i} \qquad \uparrow\downarrow \qquad 2i^2\binom{\alpha_i+2}{2}\\
1^{\beta_1}\dots -i^{\alpha_i+2}\dots (2i)^{\beta_{2i}-1}\dots n^{\beta_n}\\
\text{(f)}
\end{array}$$

Fig. 2. Action by multiplication of R on conjugacy classes of type (λ,μ) in B_n and D_n.

We give a weight to each oriented edge (C_i, C_j) of $\mathfrak{T}(W)$ equal to the coefficient $C_i * R|_{C_j}$ when C_i covers C_j or C_j covers C_i and the weight of a walk from the identity class to the class C_i is defined as the product of the weights of each one of its oriented edge. Thus the *weight* $p(C_i)$ of a conjugacy class C_i described in Eq. (1.2) is the sum of all weighted chains in $\mathfrak{T}(W)$ from the identity class to C_i. In the case of the symmetric group, the poset $\mathfrak{T}(\mathfrak{S}_n)$ is isomorphic to the poset of partitions ordered by refinement [2] and it has been used and described in more details in [9, 1].

2.1. The symmetric group

In $\mathfrak{S}_n$, it is well known [3] that the number $p(1^{n-k}k;\mathfrak{S}_n)$ of decompositions of a k-cycle as product of a minimum number of transpositions is k^{k-2}. If a permutation consists of two disjoint cycles of length k_1 and k_2, the number of its decompositions is given by

$$p(1^{n-k_1-k_2}k_1k_2,\mathfrak{S}_n)=k_1^{k_1-2}\binom{k_1+k_2-2}{k_1-1} \tag{2.4}$$

because building the two disjoint cycles is done with two independent sets of transpositions of cardinality k_1-1 and k_2-1. The product of a transposition from one set with a transposition from the other set commutes. So any two minimal sequences of transpositions, one giving a k_1-cycle and the other a k_2-cycle, can be *shuffled* without changing the result. The number of shuffles is counted by the binomial coefficient in (2.4). This *shuffle principle* can be used to obtain an expression for $p(\mu,\mathfrak{S}_n)$ where μ is any partition of n [1] and it will also be used throughout the text for the groups B_n and D_n.

3. The group D_n

We first consider the case where the conjugacy class $C(\lambda,\mu;D_n)$ contains no negative cycle, i.e. $\mu=\emptyset$.

Proposition 3.1. *For any partitions $\lambda\vdash k$ and $\mu\vdash n-k$, we have*

$$p(\lambda,\emptyset;D_k)=p(\lambda,\emptyset;B_k)=p(\lambda;\mathfrak{S}_k), \tag{3.1}$$

$$p(\lambda,\mu;D_n)=p(\lambda,\mathfrak{S}_n)p(\emptyset,\mu;D_n)\binom{r(\lambda,\mu;D_n)}{r(\lambda,\mathfrak{S}_n)}, \tag{3.2}$$

$$p(\lambda,\mu;B_n)=p(\lambda,\mathfrak{S}_n)p(\emptyset,\mu;B_n)\binom{r(\lambda,\mu;B_n)}{r(\lambda,\mathfrak{S}_n)}. \tag{3.3}$$

Proof. Identity (3.1) is a straightforward consequence of the comparison of cases (a) and (b) in Figs. 1 and 2. Identities (3.2) and (3.3) are obtained by using the shuffle principle on λ and μ. □

In order to obtain a complete picture of reflection decompositions, the remaining cases to be studied are the ones with conjugacy classes of D_n and B_n that have no positive k-cycle with $k>1$ and the generic situation is the conjugacy class $C((1^{n-k_1-k_2}),(k_1,k_2);D_n)$ containing two disjoint negative cycles of length k_1,k_2. We begin our study with a technical result.

Lemma 3.1. *Let k_1 and k_2 be two positive integers, then we have*

$$\sum_{i=1}^{k_1-1} \frac{\binom{k_1+k_2-1}{i-1}\binom{k_1+k_2-i}{k_2} i^{i-2}(k_1-i)^{k_1-i+1}}{k_1+k_2-i}$$

$$=\frac{\binom{k_1+k_2-1}{k_2}k_1^{k_1}-\sum_{j=0}^{k_1-1}\binom{k_1+k_2-1}{j}k_1^j k_2^{k_1-j+1}}{k_1+k_2}. \tag{3.4}$$

Proof. (*Gessel* [4]). We transform the sum on the left-hand side of (3.4) as follows:

$$\sum_{i=1}^{k_1-1} \frac{\binom{k_1+k_2-1}{i-1}\binom{k_1+k_2-i}{k_2} i^{i-2}(k_1-i)^{k_1-i+1}}{k_1+k_2-i}$$

$$=\frac{1}{k_1+k_2}\sum_{i=1}^{k_1}\binom{k_1+k_2}{i}\binom{k_1+k_2-i-1}{k_2} i^{i-1}(k_1-i)^{k_1-i}$$

$$=\frac{1}{k_1+k_2}\sum_{i=0}^{k_1-1}\binom{k_1+k_2}{k_1-i}\binom{k_2+i-1}{k_2} i^{i}\sum_{j=0}^{k_1-i-1}\binom{k_1-i-1}{j}k_1^j(-i)^{k_1-i-j-1}$$

$$=\frac{1}{k_1+k_2}\sum_{j=0}^{k_1-1}(-k_1)^j\sum_{i=0}^{k_1-j-1}(-1)^{k_1-i-1}\binom{k_1+k_2}{k_2+i}\binom{k_2+i-1}{k_2}$$

$$\times\binom{k_1-i-1}{j}(i)^{k_1-j-1}$$

$$=\frac{1}{k_1+k_2}\sum_{j=0}^{k_1-1}(-k_1)^j\sum_{i=k_2}^{k_1+k_2-j-1}(-1)^{k_1+k_2-i-1}\binom{k_1+k_2}{i}$$

$$\times\binom{i-1}{k_2}\binom{k_1+k_2-i-1}{j}(i-k_2)^{k_1-j-1}. \tag{3.5}$$

Now the expression in identity (3.5)

$$\binom{i-1}{k_2}\binom{k_1+k_2-i-1}{j}(i-k_2)^{k_1-j-1}$$

is a polynomial in i of degree k_1+k_2-1. So if the second sum in the right-hand side of (3.5) were from $i=0$ to k_1+k_2 instead of from $i=k_2$ to k_1+k_2-j-1 it would be equal to zero. But the only nonzero terms added when the sum is taken from $i=0$ to k_1+k_2 occur when $i=0$ and $i=k_1+k_2$. These terms are respectively

$$(-1)^j\binom{k_1+k_2-1}{j}k_2^{k_1-j-1},\ (-1)^{j-1}\binom{k_1+k_2-1}{k_2}k_1^{k_1-j-1}.$$

Thus the value of the sum $\sum_{i=k_2}^{k_1+k_2-j-1}$ in (3.5) must be the negative of the sum of these two terms and the left-hand side of (3.4) becomes

$$\sum_{i=1}^{k_1-1} \frac{\binom{k_1+k_2-1}{i-1}\binom{k_1+k_2-i}{k_2} i^{i-2}(k_1-i)^{k_1-i+1}}{k_1+k_2-i}$$

$$=\frac{1}{k_1+k_2}\sum_{j=0}^{k_1-1}(-k_1)^j\left[(-1)^{j+1}\binom{k_1+k_2-1}{j}k_2^{k_1-j-1}+(-1)^j \times\binom{k_1+k_2-1}{k_2}k_1^{k_1-j-1}\right]$$

$$=\frac{1}{k_1+k_2}\sum_{j=0}^{k_1-1}\left[-\binom{k_1+k_2-1}{j}k_1^j k_2^{k_1-j-1}+\binom{k_1+k_2-1}{k_2}k_1^{k_1-1}\right],$$

which is equal to the right-hand side of (3.4) $\square$

Theorem 3.1. *Let $\mu=(k_1,k_2)$ be a partition with two nonzero parts, then we have*

$$p(1^{n-k_1-k_2},\mu;D_n)=\frac{2k_1^{k_1+1}k_2^{k_2+1}\binom{k_1+k_2}{k_1}}{k_1+k_2}. \tag{3.6}$$

Proof (*By induction on the rank of μ*). The action of reflections by multiplication described in Fig. 2 and induction hypothesis on the length of the cycles provides the following recurrence:

$$\begin{aligned} &p(1^{n-k_1-k_2},\mu;D_n)\\ &=2k_1k_2(k_1+k_2)^{k_1+k_2-2}+k_1\sum_{i=1}^{k_1-1}p((1^{n-k_1-k_2}i),(k_1-i),k_2);D_n)\\ &\quad+k_2\sum_{j=1}^{k_2-1}p((1^{n-k_1-k_2}j),(k_1,k_2-j);D_n)\\ &=2k_1k_2(k_1+k_2)^{k_1+k_2-2}+2k_1\sum_{i=1}^{k_1-1}\frac{\binom{k_1+k_2-1}{i-1}\binom{k_1+k_2-i}{k_2}i^{i-2}(k_1-i)^{k_1-i+1}k_1^{k_2+1}}{k_1+k_2-i}\\ &\quad+2k_2\sum_{j=1}^{k_2-1}\frac{\binom{k_1+k_2-1}{j-1}\binom{k_1+k_2-j}{k_1}j^{j-2}(k_2-j)^{k_2-j+1}k_2^{k_1+1}}{k_1+k_2-j}. \end{aligned} \tag{3.7}$$

Using Lemma 3.1 and the binomial theorem, we transform the second term in the right-hand side of (3.7) into

$$\frac{2k_1k_2}{k_1+k_2}\left[k_1^{k_1}k_2^{k_2}\binom{k_1+k_2-1}{k_2}-\sum_{j=0}^{k_1-1}\binom{k_1+k_2-1}{j}k_1^j k_2^{k_1+k_2-j-1}\right]. \tag{3.8}$$

Exchanging k_1 and k_2 we get for the third term of the right-hand side of (3.7):

$$\frac{2k_1k_2}{k_1+k_2}\left[k_1^{k_1}k_2^{k_2}\binom{k_1+k_2-1}{k_1}-\sum_{j=0}^{k_2-1}\binom{k_1+k_2-1}{j}k_2^jk_1^{k_1+k_2-j-1}\right]$$

$$=\frac{2k_1k_2}{k_1+k_2}\left[k_1^{k_1}k_2^{k_2}\binom{k_1+k_2-1}{k_1}-\sum_{j=k_1}^{k_1+k_2-1}\binom{k_1+k_2-1}{j}k_1^jk_2^{k_1+k_2-j-1}\right]. \quad (3.9)$$

Then adding the expressions in (3.8) and (3.9) we obtain

$$\frac{2k_1k_2}{k_1+k_2}\left[k_1^{k_1}k_2^{k_2}\left[\binom{k_1+k_2-1}{k_1}+\binom{k_1+k_2-1}{k_2}\right]\right.$$

$$\left.-\sum_{j=0}^{k_1+k_2-1}\binom{k_1+k_2-1}{j}k_1^jk_2^{k_1+k_2-j-1}\right]$$

$$=\frac{2k_1k_2}{k_1+k_2}\left[k_1^{k_1}k_2^{k_2}\binom{k_1+k_2}{k_1}-(k_1+k_2)^{k_1+k_2-1}\right]. \quad (3.10)$$

Finally we add the first term of the right-hand side of (3.7) to the right-hand side of (3.10) and the proof is complete. □

Next we show how the weight of other conjugacy classes in D_n are obtained from the one given in Theorem 3.1.

Theorem 3.2. *Let $\mu=(k_1,k_2,\cdots,k_{2r})\vdash k$ be a partition with an even number of parts, then we have*

$$p(1^{n-k},\mu;D_n)$$

$$=\sum_{i=2}^{2r}p(1^{n-k_1-k_i},(k_1,k_i);D_n)p(1^{n-k+k_1+k_i},(k_2,\cdots,k_i-1,k_i+1,\cdots,k_{2r});D_n)$$

$$\times\binom{k}{k_1+k_i}.$$

Proof. The proof is straightforward when we observe that the negative cycles of D_n are constructed by pairs using rules (e) and (f) in Fig. 2 with the shuffle principle. □

Observe that we can rewrite theorem 3.2 in the following form:

$$p(1^{n-k},\mu;D_n)=2^r\prod_{i=1}^{2r}k_i^{k_i+1}\binom{k}{k_1,\cdots,k_{2r}}\sum_{\mu'}\frac{1}{(k_{i_1}+k_{i_2})\cdots(k_{i_{2r-1}}+k_{i_{2r}})}, \quad (3.11)$$

where the sum is over all partitions μ' in r parts, of two elements each, of the set $\{k_1,k_2,\cdots,k_{2r}\}$. For example if $\mu=(k_1,k_2,k_3,k_4)$ have four parts, the three partitions μ' would be $\{(k_1,k_2),(k_3,k_4)\}$, $\{(k_1,k_3),(k_2,k_4)\}$, $\{(k_1,k_4),(k_2,k_3)\}$ and we would have

$$p(1^{n-k},(k_1,k_2,k_3,k_4);D_n)$$
$$=4\prod_{i=1}^{4}k_i^{k_i+1}\binom{k}{k_1,k_2,k_3,k_4}$$
$$\times\left[\frac{1}{(k_1+k_2)(k_3+k_4)}+\frac{1}{(k_1+k_3)(k_2+k_4)}+\frac{1}{(k_1+k_4)(k_2+k_3)}\right] \tag{3.12}$$

and if $\mu=1^{2r}$ consists of only fixed points, we obtain

$$p(1^{n-2r},1^{2r};D_n)=\frac{[(2r)!]}{r!2^r}. \tag{3.13}$$

Observation. Proving Theorem 3.1 is equivalent to proving the interesting new identity

$$(k_1+k_2)^{k_1+k_2}$$
$$=k_2^{k_2+1}\binom{k_1+k_2}{k_1}\sum_{i=1}^{k_1}\frac{i^i(k_1-i)^{k_1-i}\binom{k_1}{i}}{k_1+k_2-i}+k_1^{k_1+1}\binom{k_1+k_2}{k_2}$$
$$\times\sum_{j=1}^{k_2}\frac{j^j(k_2-j)^{k_2-j}\binom{k_2}{j}}{k_1+k_2-j} \tag{3.14}$$

for which a combinatorial proof would be interesting. A proof of (3.14) was presented to the author by Gessel [4] who essentially proved Lemma 3.1 and we have reproduced his argument here. The link between Eq. (3.14) and Theorem 3.1 is the following identity:

$$k_1^{k_1}k_2^{k_2}\binom{k_1+k_2}{k_1}=k_2^{k_2}\binom{k_1+k_2-1}{k_1}\sum_{i=1}^{k_1}i^{i-1}(k_1-i)^{k_1-i}\binom{k_1}{i}$$
$$+k_1^{k_1}\binom{k_1+k_2-1}{k_1}\sum_{j=1}^{k_2}j^{j-1}(k_2-j)^{k_2-j}\binom{k_j}{j}, \tag{3.15}$$

which is obtained using the famous Abel identity (see for instance [7]).

4. The group B_n

To decompose the elements of B_n, we first establish the following fact.

Proposition 4.1. *For any positive integer $k<n$, we have*

$$p((1^{n-k}),(k);B_n)=k^k. \tag{4.1}$$

Proof. We use the fact that in the symmetric group, the expression

$$|C(1^{n-k}k;\mathfrak{S}_n)|p(1^{n-k}k;\mathfrak{S}_n)=\binom{n}{k}(k-1)!k^{k-2} \tag{4.2}$$

gives the number of ways of choosing k elements in a set of n elements and then build labelled trees on these k vertices with the $k-1$ edges labelled. It is possible to extend this interpretation to B_n and we see that we have

$$|C((1^{n-k},(k);B_n)|p((1^{n-k},(k);B_n)=\binom{n}{k}(k-1)!k^{k-2}k^2 2^{k-1} \tag{4.3}$$

because each sequence of $k-1$ transpositions counted by (4.2) can be transformed into a negative k-cycle by adding one negative reflection chosen among k anywhere in the k positions of the sequence. Then each transposition in the sequence can be replaced by any of two positive reflections of B_n. Using (2.2), we divide (4.3) by $|C((1^{n-k},(k);B_n)|$ to obtain the right-hand side of (4.1). □

Proposition 4.2. *Let $\mu=k_1,k_2,\cdots,k_r$ be a partition with $|\mu|=k$ and let $[r]$ be the set $\{1,2,\cdots,r\}$, then we have*

$$p(1^{n-k},\mu;B_n)=\sum_{\substack{S\subseteq[r]\\|S|=2t}} p(1^{k_s},k_{s_1},k_{s_2},\cdots,k_{s_{2t}};D_n)\prod_{i\in[r]\setminus S}k_i^{k_i}\binom{k}{k_s,k_{s_{2t+1}},\cdots,k_{s_r}}, \tag{4.4}$$

where the sum is over all subsets $S=\{s_1,s_2,\cdots,s_{2t}\}$ of $[r]$ with $\sum_{i=1}^{2t}k_{s_i}=k_s$ and $[r]\setminus S=\{s_{2t+1},\cdots,s_r\}$.

Proof. Proposition 4.2 says that in order to build a set of negative k_i-cycles in B_n with $i\in[r]$, we partition the set $[r]$ in two subsets S and $[r]\setminus S$. In the subset S, we build the cycles using only positive reflections, i.e. we build the cycles in D_n. In the other part, we use a negative reflection to build each k_i*-cycle* in $k_i^{k_i}$ ways. Then we shuffle the two sequences of reflections with the appropriate multinomial coefficient. □

For example if $\mu=(k_1,k_2)$, we obtain

$$p(1^{n-k},\mu;B_n)=k_1^{k_1}k_2^{k_2}\binom{k_1+k_2}{k_1}\left[\frac{2k_1k_2}{k_1+k_2}+1\right] \tag{4.5}$$

and if $\mu=k_1,k_2,k_3$ we have

$$p(1^{n-k},\mu;B_n)=k_1^{k_1}k_2^{k_2}k_3^{k_3}\binom{k}{k_1,k_2,k_3}\left[2\left[\frac{k_1k_2}{k_1+k_2}+\frac{k_1k_3}{k_1+k_3}+\frac{k_2k_3}{k_2+k_3}\right]+1\right]. \tag{4.6}$$

5. Applications

We now use Theorem 3.1 to count the number of factorizations of elements of $\mathfrak{S}_n$ as products of more than the minimum number of transpositions.

Theorem 5.1. *Let $R(n)$ be the conjugacy class of transpositions of S_n. We have*

$$\begin{aligned} & p(\emptyset,(k_1,k_2);D_n)|C(\emptyset,(k_1,k_2);D_n)| \\ &\quad = 2^{n-1}|C(k_1,k_2;\mathfrak{S}_n)|\Big[R(n)^n|_{C(k_1,k_2,\mathfrak{S}_n)} \\ &\qquad - R(k_1)^{k_1+1}|_{C(k_1;\mathfrak{S}_{k_1})}p(1^{n-k_2}k_2;S_n)\binom{n}{k_1+1} \\ &\qquad - R(k_2)^{k_2+1}|_{C(k_2;\mathfrak{S}_{k_2})}p(1^{n-k_1}k_1;S_n)\binom{n}{k_2+1}\Big]. \end{aligned} \tag{5.1}$$

Proof. The left-hand side of (5.1) counts the number of factorizations as products of k_1+k_2 reflections of all elements $\sigma\in C(\emptyset,(k_1,k_2);D_n)$. Observe that when we remove the "minus signs" from the elements in these sequences of reflections, we obtain elements of $\mathfrak{S}_n$ in the class $C(k_1,k_2;\mathfrak{S}_n)$. Since there are two reflections of D_n that correspond to each transposition in $\mathfrak{S}_n$ after we remove the sign and since we have the same number of choices in $\mathfrak{S}_n$ and in D_n for the reflection that breaks a cycle in two, we add the factor 2^{n-1} to the right-hand side of (5.1).

To obtain an element $\sigma\in C(k_1,k_2;\mathfrak{S}_n)$ as a product of n transpositions we can also build a k_1-*cycle* with k_1+1 transpositions and a k_2-*cycle* with k_2-1 transpositions that commute with the transpositions in the first sequence or vice versa. We apply the shuffle principle and remove these possibilities from $R(n)^n|_{C(k_1,k_2;\mathfrak{S}_n)}$ in the right-hand side of (5.1) to obtain the left-and side. □

Corollary 5.1. *Let $R(n)$ be the class of transpositions of $\mathfrak{S}_n$ and suppose that k_1,k_2 are positive integers with $k_1+k_2=n$. Then the number of ways to factor an element of the class $C(k_1,k_2;\mathfrak{S}_n)$ as a product of the minimum number of transpositions plus two is*

$$\begin{aligned} & R(n)^n|_{C(k_1,k_2;\mathfrak{S}_n)} \\ &\quad = \frac{k_1^{k_1+1}k_2^{k_2+1}}{k_1+k_2}\binom{k_1+k_2}{k_1} + \frac{k_1^{k_1+1}(k_1^2-1)k_2^{k_2+2}\binom{k_1+k_2}{k_1+1}}{4!} \\ &\qquad + \frac{k_2^{k_2+1}(k_2^2-1)k_1^{k_1+2}\binom{k_1+k_2}{k_2+1}}{4!}. \end{aligned}$$

Proof. We need two identities

$$R(n)^{n+1}|_{C((n),\mathfrak{S}_n)} = \frac{n^{n+1}(n^2-1)}{4!}, \tag{5.2}$$

$$|C(\emptyset,(k_1,k_2);D_n)| = 2^{n-2}|C((k_1,k_2);\mathfrak{S}_n)|. \tag{5.3}$$

Identity (5.2) in fact is a special case of

$$R(n)^a|_{C((n),\mathfrak{S}_n)} = \frac{(n/2)^a}{n!} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (n-2k-1)^a, \tag{5.4}$$

which is derived from the character formula (see for instance [8]) for the product of conjugacy classes in $\mathfrak{S}_n$ (see also [5])

$$R(n)^a|_{C((n),\mathfrak{S}_n)} = \frac{|C(1^{n-2}2,\mathfrak{S}_n)|^a}{n!} \sum_{r=0}^{n-1} \frac{(\chi_{1^{n-2}2}^{1^r n-r})^a \chi_{(n)}^{1^r n-r}}{(f^{1^r n-r})^{a-1}}, \tag{5.5}$$

where $f^{1^r n-r}$ is the dimension of the irreducible representation indexed by $1^r n-r$. Hence the result follows from (3.6) and (5.1). □

The next theorem extends the result in Theorem 5.1 to arbitrary conjugacy classes of $\mathfrak{S}_n$.

Theorem 5.2. *Let $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_k)\vdash n$ and $r(\lambda)$ be the minimal number of transpositions in a decomposition* (1.1) *of an element of $C(\lambda;\mathfrak{S}_n)$. Then we have*

$$\begin{aligned} &R(n)^{r(\lambda)+2}|_{C(\lambda;\mathfrak{S}_n)} \\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{k} R(\lambda_i+\lambda_j)^{\lambda_i+\lambda_j}|_{C(\lambda_i,\lambda_j;\mathfrak{S}_n)} p(\lambda-\lambda_i-\lambda_j;\mathfrak{S}_{n-\lambda_i-\lambda_i}) \binom{r(\lambda)+2}{\lambda_i+\lambda_j} \\ &\quad -(k-2)\sum_{i=1}^{k} R(\lambda_i)^{\lambda_i+1}|_{C((\lambda_i);\mathfrak{S}_{\lambda_i})} p(\lambda-\lambda_i;\mathfrak{S}_{n-\lambda_i}) \binom{r(\lambda)+2}{\lambda_i+1}. \end{aligned} \tag{5.6}$$

Proof. Theorem 5.2 states that the number of decompositions of a permutation of cycle type λ as product of $r(\lambda)$ transpositions plus two is done by breaking a cycle in two smaller cycles at one point in the sequence of transpositions and then constructing on the new set of cycles the remaining part of λ in two different ways: one is to construct two disjoint cycles of λ using the two broken parts, the other is to join the two broken cycles into a cycle of λ that contains the elements of the broken cycle. These two processes are actually counted in the first sum of the right-hand side of (5.6) but the second one is counted $k-1$ times. We remove the surplus in the second sum. □

In the next corollary apply Theorem 5.2 to compute the number of decompositions for specific conjugacy classes of $\mathfrak{S}_n$.

Corollary 5.2. *Let k_1, k_2 and k be positive integers with $0<k\leqslant n=k_1+k_2$ and let $R(n)$ be the class of transpositions of $\mathfrak{S}_n$. Then*

(i) $$R(n)^{k+1}|_{C(1^{n-k}k;\mathfrak{S}_n)}=\binom{n-k}{2}\binom{k+1}{2}k^{k-2}+(n-k)R(k+1)^{k+1}|_{C(1,k;\mathfrak{S}_{k+1})}$$

$$-(n-k-1)R(k)^{k+1}|_{C(k;\mathfrak{S}_k)}$$

$$=\binom{n-k}{2}\binom{k+1}{2}k^{k-2}+k^{k+1}\left((n-k)+\frac{k^2-1}{4!}\right),$$

(ii) $$R(n)^n|_{C(1^{\alpha_1}k_1k_2;\mathfrak{S}_n)}=\binom{\alpha_1}{2}\binom{k_1+k_2-2}{k_1-1}\binom{k_1+k_2}{2}k_1^{k_1-2}k_2^{k_2-2}$$

$$+\alpha_1\left[k_1^{k_1+1}k_2^{k_2-2}\binom{k_1+k_2}{k_1+1}+k_2^{k_2+1}k_1^{k_1-2}\binom{k_1+k_2}{k_2+1}\right]$$

$$+\frac{k_1^{k_1+1}k_2^{k_2+1}\binom{k_1+k_2}{k_1}}{k_1+k_2}+\frac{k_1^{k_1-1}k_2^{k_2-1}\binom{k_1+k_2}{k_1}}{4!}$$

$$\times[k_1^2(k_1-1)+k_2^2(k_2-1)],$$

(iii) $$R(n)^{n-\alpha_k+2}|_{C(k^{\alpha_k};\mathfrak{S}_n)}=\frac{\alpha_k k^{\alpha_k(k-1)+2}(\alpha_k(k-1)+2)!}{(4!)(k!)^{\alpha_k}}[6\alpha_k k-5k-1].$$

Proof. These formulas are direct consequences of Theorem 5.2. $\square$

In order to extend Theorem 5.1 to decompositions of permutations into an arbitrary number of transpositions, we introduce the following notation. For W equals to $\mathfrak{S}_n$ or D_n denote by $q\text{-}zig(C_i, W)$ the number of decompositions of $\sigma\in C_i$ as a product of $r(C_i)+2q$ reflections. Each such decomposition can be represented in $\mathfrak{T}(W)$ by a path with q *zigzags* from the identity class to C_i. Let $q\text{-}mix(k_1,k_2)$ be the number of decompositions of $\sigma\in C(k_1,k_2;\mathfrak{S}_n)$ as a product of $r(C(k_1,k_2;\mathfrak{S}_n))+2q$ transpositions that contain at least one transposition (i,j) with i in the k_1-*cycle* and j in the k_2-*cycle*. In order to count $q\text{-}zig(k_1,k_2;\mathfrak{S}_n)$ we first observe that

$$q\text{-}zig(k_1,k_2;\mathfrak{S}_n)=\sum_{i=0}^{q}((q-i)-zig(k_1))(i-zig(k_2))$$

$$\times\binom{k_1+k_2+2q-2}{k_2+2i-1}+q\text{-}mix(k_1,k_2). \tag{5.7}$$

Since we know how to count $q\text{-}zig(k_1)$, for any positive integers q and k_1, by using formula (5.5), it remains to evaluate $q\text{-}mix(k_1,k_2)$. The next conjecture gives a partial answer to that question.

Conjecture 5.1. *Let k_1, k_2 be positive integers with $k_1+k_2=n$, for any positive integer q, we have*

$$q-zig(\emptyset,(k_1,k_2);D_n)=2^{2q+1}[(q+1)-mix(k_1,k_2)]. \tag{5.8}$$

With the zigzag notation, identity (3.6) can be written in the form

$$0\text{-}Zig(\emptyset,(k_1,k_2);D_n)=2(1-mix(k_1,k_2)), \tag{5.9}$$

which is our conjecture for $q=0$. Next we observe that for each walk in $\mathfrak{T}(\mathfrak{G}_n)$ with q zigzags, there corresponds a set of walks with q-1 zigzags in $\mathfrak{T}(D_n)$. There is numerical evidence and we have to show that the weight of this setof $(q-1)$-walks in D_n is equal to $2q+1$ times the weight of the q-walk in $\mathfrak{G}_n$. This could be tried using induction on the number q of zigzags. The problem of computing q-$mix(k_1,k_2)$ is open but we give here without proof the expressions for q-$mix(k_1,k_2)$ when q equals two and three.

$$2\text{-}mix(k_1,k_2)$$
$$=k_1^{k_1+1}k_2^{k_2+1}\left[\binom{k_1+2}{4}\binom{n+2}{k_1+2}+\binom{k_1+3}{4}\binom{n+2}{k_1+3}+\binom{k_1+4}{4}\binom{n+2}{k_1+4}\right]. \tag{5.10}$$

$$3\text{-}mix(k_1,k_2)$$
$$\begin{aligned}=k_1^{k_1+1}k_2^{k_2+1}\frac{(n+4)!}{6!k_1!k_2!}\Bigg[&k_1k_2+21\left(\binom{k_1}{2}k_2+k_1\binom{k_2}{2}\right)+62\binom{k_2}{2}\binom{k_1}{2}\\&+3\left(\binom{k_1}{2}+\binom{k_2}{2}\right)+45\left(\binom{k_1}{3}+\binom{k_2}{3}\right)+\frac{249}{4}\left(\binom{k_1}{3}k_2+k_1\binom{k_2}{3}\right)\\&+\frac{87}{2}\left(\binom{k_1}{3}\binom{k_2}{2}+\binom{k_1}{2}\binom{k_2}{3}\right)+114\left(\binom{k_1}{4}+\binom{k_2}{4}\right)\\&+75\left(\binom{k_1}{5}+\binom{k_2}{5}\right)+45\left(\binom{k_1}{4}k_2+k_1\binom{k_2}{4}\right)\Bigg]\end{aligned} \tag{5.11}$$

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