Riemann, Hurwitz, and Branched Covering Spaces

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Abstract

We will consider spaces with nice connectedness properties, and the groups that act on them in such a way that the topology is preserved; we consider looking at the symmetry groups of a surface of genus g. Restricting our view to finite groups, we will develop the concept of covering spaces and illustrate its usefulness to the study of these group actions by generalizing this development to the theory of branched coverings. This theory lets us develop the famous Riemann-Hurwitz Relation, which will in turn allow us to develop Hurwitz's Inequality, an upper bound on the order of a symmetry group of a given surface. We then follow Kulkarni and use the Riemann-Hurwitz Relation to construct a congruence relating the genus g of a surface to the cyclic deficiencies of the symmetry groups that can act on it. These developments will then be applied to study a special case of branched coverings, those in which there is only one branch point, yielding a lower bound on the genus of both surfaces involved.

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1 Preliminaries

First some clarity on definitions. A *space* X will be a closed topological space which is nicely connected, by which is meant: path connected, locally path connected, and semi-locally simply connected. A continuous function between spaces will be called a *map*. Mappings of $I = [0, 1]$ into X will be called *paths*, an example of a which is the map $\sigma : I \to X$. We will call the image of this path σ. Running σ in the opposite direction will be denote by σ^{-1}, and the path that goes nowhere will be denoted by x_0 (a fixed point). One more, define a *loop* as a path whose initial and terminal points are the same.

Take $\sigma, \tau : I \to X$ to be two paths with $\sigma(1) = \tau(0)$, then the *run-together* of σ and τ will be understood as the bifurcation,

$$\sigma\tau = \begin{cases} \sigma(2t), & t \in [0, \frac{1}{2}] \\ \tau(2t-1), & t \in [\frac{1}{2}, 1] \end{cases}.$$

With that said, note the following property of maps which lets us know that connecting maps via their points of intersection yields another map (under a certain assumption).

Pasting Lemma. *Let $X = \bigcup_i U_i$, an arbitrary union of open sets, and let the maps $f_i : U_i \to Y$ be such that $f_i(x) = f_j(x)$ for all $x \in U_i \cap U_j$. Then $f : X \to Y$ defined by $f|_{U_i} = f_i$ for all i, is a well-defined map. Moreover, if this union is finite and the sets are closed, then the result still holds.*

Proof. Consider the function f as is defined above. Set-wise, f is well-defined. Let $V \subsetneq Y$ be an open set, then $f^{-1}(V)$ can be written as $\bigcup_i \left(f^{-1}(V) \cap U_i\right)$. By assumption this last equation can be written as $\bigcup_i \left(f_i^{-1}(V)\right)$. Since f_i is continuous, $f_i^{-1}(V)$ is open for each i, and the arbitrary union of open sets is open. Therefore f is continuous, hence f is a map.

Noting that a finite union of closed sets is closed and that a continuous pullback of a closed set is closed gives the last part.

As a consequence, the run-together of two loops based at x_0 is another loop based at x_0. In fact taking the collection of loop classes (under a homotopy equivalence relation) with the run-together operation form a group commonly called the *fundamental group*. This group is denoted by $\pi_1(X, x_0)$ where x_0 is the chosen base point for the loops. A terse development of fundamental groups can be found in [10]. It should be noted that, since our spaces are path connected, picking a base point to work with can be thought of as arbitrary. More specifically, choosing another base point will yield an isomorphic fundamental group.

1.1 Theorem. *Let X be any space. For any $x_0, x_1 \in X$ we have $\pi_1(X, x_1) \cong \pi_1(X, x_0)$.*

Proof. Take any space X. Pick $x_0, x_1 \in X$ and let γ be a path from x_0 to x_1. Consider the functions $\Phi_\gamma : \pi_1(X, x_1) \to \pi_1(X, x_0)$, $[\sigma] \mapsto [\gamma^{-1}\sigma\gamma]$, and $\Phi_{\gamma^{-1}} : \pi_1(X, x_0) \to \pi_1(X, x_1)$, $[\tau] \mapsto [\gamma\tau\gamma^{-1}]$. These are well-defined as one can easily check. Moreover, the composition $\Phi_{\gamma^{-1}}\Phi_\gamma$ and $\Phi_\gamma\Phi_{\gamma^{-1}}$ are clearly identity maps, thus Φ_γ is a bijection. Furthermore, $[\gamma^{-1}\sigma\sigma'\gamma] = [\gamma^{-1}\sigma\gamma][\gamma^{-1}\sigma'\gamma]$ hence Φ_γ is homomorphism, thus we have the isomorphism we were after.

The above theorem allows us to leave out the choice of the base point for the fundamental group, in this case we denote the fundamental group by $\pi_1(X)$. As an example of a space and its fundamental group, we consider the fundamental group of the circle.

1.2 Example. View the circle as being embedded in the complex plane via polar coordinates, $S^1 := \{(r, \theta) \in \mathbb{C} \big| r = 1,\ \theta \in \mathbb{R}\}$. Consider the function $\exp(x) : \mathbb{R} \to S^1$ where $\exp(x) = e^{2\pi i x}$. This is the exponential function hence well-defined, continuous, and maps $\mathbb{R}$ onto S^1. We mod out the image of exp by the equivalence relation, $x \sim y$ if for some $k \in \mathbb{Z}$ we have $\exp(x+k) = \exp(y)$, to identify S^1 with the quotient space $\mathbb{R}/\mathbb{Z}$. Thus there is a homeomorphism between $\mathbb{R}/\mathbb{Z}$ and S^1. If two spaces are homeomorphic then their fundamental groups are isomorphic [5, 11], thus $\pi_1(\mathbb{R}/\mathbb{Z}) \cong \pi_1(S^1)$. We appeal to *(4.5) Theorem* from [4] which shows, since $\mathbb{Z}$ is a discrete subgroup of the simply connected topological group $\mathbb{R}$, that $\pi_1(\mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}$. Therefore, $\pi_1(S^1) \cong \mathbb{Z}$.

We make note that any arbitrary group considered in this paper is assumed to be of finite order unless mentioned otherwise, such as when we consider the group of integers. Now let us recall the definition and some immediate properties associated with the algebraic concept of group actions – we begin with the definition.

Definition. A group G induces a *right group action* on S (a nonempty set) if there is a binary operation $S \times G \to S$ such that:

(i) For any $s \in S$, $(s, 1) \mapsto s$ where $1 \in G$ is the identity element,

(ii) For any $s \in S$ and for any $g_1, g_2 \in G$, $(s, g_1 g_2) = \big((s, g_1), g_2)\big)$.

Similarly, a group H induces a *left group action* on S if there is a binary operation $H \times S \to S$ such that for any $s \in S$ and any $h_1, h_2 \in H$, $(1, s) \mapsto s$ and $(h_1 h_2, s) = \big(h_1, (h_2, s)\big)$.

All groups are assumed to act *effectively*, which means for every two distinct elements in the group there is some point in the space at which they differ. Now assume that G acts on the right of S (we also assume these sets are nonempty), let us run through some terms that are associated with this group action. The group G acts *transitively* on S if for any $s_1, s_2 \in S$ there is a $g \in G$ such that $s_1 g = s_2$, and is called *free* if $sg = s$ for any $s \in S$ implies g is the identity. Moreover, we call an action *regular* if it is both transitive and free. Denote the *orbit* of $s \in S$ by $sG := \{sg | g \in G\}$, and denote the collection of all orbits by $S/G := \{sG | s \in S\}$, called the *orbit space*. The *isotropy subgroup*, also called the *stabilizer subgroup*, of a point $s \in S$ is denoted by $G_s := \{g \in G | sg = s\}$. With these terms made clear let us now show a connection between elements in the same orbit and there respective isotropy subgroups.

1.3 Proposition. *Let G act on the right of S. Choose any $s \in S$, then for $s_1, s_2 \in sG$ we have that $G_{s_1} = g G_{s_2} g^{-1}$ for some $g \in G$. Moreover, $|G_{s_1}| = |G_{s_2}|$.*

Proof. Let G and S be as above. Take any $s_1, s_2 \in sG$ for some $s \in S$. Since these elements are in the same orbit, there is a $g \in G$ in which $s_2 = s_1 g$. Consider $h \in G_{s_1}$, so $s_2 g^{-1} h = s_2 g^{-1}$. This relation can be rewritten as $s_2 g^{-1} h g = s_2$, which is to say $h \in G_{s_1}$ if and only if $g^{-1} h g \in G_{s_2}$ if and only if $h \in g G_{s_2} g^{-1}$. Therefore, elements in the same orbit have conjugate isotropy subgroups.

Moreover, let $\Phi_g : G_{s_2} \to g G_{s_2} g^{-1}$ by $h \mapsto g h g^{-1}$. This is clearly a group conjugacy homomorphism and has an inverse given by $h \mapsto g^{-1} h g$. Hence these subgroups are isomorphic, thus $|G_{s_1}| = |G_{s_2}|$.

There is one more important relationship between the isotropy subgroup of a point and the orbit of which the point is in that we should point out. This relationship is derived from the following two famous group theory theorems.

Orbit-Stabilizer Theorem. *Let G act on the right of S. Then, for $s \in S$, $sG \cong G/G_s$.*

Lagrange's Theorem. *Let G be a group and let H be a subgroup, then $|G|=|H|[G:H]$.*

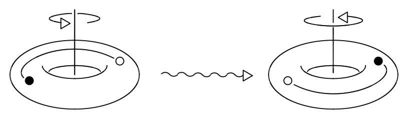
The development of these two theorems is straight forward and can be found in [3, 14]. To apply these theorems to group actions first recall that all groups are considered to be of finite order, also recall that G_s is a subgroup of G. Next note that the *index* of this subgroup is defined by $[G : G_s] := |G/G_s|$. Thus, when we apply *Lagrange's Theorem* to the *Orbit-Stabilizer Theorem*, we yield the following equality,

$$|sG| = [G : G_s].$$

Finally, we note the property that connects a left and a right group action on a set. Let H and G act on S (on the left and right respectively), these actions are *compatible* if for any $g \in G$, any $s \in S$, and any $h \in H$ we have associativity, *i.e.* $\big((h, s), g\big) = \big(h, (s, g)\big)$.

A *symmetry group* of a space can be defined as a group which acts on the space in such a way that this action preserves the topology of the space. To help illustrate this definition we will conclude this section with a straight forward example of a symmetry group.

1.4 Example. Take the torus. The action of counterclockwise rotation by angle π, about the indicated axis as pictured below, describes a symmetry of the torus. As can be seen, the two tori below both have the same shape (hence topology) even though the points on the torus are being moved.



The collection of all rotations under this action can be regarded as the group $\mathbb{Z}/2\mathbb{Z}$ since two rotations yields the same outcome as no rotation at all. So we have that $\mathbb{Z}/2\mathbb{Z}$ is a symmetry group of the torus.

2 Covering Spaces

Definition. A *covering space* of X is a pair (E, ρ) with $\rho : E \to X$ where ρ has the property that for each $x \in X$ there is an open connected neighborhood of x, U_x, such that $\rho^{-1}(U_x)$ is a disjoint union of open connected subsets S_j each of which is mapped homeomorphically onto U_x by $\rho|_{S_j}$. We will denote a covering space of E over X by $E \overset{\rho}{\to} X$.

If a base point is to be kept track of, then the notation $(E, e_0) \overset{\rho}{\to} (X, x_0)$ is used to refer to a covering space in which $\rho(e_0) = x_0$. Also, we call X the *base space*, E the *total space*, ρ the *covering map*, and refer to the open neighborhood U_x as being *evenly covered* by the *sheets* S_j. Take $\phi : X \to X$, any homeomorphism, as a trivial example of a covering space. In this case: X is the base space, X is the total space, ϕ is the covering map, and any proper open neighborhood of a point is an evenly covered neighborhood which is covered by one sheet.

2.1 Example. Does the map $\exp : \mathbb{R} \to S^1$, $\exp(x) = e^{2\pi i x}$, induce a covering space? This map is clearly onto and takes 0 to $(1, 0)$. We assign to each point $(1, \theta) \in S^1$ an open neighborhood by letting $U_{(1,\theta)} = \{(1, \theta + k)\big|k \in (-\pi, \pi)\}$. Let $x' \in \mathbb{R}$ such that $\theta = 2\pi x'$, this lets us write the pullback as $\exp^{-1}(U_{(1,\theta)}) = \{S_j \subset \mathbb{R}\big|j \in \mathbb{Z}\}$ where $S_j = \{x \in \mathbb{R}\big|x \in (x' - \frac{1-2j}{2}, x' + \frac{1+2j}{2})\}$, note S_j is open and pairwise disjoint to S_ℓ for $\ell \neq j$. Also, $\exp|_{S_j}$ homeomorphically maps S_j onto $U_{(1,\theta)}$, so $U_{(1,\theta)}$ is evenly covered. Therefore $(\mathbb{R}, 0) \overset{\exp}{\to} \big(S^1, (1, 0)\big)$ has been shown to be a covering space.

Several immediate consequences of the definition of a covering space should be noted: (i) ρ surjectively maps E onto X, (ii) ρ is a local homeomorphism hence an open map, (iii) $\rho^{-1}(x)$, the *fiber*, is a discrete set for all $x \in X$. It is good to note that if $E \xrightarrow{\rho} X$ is a covering space then ρ is a local homeomorphism, but that the converse is not necessarily true as is pointed out by [2]. The following gives an illustration of this.

2.2 Example. Does the map $\rho : (0,3) \to S^1$, $\exp(x) = e^{2\pi i x}$, induce a covering space? Clearly exp is a local homeomorphism but no neighborhood of the point $(1,0)$ is evenly covered. To see this take any open neighborhood $U_{(1,0)}$, its pullback contains a connected open set which is not homeomorphically mapped onto $U_{(1,0)}$ by the restriction of exp. In particular, the set $S = (0,a) \subset \exp^{-1}\left(U_{(1,0)}\right)$, for some $a \in (0,1)$, has an image under $\exp|_S$ that does not even contain the point $(1,0)$. Hence $U_{(1,0)}$ is not evenly covered, so in this case exp does not induce a covering space.

A useful outcome of covering spaces is how commuting diagrams involving a covering space can help determine if another map in the diagram induces a covering space. We show this connection first between two covering spaces of which share the same base space.

2.3 Proposition. *Given two covering spaces, $E \xrightarrow{\rho} X$ and $Y \xrightarrow{\nu} X$. If there is a map μ such that the following diagram commutes,*

$$\begin{tikzcd} E \arrow[rr, "\mu"] \arrow[dr, "\rho"'] & & Y, \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

then $E \xrightarrow{\mu} Y$ is a covering space.

Proof. Consider the above diagram. First note that μ is surjective by the commutativity of the diagram and since both ρ and ν are surjective. We aim to show every $y \in Y$ is has an evenly covered neighborhood. Take $y \in Y$ and let $x = \nu(y)$. Let U_x be an evenly covered neighborhood of x with regards to ρ, and let V_x be an evenly covered neighborhood of x with regards to ν. Now define $W_x := U_x \cap V_x$, clearly this is an evenly covered neighborhood of x with regards to both ρ and ν. Now let $T \subset \nu^{-1}(W_x)$ be the sheet above W_x which contains y. We claim that this T is an evenly covered neighborhood of y with regards to μ.

Consider $\rho^{-1}(W_x) = \bigcup S_j$ where each S_j is a sheet over W_x. Since $\nu\mu(S_j) = \rho(S_j) = W_x$ for each j, we have $\mu(S_j) \subset \nu^{-1}(W_x)$. Now fix a j, by the continuity of μ either $\mu(S_j) \subset T$ or $\mu(S_j) \cap T = \emptyset$. So we can write $\mu^{-1}(T) = \bigcup S_j$, a union is of disjoint sets. Also, we are led to the following commuting diagram,

$$\begin{tikzcd} S_j \arrow[rr, "\mu|_{S_j}"] \arrow[dr, "\rho|_{S_j}"'] & & T. \arrow[dl, "\nu|_T"] \\ & W_x & \end{tikzcd}$$

Since $\rho|_{S_j}$ and $\nu|_T$ are homeomorphisms, $\mu|_{S_j}$ must be a homeomorphism, and this is true for each j. Hence T is an evenly covered neighborhood of $y \in Y$ with regards to μ. Therefore, μ induces a covering space.

The next proposition shows another connection between covering spaces and how one covering space can be used to determine if another map induces a covering space.

2.4 Proposition. *Given a covering space $E \xrightarrow{\rho} X$, two homeomorphisms $\Phi : E \to E'$ and $\Psi : X \to X'$, and a map $\nu : E' \to X'$ such that the following diagram commutes,*

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\rho"'] & E' \arrow[d, "\nu"] \\ X \arrow[r, "\Psi"'] & X', \end{tikzcd}$$

then $E' \xrightarrow{\nu} X'$ is a covering space.

Proof. Showing ν is surjective is straight forward since the diagram commutes and Φ and Ψ are homeomorphisms. Now consider the following commuting diagram,

$$\begin{tikzcd} E \arrow[d, "\rho"'] \arrow[dr, "\nu\Phi"] & \\ X & X'. \arrow[l, "\Psi^{-1}"] \end{tikzcd}$$

It follows from *Proposition 2.3* that $\nu\Phi$ induces a covering space. Now consider the composition $E \xrightarrow{\Phi} E' \xrightarrow{\nu} X'$. Let U be evenly covered by $\nu\Phi$, so $\Phi^{-1}\nu^{-1}(U) = \bigcup S_j$ where the S_j are sheets over U. Since Φ is a homeomorphism, we get $\nu^{-1}(U) = \bigcup \Phi(S_j)$. Note $\Phi(S_j) \cap \Phi(S_\ell) = \emptyset$ for all $j \neq \ell$. Furthermore, for each j, $\nu\big(\Phi(S_j)\big) = U$, therefore ν induces a covering space.

We conclude this section with an example which shows, that between two spaces, more than one map can induce a covering space. Then we will follow with an example of another well known covering space.

2.5 Example. The mapping $\rho_n : S^1 \to S^1$, $(1, \theta) \mapsto (1, n\theta)$, induces a covering space for all nonzero $n \in \mathbb{Z}$. Clearly this map is continuous and surjective. For any U_z, an open neighborhood of $z \in S^1$ (the range), we see that $\rho_n^{-1}(U_z) = \bigcup S_j$ is a collection of n disjoint open neighborhoods in S^1 (the domain). Moreover, $\rho_n|_{S_j}$ is clearly a homeomorphism for all j. Therefore ρ_n induces a covering space.

2.6 Example. S^n is a covering space of RP^n (the real projective n-space), for $n \geq 2$, under the antipodal-identification map, $\mathfrak{a}$. It's clear that $\mathfrak{a}$ is surjective and a local homeomorphism. By properties of local homeomorphism each point in the domain has an open neighborhood, U, such that $\mathfrak{a}|_U : U \to \mathfrak{a}(U)$ homeomorphically. Let $W = \mathfrak{a}(U)$, then one can easily check that W is evenly covered by two sheets. Thus $\mathfrak{a}$ induces a covering space.

3 Map Lifting Theorems

Covering maps are local homeomorphisms, and because of this the total space and base space must share local properties, such as the local connectedness as is assumed of our spaces. Furthermore, these local connectedness properties allows for a mapping into the base space to be uniquely lifted into the total space. For a covering space $E \xrightarrow{\rho} X$ and a map $f : Y \to X$, by a *lift* we mean a map $f' : Y \to E$ such that $\rho f' = f$. We will skip the proof for existence of lifts at first so as to prove the uniqueness. The following proof is adapted from an argument in [5] and is as follows.

3.1 Proposition (Uniqueness of Lifted Maps)**.** *Given a covering space $E \xrightarrow{\rho} X$ and any map $f : Y \to X$ of which has two lifts $f_1', f_2' : Y \to E$ that agree at a point of Y. Then these lifts must agree on all of Y.*

Proof: Let $y \in Y$ and denote $U_{f(y)}$ as an evenly covered neighborhood of $f(y)$. Let S_1 and S_2 be the sheets above $U_{f(y)}$ which contain $f_1'(y)$ and $f_2'(y)$ respectively. Take an open neighborhood $N_y \subset {f_1'}^{-1}(S_1) \cap {f_2'}^{-1}(S_2)$ of y and note that $f_j'(N_y) \subset S_j$ for $j = 1, 2$. Assume $f_1'(y) \neq f_2'(y)$, then $S_1 \neq S_2$ and since these are disjoint sets, $f_1' \neq f_2'$ on N_y. Conversely, assume $f_1'(y) = f_2'(y)$, then $S_1 = S_2$ and, since $\rho f_1' = \rho f_2'$ and $\rho|_{S_1}$ is a homeomorphism, we must have that $f_1' = f_2'$ on N_y.

Now create an open covering of Y using these open neighborhoods N_y for all $y \in Y$. The *Pasting Lemma* gives that the image of this cover is connected since the intersection of these neighborhoods must agree under f_1' and f_2'. Thus the set of all $y \in Y$ in which $f_1'(y) = f_2'(y)$ is either empty or all of Y. Since we are assuming f_1' and f_2' agree at one point of Y, it follows that the two lifts agree on the entire space Y.

Now that the uniqueness of a lifted map has been shown we move on to showing homotopies can be lifted. This is of interest since homotopies are a necessary relationship if one is to define the fundamental group of a space. In the following theorem, due to [4, 5, 9], it will be shown that, if given a covering space and a mapping into the base space which has a lift, any homotopy of this map in the base space can be lifted to a homotopy in the total space. Furthermore, this lifted homotopy will be unique.

3.2 Theorem (Homotopy Lifting Theorem)**.** *Given a covering space $E \xrightarrow{\rho} X$, a space Y with a map $f : Y \to X$ that has a lift $f' : Y \to E$. If there is a homotopy $F : Y \times I \to X$ with $F(y, 0) = f(y)$ for all $y \in Y$, then it can be uniquely lifted to a homotopy $F' : Y \times I \to E$ with $F'(y, 0) = f'(y)$ for all $y \in Y$. Hence, letting the map $\imath$ take $y \mapsto (y, 0)$, we have the following commuting diagram,*

$$\begin{tikzcd}
Y \arrow[r, "f'"] \arrow[d, "\imath"'] & E \arrow[d, "\rho"] \\
Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "F'"] & X.
\end{tikzcd}$$

Proof. Since F is continuous, each $(y_0, t) \in Y \times I$ has an open neighborhood $N_t \times (a_t, b_t)$ such that F maps $N_t \times (a_t, b_t)$ into some evenly covered neighborhood. By the compactness of $\{y_0\} \times I$ we can choose a single neighborhood N of y_0 and a particular partition of I, $0 = t_0 < t_1 < \ldots < t_n = 1$, such that $F\big(N \times [t_j, t_{j+1}]\big) \subset U_j$ where U_j is evenly covered. Define $F_j' = p|_{S_j}^{-1} F$ where S_j is the sheet above U_j that contains $F'(y_0, t)$. Clearly this is well defined for each j. Now note that $F_j'(N \times \{t_{k+1}\}) = F_{j+1}'(N \times \{t_{k+1}\})$, therefore the *Pasting Lemma* gives a well-defined lift $F' : N \times I \to E$ which is also unique by *Proposition 3.1.*

Finally, we can extend the lift F' to $Y \times I$ by noting $\{y\} \times I$ lifts uniquely. So F' must agree on the intersection of any two such sets $N \times I$ that overlap, thus the *Pasting Lemma* gives us that $F' : Y \times I \to E$ is a well-defined map. Furthermore, $Y \times I$ is connected, thus *Proposition 3.1* shows F' is unique.

The above theorem allows us to easily show the existence of lifted paths. The value of this is that, given a covering space, the fundamental group of the base space will be able to be lifted uniquely into a collection of paths in the total space.

3.3 Theorem (Unique Path Lifting Theorem)**.** *Given a covering space $E \xrightarrow{\rho} X$. If σ is a path in X with initial point x_0, then there is a path σ' in E such that $p\sigma' = \sigma$, hence the following diagram commutes,*

$$\begin{tikzcd} & E \arrow[d, "\rho"] \\ I \arrow[ur, dashed, "\sigma'"] \arrow[r, "\sigma"'] & X. \end{tikzcd}$$

Moreover, this lift is unique once an initial point $e_0 \in \rho^{-1}(x_0)$ is chosen for σ'.

Proof. Let $E \xrightarrow{\rho} X$ be a covering space. The map $f : \{x_0\} \to X$, $x_0 \mapsto x_0$, clearly has a lift $f' : \{x_0\} \to E$, $x_0 \mapsto e_0$ for some chosen $e_0 \in \rho^{-1}(x_0)$. Now assume $\sigma : I \to X$ is a path with initial point x_0 and terminal point x_1. We can view σ as a homotopy by setting $\sigma : \{x_0\} \times I \to X$ with $\sigma(x_0, 0) = f(x_0)$. Thus allowing us to use *Theorem 3.2* to show that there exists a unique homotopy (hence a path) $\sigma' : \{x_0\} \times I \to E$ with $\sigma'(x_0, 0) = f'(x_0)$ and such that $\rho\sigma' = \sigma$. Therefore paths lift uniquely.

Uniquely lifted paths, as in the above theorem, will be denoted by σ'_e, which is read as the unique lift of σ starting at the point $e \in E$. Now that the fundamental group of a base space has been shown to lift uniquely into the total space, let us question what the converse might mean. In other words let us look at the mapping of the fundamental group of a total spaces into the base space under a covering map.

3.4 Corollary. *Given a covering space $(E, e_0) \xrightarrow{\rho} (X, x_0)$, the homomorphism induced by the covering map, $\rho_\# : \pi_1(E, e_0) \to \pi_1(X, x_0)$, is a monomorphism.*

Proof. Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a covering space. For an element to be in the kernal of $\rho_\#$ we must have $\rho_\#([\sigma'_{e_0}]) \in [x_0]$. Which is to say $[\rho\sigma'_{e_0}] \simeq [x_0]\text{rel}\{0,1\}$ and since $[\rho\sigma'_{e_0}] = [\sigma]$, we have $[\sigma] \simeq [x_0]\text{rel}\{0,1\}$. By *Theorem 3.2*, this homotopy lifts uniquely into E. So now we have $[\sigma'_{e_0}] \simeq [e_0]\text{rel}\{0,1\}$, hence $\rho_\#$ has a trivial kernel. Therefore $\rho_\#$ is monic.

Thus it has been shown that, given a covering space, the fundamental group of the total space is isomorphic to a subgroup of the fundamental group of the base space. Now another question arises: which subgroups of the fundamental group of the base space are realized as the fundamental group for some total space? It is not yet clear, and developing the answer to this question amounts to knowing if the trivial subgroup of $\pi_1(X)$ is obtained by some covering space of E over X. This will be discussed more in *Section 6*.

We now solidify the above lifting theorems by giving a statement, made clear by [4, 5, 11], that proves lifts exist and also governs when a map can be lifted in the first place.

3.5 Theorem (Map Lifting Criteria)**.** *Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a covering space and any map $f : (Y, y_0) \to (X, x_0)$ for some space Y. Then there exists a unique lift of f if and only if $f_\#\pi_1(Y, y_0) \subseteq \rho_\#\pi_1(E, e_0)$.*

Proof. Assume f' is the unique lift of f then $\rho f' = f$, so $(\rho f')_\# = f_\#$. This leads us to note that the image of $f_\#$ is contained in the image of $\rho_\#$, hence $f_\#\pi_1(Y, y_0) \subseteq \rho_\#\pi_1(E, e_0)$.

Conversely, assume $f_\#\pi_1(Y, y_0) \subseteq \rho_\#\pi_1(E, e_0)$. Let σ to be a path in Y starting at y_0 and ending at y, then $f\sigma$ is a path in X starting at x_0 and ending at $f(y)$. *Theorem 3.3* gives that this lifts uniquely to a path $(f\sigma)'$ in E starting at e_0. We will define the endpoint of this lifted path by $f'(y)$, *i.e.* $f'(y) := (f\sigma)'(1)$. First we show that this is a well-defined function from Y to E taking y_0 to e_0.

Assume γ is another path in Y from y_0 to y. Then the run-together, $\tau := (f\gamma)(f\sigma)^{-1}$, is a loop in X based at x_0. So $[\tau] \in f_\# \pi_1(Y, y_0)$ thus, by assumption, $[\tau] \in \rho_\# \pi_1(E, e_0)$, hence there is a homotopy $[\tau] \simeq [\delta]\mathrm{rel}\{0,1\}$ such that δ' is a loop in E based at e_0. *Theorem 3.2* gives that this homotopy lifts uniquely to the homotopy $[\tau'] \simeq [\delta']\mathrm{rel}\{0,1\}$, thus τ' is a loop in E based at e_0. But, by uniquness, this means the first half of τ' is $(f\gamma)'$ and the second half is $(f\sigma)'$ run backwards. So they share a point in common, in other words $(f\gamma)'(1) = (f\sigma)'(1)$. Hence f' is well-defined.

To show f' is continuous we consider an evenly covered neighborhood $U_{f(y)}$ for $y \in Y$. Let S_1 be the sheet above $U_{f(y)}$ such that $f'(y) \in S_1$. We may choose an open $V_y \subset Y$ with $f(V_y) \subset U_{f(y)}$ since f is continuous. For any $y_1 \in V_y$ we can choose a fixed path, σ, from y_0 to y, and then follow it by a path γ (in V_y), from y to y_1. The run-together lifts to $(f\sigma)'(f\gamma)'$ in E and $(f\gamma)' = \rho|_{S_1}^{-1} f\gamma$. Thus $f'(V_y) \subset S_1$, therefore f' is continuous hence a lift of f. Furthermore, this lift is unique by *Proposition 3.1.*

4 Group Action Induced by $\pi_1(X, x_0)$

Within the proof of *Theorem 3.5* a function was created which represented the endpoint of a lifted path, recall that $f'(y) := (f\sigma)'(1)$. Let us restrict this idea to loops so as to develop an action involving these loops and the fiber of a covering space. The property to note is that lifted loops will all start at the chosen base point but they may end up at some other point, but this point will be in the fiber. Thus this action would take a loop based at $x_0 \in X$, lift it to a path based at $e \in \rho^{-1}(x_0)$, and then yields this uniquely lifted paths end point.

4.1 Proposition. *Given a covering space $(E, e_0) \xrightarrow{\rho} (X, x_0)$. If we define the binary operation $\rho^{-1}(x_0) \times \pi_1(X, x_0) \to \rho^{-1}(x_0)$ by $(e, [\sigma]) \mapsto e[\sigma]$ where $e[\sigma] := \sigma'_e(1)$, then this defines a transitive right action of $\pi_1(X, x_0)$ on $\rho^{-1}(x_0)$.*

Proof. The above binary operation is well-defined in light of *Theorems 3.3.* Note, for the identity $[x_0] \in \pi_1(X, x_0)$, we have $e[x_0] = (x_0')_e(1) = (e)_e(1) = e$. Thus the first property of a group action holds, the second property is to show associativity. Choose any $e \in \rho^{-1}(x_0)$ and any $[\sigma], [\tau] \in \pi_1(X, x_0)$, we have

$$\big(e[\sigma]\big)[\tau] = \big(\sigma'_e(1)\big)[\tau] = \tau'_{\sigma'_e(1)}(1).$$

But we also have

$$e\big([\sigma][\tau]\big) = e[\sigma\tau] = (\sigma\tau)'_e(1) = (\sigma'\tau')_e(1) = \tau'_{\sigma'_e(1)}(1).$$

Thus $(e[\sigma])[\tau] = e([\sigma][\tau])$. So both properties of a group action hold, hence $\pi_1(X, x_0)$ induces a right group action on $\rho^{-1}(x_0)$. Furthermore, recall the space E is connected, so there is a path between any $e_1, e_2 \in \rho^{-1}(x_0)$, say τ', which is a lift of a loop τ and $e_1[\tau] = \tau'_{e_1}(1) = e_2$. Hence $\pi_1(X, x_0)$ acts transitively on $\rho^{-1}(x_0)$.

Thus we will treat $\rho^{-1}(x_0)$ as a transitive right $\pi_1(X, x_0)$-set, and as an immediate result we can see that the isotropy subgroup of $e \in \rho^{-1}(x_0)$, $\pi_1(X, x_0)_e$, can be realized as the projection of the fundamental group of the total space based at e.

4.2 Proposition. *Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a covering space, and view $\rho^{-1}(x_0)$ as a transitive right $\pi_1(X, x_0)$-set. Then for each point $e \in \rho^{-1}(x_0)$, we have that $\pi_1(X, x_0)_e = \rho_\# \pi_1(E, e)$.*

Proof. Both $\pi_1(X,x_0)_e$ and $\rho_\#\pi_1(E,e)$ are subgroups of $\pi_1(X,x_0)$ so we will show equality by containment. Let $e \in \rho^{-1}(x_0)$ and consider $e[\sigma] = e$, $[\sigma] \in \pi_1(X,x_0)$. In terms of actions this equality means $[\sigma] \in \pi_1(X,x_0)_e$, but by definition $e[\sigma] = \sigma'_e(1)$ and if this is to equal e then it must be that σ' is a loop based at e, thus $[\sigma] \in \rho_\#\pi_1(E,e)$. Therefore, $\pi_1(X,x_0)_e = \rho_\#\pi_1(E,e)$.

Applying the *Orbit-Stabilizer Theorem* and *Lagrange's Theorem* to the set $\rho^{-1}(x_0)$ and the group $\pi_1(X,x_0)$, which acts transitively on $\rho^{-1}(x_0)$, we get,

$$\left|\rho^{-1}(x_0)\right| = \left[\pi_1(X,x_0) : \pi_1(X,x_0)_e\right]. \tag{1}$$

If $\pi_1(X,x_0)_e$ is of finite index, then we say the number of sheets evenly covering x_0 is equal to this index. We are now able to show the following, taken from [4, 13], which gives a useful and necessary condition for a covering space.

4.3 Proposition. *Let $E \xrightarrow{\rho} X$ be a covering space. For any $x_0, x_1 \in X$, $|\rho^{-1}(x_0)| = |\rho^{-1}(x_1)|$.*

Proof. Let $e_0 \in \rho^{-1}(x_0)$, $e_1 \in \rho^{-1}(x_1)$. We have the following commutative diagram,

$$\begin{tikzcd}
\pi_1(E,e_0) \arrow[r, "\Phi"] \arrow[d, "\rho_\#"'] & \pi_1(E,e_1) \arrow[d, "\rho_\#"] \\
\pi_1(X,x_0) \arrow[r, "\Psi"'] & \pi_1(X,x_1),
\end{tikzcd}$$

where Φ and Φ are the isomorphisms given to us by *Theorem 1.1*. Also $\rho_\#$ is a monomorphism as was shown in *Corollary 3.4*. Thus there is a bijection between cosets, hence *equation 1* gives us the result we were after, $|\rho^{-1}(x_1)| = |\rho^{-1}(x_2)|$.

As a consequence of the above, the fibers of a covering space are homeomorphic to one another [2]. One more property of covering spaces we should show is that the fiber corresponds to the conjugates of a collection of subgroups, particularly to all the conjugates of the isotropy subgroups.

4.4 Proposition. *Let $E \xrightarrow{\rho} X$ be a covering space. If $e_0, e_1 \in \rho^{-1}(x_0)$ then $\rho_\#\pi_1(E,e_0)$ and $\rho_\#\pi_1(E,e_1)$ are conjugate subgroups. Moreover, all conjugates of $\rho_\#\pi_1(E,e_0)$ are realized as $\rho_\#\pi_1(E,e_1)$ for some $e_1 \in \rho^{-1}(x_0)$.*

Proof. The first part follows from *Proposition 1.3* since $\rho_\#\pi_1(E,e_0) = \pi_1(X,x_0)_{e_0}$ and since $e_0\pi_1(X,x_0) = \rho^{-1}(x_0)$. Moreover, take any $[\gamma] \in \pi_1(X,x_0)$ and let $H = [\gamma]\rho_\#\pi_1(E,e_0)[\gamma^{-1}]$. Then γ'_{e_0} has a terminal point in $\rho^{-1}(x_0)$, say e_1, since $\rho\gamma' = \gamma$. We consider the diagram from *Proposition 4.3* with $x_1 = x_0$ and use the commutativity to show that,

$$H = \Psi\rho_\#\pi_1(E,e_0) = \rho_\#\Phi\pi_1(E,e_0) = \rho_\#\pi_1(E,e_1).$$

Therefore H is the isotropy subgroup of $e_1 \in \rho^{-1}(x_0)$.

For a covering space $(E,e_0) \xrightarrow{\rho} (X,x_0)$ let us state one particular subset of $\rho^{-1}(x_0)$ that will become of importance later on. We define the set,

$$S := \{e \in \rho^{-1}(x_0) | \pi_1(X,x_0)_e = \pi_1(X,x_0)_{e_0}\}. \tag{2}$$

In other words, S is the collection of elements from $\rho^{-1}(x_0)$ of which have the same isotropy subgroup as our chosen base point e_0. Equivalently, by *Proposition 4.2*, we can define this set by,

$$S := \{e \in \rho^{-1}(x_0) | \rho_\#\pi_1(E,e) = \rho_\#\pi_1(E,e_0)\}. \tag{3}$$

Let us move on to yet another action that can be defined on the fiber of a covering space.

5 Group Action Induced by $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$

Given a covering space, recall the fiber over a point is a discreet set of points so talking about permutations of this set makes sense. We now follow the development laid out by [9, 13] of which defines a particular subgroup of the group of permutations of the fiber.

Definition. A function $f : \rho^{-1}(x_0) \to \rho^{-1}(x_0)$ is called $\pi_1(X, x_0)$-*equivariant* if given any $e \in \rho^{-1}(x_0)$, $f(e[\sigma]) = f(e)[\sigma]$ for any $[\sigma] \in \pi_1(X, x_0)$. Furthermore, if f is $\pi_1(X, x_0)$-equivariant and is also a set-theoretic bijection, then we call f an *automorphism* of the right $\pi_1(X, x_0)$-set $\rho^{-1}(x_0)$.

The collection of these automorphisms form a group under composition of which acts on the left of $\rho^{-1}(x_0)$, both easily checked, and is also compatible with the $\pi_1(X, x_0)$ action by definition. We denote this group by $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ and make note that if two automorphisms agree at one point $e \in \rho^{-1}(x_0)$, then these automorphisms are equal on the entire set $\rho^{-1}(x_0)$. In other words, an automorphism of the right $\pi_1(X, x_0)$-set $\rho^{-1}(x_0)$ is uniquely determined by what it does to a single point.

5.1 Lemma. *Given a covering space* $(E, e_0) \xrightarrow{\rho} (X, x_0)$. *If* $f, g \in Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ *such that* $f(e_1) = g(e_1)$ *for some* $e_1 \in \rho^{-1}(x_0)$, *then* $f = g$.

Proof. Take any $e \in \rho^{-1}(x_0)$ and let $[\sigma] \in \pi_1(X, x_0)$ be such that $e = e_1[\sigma]$, we can do this since $\pi_1(X, x_0)$ acts transitively on the fiber. So $f(e) = f(e_1[\sigma]) = f(e_1)[\sigma]$, but $f(e_1) = g(e_1)$ by assumption, and $g(e_1)[\sigma] = g(e_1[\sigma]) = g(e)$. Therefore $f(e) = g(e)$ for all $e \in \rho^{-1}(x_0)$.

Now we utilize the set S as defined by *formulas 2* and *3*. We note that the group of automorphisms of $\rho^{-1}(x_0)$ acts transitively on the set S, moreover, this action is regular.

5.2 Proposition. *Let* $(E, e_0) \xrightarrow{\rho} (X, x_0)$ *be a covering space. Then* $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ *induces a regular left action on the set* S.

Proof. Assume $e_1 \in S$, and $f \in Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$. Then a simple check using the properties of f will show that,

$$\pi_1(X, x_0)_{e_1} = \pi_1(X, x_0)_{f(e_1)}.$$

But $\pi_1(X, x_0)_{e_1} = \pi_1(X, x_0)_{e_0}$ by assumption, hence $f(e_1) \in S$ by *formula 2*. Thus $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ acts on the left of S.

Now assume $e_1, e_2 \in S$. Since $\rho^{-1}(x_0)$ is a transitive right $\pi_1(X, x_0)$-set we can find a $[\sigma] \in \pi_1(X, x_0)$ such that $e_1[\sigma] = e$ for $e \in \rho^{-1}(x_0)$. Define $f(e) := e_2[\sigma]$. This definition is independent of the choice of $[\sigma]$, for if $e_2[\sigma] = e_2[\tau]$ then $[\sigma][\tau^{-1}]$ is in $\pi_1(X, x_0)_{e_2}$ which by our assumption is equal to $\pi_1(X, x_0)_{e_1}$, it then follows that $e_1[\sigma] = e_1[\tau]$. Therefore f is well-defined and $f(e_1) = e_2$. Furthermore, f is $\pi_1(X, x_0)$-equivarient since for any $[\tau] \in \pi_1(X, x_0)$ we have $e_1[\sigma][\tau] = e[\tau]$ so,

$$f(e[\tau]) = f(e_1[\sigma][\tau]) = e_2[\sigma][\tau] = f(e)[\tau].$$

Now define a similar function $g(e) := e_1[\tau]$ where, for $e \in \rho^{-1}(x_0)$, $[\tau] \in \pi_1(X, x_0)$ is such that $e_2[\tau] = e$. It then follows, by the same method as above, that g is a well-defined,

$g(e_2) = e_1$, and g is $\pi_1(X, x_0)$-equivarient. Now note the following two equalities.

$$\begin{aligned}
\big(gf\big)(e) &= g\big(f(e)\big) & \qquad \big(fg\big)(e) &= f\big(g(e)\big)\\
&= g\big(e_2[\sigma]\big) & &= f\big(e_1[\tau]\big)\\
&= \big(g(e_2)\big)[\sigma] & &= \big(f(e_1)\big)[\tau]\\
&= e_1[\sigma] & &= e_2[\tau]\\
&= e & &= e
\end{aligned}$$

So we have that $g = f^{-1}$, thus f is a set-theoretic bijection on $\rho^{-1}(x_0)$. Therefore it has been shown that $f \in Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$. It follows that this group acts transitively on the set S, applying *Lemma 5.1* we see that this action is in fact free. Therefore we have a regular left action on S.

A useful fact comes from the above proposition if we assume this group of automorphisms acts transitively on $\rho^{-1}(x_0)$. It will be proven later but is stated as follows: $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ inducing a regular left action on $\rho^{-1}(x_0)$ is equivalent to having the isotropy subgroups of the fiber be normal subgroups. This statement is also true for another group which induces an action on the fiber of a covering space, as will be developed in the next section.

6 Group Action Induced by $Cov(E|X)$

Consider the covering space $E \overset{\rho}{\to} X$. We shall consider the homeomorphisms on the total space with the property of preserving the covering map ρ. This leads to the definition of a particular subgroup of the group of homeomorphism of E.

Definition. Given a covering space $E \overset{\rho}{\to} X$, a *covering transformation* is a homeomorphism $\phi : E \to E$ with the property that $\rho\phi = \rho$. Hence the following diagram commutes,

$$\begin{tikzcd}
E \arrow[rr, "\phi"] \arrow[dr, "\rho"'] & & E \arrow[dl, "\rho"] \\
& X. &
\end{tikzcd}$$

The collection of all such elements under composition is called the *group of covering transformations* and will be denoted by $Cov(E|X)$. This group is also known as the group of *deck transformations*.

6.1 Example. What is $Cov(\mathbb{R}|S^1)$? Showing $\mathbb{R} \overset{\exp}{\to} S^1$ is a covering space was done in *example 2.1*. Note $f \in Cov(\mathbb{R}|S^1)$ if and only if $\exp\big(f(x)\big) = \exp(x)$, hence $e^{2\pi i f(x)} = e^{2\pi i x}$. By the periodicity of exp we know that for this equality to hold $f(x) = x + n$ for some $n \in \mathbb{Z}$. Moreover, if $f, g \in Cov(\mathbb{R}|S^1)$ with $f(x) = x + n$ and $g(x) = x + m$ then the composition is $(f \circ g)(x) = x + (n + m)$, hence $Cov(\mathbb{R}|S^1) \cong \mathbb{Z}$.

With covering transformations defined, it can be noted that this group induces a left group action on $\rho^{-1}(x_0)$. This can be justified by noting that the properties of a group action are satisfied, this is easily verified. Thus, $\rho^{-1}(x_0)$ is a left $Cov(E|X)$-set. Moreover, and as with the automorphisms in *Section 5*, a covering transformation is uniquely determined by what it does to a single point.

6.2 Lemma. *Given a covering space $(E, e_0) \overset{\rho}{\to} (X, x_0)$. If $\phi, \psi \in Cov(E|X)$ are such that $\phi(e_1) = \psi(e_1)$ for some $e_1 \in \rho^{-1}(x_0)$, then $\phi = \psi$.*

Proof. Take any $e_1 \in \rho^{-1}(x_0)$. If $\phi, \psi \in Cov(E|X)$ are such that $\phi(e_1) = e_2$ and $\psi(e_1) = e_2$, then $\phi, \psi : (E, e_1) \to (E, e_2)$ are both lifts of ρ since $\rho\phi = \rho$ and $\rho\psi = \rho$. By *Proposition 3.1* these lifts are unique, hence $\phi = \psi$.

Now we show, as we did with $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$, that $Cov(E|X)$ induces a regular left action on the set S as defined in *formulas 2* and *3*.

6.3 Proposition. *Let* $(E, e_0) \xrightarrow{\rho} (X, x_0)$ *be a covering space. Then* $Cov(E|X)$ *induces a regular left action on the set* S*.*

Proof. Assume $e_1 \in S$ and $\phi \in Cov(E|X)$. By the properties of ϕ we have the following,

$$(\rho\phi)_{\#}\pi_1(E, e_1) = \rho_{\#}\pi_1(E, e_1).$$

By assumption we have that $\rho_{\#}\pi_1(E, e_1) = \rho_{\#}\pi_1(E, e_0)$, also by the properties of ϕ we get $(\rho\phi)_{\#}\pi_1(E, e_1) = \rho_{\#}\pi_1(E, \phi(e_1))$. So $\phi(e_1) \in S$ by *formula 3* hence $Cov(E|X)$ acts on the left of S.

Now assume that $e_1, e_2 \in S$. We consider the following diagram,

$$\begin{tikzcd}[column sep=small] (E, e_1) \arrow[rr, dashed, no head] \arrow[dr, "\rho"'] & & (E, e_2) \arrow[dl, "\rho"] \\ & (X, x_0). & \end{tikzcd}$$

Since, trivially, $\rho_{\#}\pi_1(E, e_1) \subseteq \rho_{\#}\pi_1(E, e_2)$, *Theorem 3.5* gives a unique lifting of ρ, say ϕ, taking (E, e_1) into (E, e_2). We also have, trivially, $\rho_{\#}\pi_1(E, e_1) \supseteq \rho_{\#}\pi_1(E, e_2)$, so *Theorem 3.5* gives a unique lifting of ρ taking (E, e_2) into (E, e_1) which must be ϕ^{-1} in light of *Proposition 3.1*. So it has been shown that $\phi : (E, e_1) \to (E, e_2)$ is a unique homeomorphism, moreover, $\phi\rho = \rho$ by construction. Therefore ϕ is a covering transformation taking e_1 to e_2. Thus $Cov(E|X)$ acts transitively on the set S, moreover, by *Lemma 6.2* this action is free. It follows that this is a regular left action.

We will now make a distinction between two types of covering spaces by defining what it is to be a regular covering space.

Definition. If $Cov(E|X)$ induces a regular left action on $\rho^{-1}(x_0)$, then the corresponding covering space $(E, e_0) \xrightarrow{\rho} (X, x_0)$ is called *regular*.

Take a regular covering spaces, it follows that the set S is equal to the fiber since transitivity implies for any $e_1, e_2 \in \rho^{-1}(x_0)$ there is a covering transformation ϕ such that $\phi(e_1) = e_2$ and *Proposition 6.3* shows the isotropy subgroup of these elements to be the same. This allows for an equivalent definition which distinguishes a regular covering space as one in which the isotropy subgroups of the fiber are normal subgroups.

6.4 Proposition. *Let* $(E, e_0) \xrightarrow{\rho} (X, x_0)$ *be a covering space. Then this is a regular covering space if and only if* $\rho_{\#}\pi_1(E, e_0)$ *is a normal subgroup of* $\pi_1(X, x_0)$*.*

Proof. Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a regular covering space, then $\rho^{-1}(x_0) = S$ (as defined by *formula 3*). Thus, by *Proposition 4.4*, it is clear that $\rho_{\#}\pi_1(E, e_0)$ is a normal subgroup of $\pi_1(X, x_0)$. Conversely, if $\rho_{\#}\pi_1(E, e_0)$ is a normal subgroup of $\pi_1(X, x_0)$, then *Proposition 4.4* shows that $S = \rho^{-1}(x_0)$. It now follows from *Proposition 6.3* that $Cov(E|X)$ acts regularly on $\rho^{-1}(x_0)$.

6.5 Example. Take $\mathbb{R} \overset{\exp}{\to} S^1$, which was shown to be a covering space in *example 2.1.* Clearly $\exp_{\#}\pi_1(\mathbb{R})$ is a normal subgroup of $\pi_1(S^1) \cong \mathbb{Z}$ since $\mathbb{Z}$ is abelian (all subgroups are normal). Thus this is a regular covering space.

In the example above one can alternatively note that $\mathbb{R}$ is simple connected, hence has a trivial fundamental group which is mapped to the trivial subgroup subgroup of $\pi_1(S^1)$ in light of *Corollary 3.4*, thus is trivially normal. This leads us to define a special type of regular covering space.

Definition. A *universal covering* is a regular covering space in which the fundamental group of the total space is trivial; $\widetilde{X} \overset{\rho}{\to} X$ is a covering space with $\pi_1(\widetilde{X}) = 1$.

The left group action on $\rho^{-1}(x_0)$, induced by the group of covering transformations, has an important property in regards to the right action induced by $\pi_1(X, x_0)$. We follow an argument from [9] which observes that covering transformations, as with the automorphisms of $\rho^{-1}(x_0)$, are $\pi_1(X, x_0)$-equivariant.

6.6 Proposition. *Let $(E, e_0) \overset{\rho}{\to} (X, x_0)$ be a covering space. For any $\phi \in Cov(E|X)$, any $e \in \rho^{-1}(x_0)$, and any $[\sigma] \in \pi_1(X, x_0)$, we have $\phi\big(e[\sigma]\big) = \big(\phi(e)\big)[\sigma]$. In other words, these actions are compatible.*

Proof. Pick any $\phi \in Cov(E|X)$, $e \in \rho^{-1}(x_0)$, and $[\sigma] \in \pi_1(X, x_0)$. Let σ'_e be the unique lift of σ starting at e. We have the terminal point of σ'_e as $e[\sigma]$. Since ϕ is continuous we can view $\phi(\sigma'_e)$ as a path in E with initial point $\phi(e)$ and terminal point $\phi\big(e[\sigma]\big)$. Now note the following,

$$\rho\big(\phi(\sigma'_e)\big) = \big(\rho\phi\big)(\sigma'_e) = \rho(\sigma'_e) = \sigma.$$

So $\phi(\sigma'_e)$ is the unique lift of σ starting at $\phi(e)$. Thus we have that $\big(\phi(e)\big)[\sigma]$ is the terminal point of $\phi(\sigma'_e)$. Hence we have the equality we were after, $\phi\big(e[\sigma]\big) = \big(\phi(e)\big)[\sigma]$.

The above proposition shows not only the compatibility of $Cov(E|X)$ and $\pi_1(X, x_0)$ but it also illustrates a third similarity between $Cov(E|X)$ and $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$. This leads us to look for a possible correspondence between $Cov(E|X)$ and $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$, when viewed as groups of operators acting on the set $\rho^{-1}(x_0)$.

7 Correlation of Group Actions

To recap, three group actions have been defined on the fiber of a covering space. We now show that these groups can all be represented by the fundamental group of the base space. The first step is to show that there is a correspondence which induces an isomorphism between the group of covering transformations, $Cov(E|X)$, and the group of automorphisms, $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$.

7.1 Theorem. *Let $(E, e_0) \overset{\rho}{\to} (X, x_0)$ be a covering space. We have the following isomorphism,*

$$Cov(E|X) \cong Aut_{\pi_1}\big(\rho^{-1}(x_0)\big).$$

Proof. *Proposition 6.6* showed that all covering transformations are $\pi_1(X, x_0)$-equvariant. Now take $\phi \in Cov(E|X)$ and send it to its restriction $\phi\big|_{\rho^{-1}(x_0)}$, this is clearly a well-defined function taking ϕ into $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$. Moreover, this function is a homomorphism, for given any two $\phi, \psi \in Cov(E|X)$ we have,

$$\psi\phi \mapsto (\psi\phi)\big|_{\rho^{-1}(x_0)} = \psi\big|_{\rho^{-1}(x_0)}\phi\big|_{\rho^{-1}(x_0)}.$$

The kernel of this function is the set,

$$\{\phi \in Cov(E|X)|\ \phi|_{\rho^{-1}(x_0)}(e) = e, \text{ for all } e \in \rho^{-1}(x_0)\},$$

which by *Lemma 6.2* must contain only the identity covering transformation. Note, for any $f \in Aut_{\pi_1}\big(p^{-1}(x_0)\big)$ and for any $e_1 \in S$, $f(e_1) \in S$. Now use *Proposition 6.3* to get the unique $\phi \in Cov(E|X)$ that takes e_1 to $f(e_1)$, which means $\phi|_{\rho^{-1}(x_0)}(e_1) = f(e_1)$. Since $\phi|_{\rho^{-1}(x_0)} \in Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ we must have that $\phi|_{\rho^{-1}(x_0)} = f$ by *Lemma 5.1*. Therefore we have an isomorphism between $Cov(E|X)$ and $Aut_{\pi_1}\big(p^{-1}(x_0)\big)$.

We can now create a correspondence between the group of covering transformations and the fundamental group $\pi_1(X, x_0)$. To do this we first define, for a subgroup $H \leq \pi_1(X, x_0)$, the *normalizer* of H to be the largest subgroup of $\pi_1(X, x_0)$ in which H is a normal subgroup of. The normalizer is denoted by,

$$N(H) := \{[\sigma] \in \pi_1(X, x_0)\big|[\sigma]H[\sigma]^{-1} = H\}.$$

Now observe, for $e_0 \in \rho^{-1}(x_0)$, that the quotient group $N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$ induces a regular right action on the set S as defined by *formulas 2* and *3*.

7.2 Lemma. *Given a covering space $(E, e_0) \xrightarrow{\rho} (X, x_0)$, view $\rho^{-1}(x_0)$ as a right $\pi_1(X, x_0)$-set. Then $N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$ induces a regular right action on the set S.*

Proof. We aim first to show that S is a transitive right $N\big(\rho_{\#}\pi_1(E, e_0)\big)$-set. Let $e \in S$ and let $[\sigma] \in \pi_1(X, x_0)$. We note the following equivalent statement: $e[\sigma] \in S$ if and only if $\pi_1(X, x_0)_{e_0} = \pi_1(X, x_0)_{e[\sigma]}$. *Proposition 1.3* shows $\pi_1(X, x_0)_{e[\sigma]} = [\sigma]\pi_1(X, x_0)_e[\sigma]^{-1}$, but since $e[\sigma]$ and e were assumed to be in S, they both have the same isotropy subgroup, hence $[\sigma]$ must be in the normalizer of $\pi_1(X, x_0)_e = \rho_{\#}\pi_1(E, e_0)$. Thus S is a transitive right $N\big(\rho_{\#}\pi_1(E, e_0)\big)$-set.

Next note that the kernel of this action is $\rho_{\#}\pi_1(E, e_0)$. Thus, if we consider the quotient group $N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$, we will yield a group which preserves the transitive right action on S, induced by the normalizer, but will also be free. Therefore this is a regular left action on the set S.

Now to the main theorem of this section. Following [9, 13], this next theorem shows that the algebraic structure of the group of covering transformations is determined by the isotropy subgroup of a chosen base point and its normalizer.

7.3 Theorem. *Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a covering space. We have the following isomorphism,*

$$Cov(E|X) \cong N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0),$$

where $N\big(\rho_{\#}\pi_1(E, e_0)\big)$ is the normalizer of the isotropy subgroup of $e_0 \in \rho^{-1}(x_0)$, $p_{\#}\pi_1(E, e_0)$.

Proof. In light of *Theorem 7.1*, it suffices to show that $N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$ is isomorphic to $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$. Since both of these groups act regularly on the set S, it is clear that each $f \in Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ is identified with a unique $[\sigma] \in N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$ by defining $f(e) := e[\sigma]$ (as was done in *Proposition 5.2*) where σ is a path between e and $f(e)$. Also, each $[\sigma]$ in $N\big(\rho_{\#}\pi_1(E, e_0)\big)/\rho_{\#}\pi_1(E, e_0)$ is identified with a unique f in

$Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ by setting $e[\sigma] := f(e)$. Hence we have a one-to-one correspondence. Furthermore, this correspondence is in fact an isomorphism, for assume the following equalities: $f(e) = e[\sigma]$ and $g(e) = e[\tau]$. Then,

$$\big(fg\big)(e) = f\big(g(e)\big) = f\big(e[\tau]\big) = f(e)[\tau] = \big((e)[\sigma]\big)[\tau] = e[\sigma][\tau],$$

which shows that products are preserved. Therefore we have an isomorphism between the quotient group $N\big(\rho_{\#}\pi_1(E, e_0)\big)\big/\rho_{\#}\pi_1(E, e_0)$ and the automorphism group $Aut_{\pi_1}\big(\rho^{-1}(x_0)\big)$ as was to be shown.

Here is a natural question to consider, when the normalizer of the isotropy subgroup of a point in the fiber is a normal subgroup what can be said about the corresponding group of covering transformations? Recall a covering space with this property is called a regular covering space. Utilizing this definition allows for the following two corollaries.

7.4 Corollary. *Let $(E, e_0) \xrightarrow{\rho} (X, x_0)$ be a regular covering space. Then we have the following,*

$$Cov(E|X) \cong \pi_1(X, x_0)/\rho_{\#}\pi_1(E, e_0).$$

Proof. By definition $N\big(\rho_{\#}\pi_1(E, e_0)\big)$ is equal to $\pi_1(X, x_0)$, after applying *Theorem 7.3* we have that $Cov(E|X)$ is isomorphic to the quotient group $\pi_1(X, x_0)\big/\rho_{\#}\pi_1(E, e_0)$.

The final corollary shows that a universal covering induces a group of covering transformations that is isomorphic to the fundamental group of the base space. First, recall that a universal covering is a spacial case of a regular covering space.

7.5 Corollary. *Let $(\widetilde{X}, \widetilde{x}_0) \xrightarrow{\rho} (X, x_0)$ be a universal covering. Then we have the following,*

$$Cov(\widetilde{X}|X) \cong \pi_1(X, x_0).$$

Proof. Universal coverings are regular covering spaces so we can apply *Corollary 7.4* to get that $Cov(\widetilde{X}|X)$ is isomorphic to $\pi_1(X, x_0)\big/\rho_{\#}\pi_1(\widetilde{X}, \widetilde{x}_0)$. Add the property that universal covering spaces have trivial fundamental groups, $\pi_1(\widetilde{X}, \widetilde{x}_0) = [\widetilde{x}_0]$, yields the needed result, $Cov(\widetilde{X}|X) \cong \pi_1(X, x_0)$.

7.6 Example. What is $Cov(S^n|RP^n)$ for $n \geq 2$? Showing $S^n \xrightarrow{\mathfrak{a}} RP^n$ is a covering space was done in *example 2.6*, one can also easily verify that this is a universal covering. Moreover, *equation 1* gives $|\rho^{-1}(x_0)| = [\pi_1(RP^n) : \rho_{\#}\pi_1(S^n)]$. Applying what we know, $|\rho^{-1}(x_0)| = 2$ and that $\pi_1(S^n)$ is trivial, gives $2 = |\pi_1(RP^n)|$. So it must be that $\pi_1(RP^n) \cong \mathbb{Z}/2\mathbb{Z}$. Furthermore, *Corollary 7.5* shows $Cov(S^n|RP^n) \cong \pi_1(RP^n)$, therefore $Cov(S^n|RP^n) \cong \mathbb{Z}/2\mathbb{Z}$.

With the developments made in this section, we are able to see that a universal covering of X has a group of covering transformations that is isomorphic to the fundamental group of X. The next section will make clear that any covering space of X can be realized as a quotient of this universal covering space by a particular subgroup of covering transformations.

8 Equivalence of Covering Spaces

We noted that *example 6.5* showed S^1 has a universal covering space, $\mathbb{R}$. In fact all spaces, assuming the nice connectedness properties, have universal coverings. In other words, universal coverings exist for all our spaces. The proof of this will be omitted but can be found in [4, 5, 9, 11]. With this fact we shall follow [9, 13] and develop an important feature of covering spaces; covering spaces are determined (up to an equivalence) by the subgroups of the base spaces fundamental group. To show this we first define what it means for two covering spaces to be equivalent.

Definition. Let $E \xrightarrow{\rho} X$ and $E' \xrightarrow{\rho'} X'$ be covering spaces. We say that these covering spaces are *equivalent* if there are homeomorphisms $\Phi : E \to E'$ and $\Psi : X \to X'$ such that $\rho'\Phi = \Psi\rho$. In other words, the following diagram commutes,

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\rho"'] & E' \arrow[d, "\rho'"] \\ X \arrow[r, "\Psi"'] & X'. \end{tikzcd}$$

The definition of equivalence above can be broken down two fold. If $E = E'$ and $\Phi = Id$ then we have an equivalence of base spaces, and if $X = X'$ with $\Psi = Id$ then we have an equivalence of total spaces. Let us consider an equivalence of base spaces, particularly let us show that, given a regular covering space, the base space is equivalent to the orbit space of the total space by the group of covering transformations.

8.1 Lemma. *Given any regular covering space $E \xrightarrow{\rho} X$ with $G = Cov(E|X)$, then there is a homeomorphism $\Psi : X \to E/G$ making the following diagram commute,*

$$\begin{tikzcd} E \arrow[d, "\rho"'] \arrow[dr, "\overline{\rho}"] & \\ X \arrow[r, dashed, "\Psi"'] & E/G. \end{tikzcd}$$

Moreover, the projection map $\overline{\rho}$ induces a covering space.

Proof. For $x \in X$, choose $e \in \rho^{-1}(x)$ and define $\Psi : X \to E/G$ by $\Psi(x) = eG\ \big(= \overline{\rho}(e)\big)$ which is well-defined since G acts transitively on $\rho^{-1}(x_0)$. Furthermore, if $U \subset E/G$ is open then $\overline{\rho}^{-1}(U) = \rho^{-1}\Psi^{-1}(U)$ is open in E, and since ρ^{-1} is a local homeomorphism, we have that $\rho\overline{\rho}^{-1}(U) = \Psi^{-1}(U)$ is open in X, hence $\Psi : X \to E/G$ is a well-defined map.

It can be noted that Ψ is surjective since $\Psi\rho = \overline{\rho}$ and since both ρ and $\overline{\rho}$ are surjective. If $\Psi(x_1) = \Psi(x_2)$, then we have $e_1 \in \rho^{-1}(x_1)$ and $e_2 \in \rho^{-1}(x_2)$ such that $e_1G = e_2G$. Thus there is a $\phi \in G = Cov(E|X)$ with $\phi(e_2) = e_1$. So $x_1 = \rho(e_1) = \rho(\phi(e_2)) = \rho(e_2) = x_2$, thus Ψ is injective, hence a bijection.

Now take an open set V in X, then $\overline{\rho}^{-1}\Psi(V) = \rho^{-1}(V)$ is open in E thus $\Psi(V) = \overline{\rho}\rho^{-1}(V)$ is open in E/G. Therefore Ψ is a homeomorphism and we have the following commuting diagram,

$$\begin{tikzcd} E \arrow[r, "Id"] \arrow[d, "\rho"'] & E \arrow[d, "\overline{\rho}"] \\ X \arrow[r, "\Psi"'] & E/G. \end{tikzcd}$$

It now follows from *Proposition 2.4* that $E \xrightarrow{\overline{\rho}} E/G$ is a covering space.

The above lemma shows that a regular covering space $E \xrightarrow{\rho} X$ is equivalent to $E \xrightarrow{\overline{\rho}} E/G$ where $G = Cov(E|X)$. Let us continue by developing an equivalence of intermediate covering spaces of which are not regular.

8.2 Lemma. *Let $E \overset{\rho}{\to} X$ and $E \overset{\nu}{\to} Y$ be regular covering spaces and let $Y \overset{\mu}{\to} X$ be a covering space (not necessarily regular) all of which make the following diagram commute,*

$$\begin{tikzcd} E \arrow[dr, "\nu"] \arrow[dd, "\rho"'] & \\ & Y. \arrow[dl, "\mu"] \\ X & \end{tikzcd}$$

Then, for $G = Cov(E|X)$ and $H = Cov(E|Y)$, we have the following commuting diagram of covering spaces,

$$\begin{tikzcd} E \arrow[dr, "\overline{\nu}"] \arrow[dd, "\overline{\rho}"'] & \\ & E/H. \arrow[dl, "\overline{\mu}"] \\ E/G & \end{tikzcd}$$

Moreover, $X \cong E/G$ and $Y \cong E/H$, so the diagrams are equivalent.

Proof. Let $G = Cov(E|X)$ and $H = Cov(E|Y)$. By *Lemma 8.1*, we have that $X \cong E/G$ and $Y \cong E/H$, so we have an equivalent diagram,

$$\begin{tikzcd} E \arrow[dr, "\overline{\nu}"] \arrow[dd, "\overline{\rho}"'] & \\ & E/H\ . \\ E/G & \end{tikzcd}$$

What is left to be shown is that E/H over E/G under some map is a covering space. Note that if $\phi \in H$ then $\nu\phi = \nu$ and so $\rho\phi = \mu\nu\phi = \mu\nu = \rho$, thus $H \leq G$. So we consider the projection map $\overline{\mu} : E/H \to E/G$ given by $eH \mapsto eG$. Furthering this, we also clearly have $\overline{\mu}\overline{\nu} = \overline{\rho}$. Showing the function $\overline{\mu}$ is continuous comes from noting, given an open set $U \subset E/G$, that $\overline{\nu}^{-1}\overline{\mu}^{-1}(U) = \overline{\rho}^{-1}(U)$ is open in E. Then, since ν is a local homeomorphism, we get $\overline{\mu}^{-1}(U) = \nu\overline{\rho}^{-1}(U)$ is open in E/H, thus $\overline{\mu}$ is continuous. Now we have the following commuting diagram,

$$\begin{tikzcd} Y \arrow[r, "\Phi"] \arrow[d, "\mu"'] & E/H \arrow[d, "\overline{\mu}"] \\ X \arrow[r, "\Psi"'] & E/G. \end{tikzcd}$$

Which, by *Proposition 2.4*, means that $\overline{\mu}$ induces a covering space.

8.3 Proposition. *Take any universal covering $\widetilde{X} \overset{\rho}{\to} X$. Then every total space over X is equivalent to $\widetilde{X}/H$ for some subgroup $H \leq Cov(\widetilde{X}|X)$.*

Proof. Let $E \overset{\nu}{\to} X$ be any covering space. By *Theorem 3.5* there is a unique lift of ρ, which makes the following diagram of commute,

$$\begin{tikzcd} \widetilde{X} \arrow[dr, "\rho'"] \arrow[dd, "\rho"'] & \\ & E. \arrow[dl, "\nu"] \\ X & \end{tikzcd}$$

Now we note that $\widetilde{X} \overset{\rho'}{\to} E$ is a covering space by *Proposition 2.3*. Furthermore, $\widetilde{X}$ is simply connected, thus ρ and ρ' induce regular covering spaces. Therefore, by *Lemma 8.2*, E is homeomorphic to $\widetilde{X}/H$ where $H = Cov(\widetilde{X}|E)$.

Given any space, X, we know that there exists a universal covering space, $\widetilde{X}$. Thus the proposition above allows us to consider *every* total space over X as a quotient space of this universal covering space by a subgroup of covering transformations. By considering all the total spaces over X to be of the form $\widetilde{X}/H$ for some $H \leq Cov(E|X)$, we are able to follow [13] and develop a classification of covering spaces.

8.4 Theorem. *Let $\widetilde{X} \overset{\rho}{\to} X$ be a universal covering. Denote $\mathcal{C} = \{\widetilde{X}/H : H \leq Cov(\widetilde{X}|X)\}$ and $\mathcal{G} = \{H : H \leq Cov(\widetilde{X}|X)\}$. There are maps,*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\Phi"] & \mathcal{G}, \arrow[l, bend left, "\Psi"] \end{tikzcd}$$

where $\Phi(\widetilde{X}/H) = Cov(\widetilde{X}|\widetilde{X}/H)$ and $\Psi(H) = \widetilde{X}/H$. Moreover, these are set-theoretic bijections and inverses of one another.

Proof. Let $H \leq Cov(\widetilde{X}|X)$, then $\Phi\Psi(H) = Cov(\widetilde{X}|\widetilde{X}/H)$ which is the group of covering transformations of the covering space $\widetilde{X} \overset{\rho}{\to} \widetilde{X}/H$. For any $\phi \in H$ and $\widetilde{x} \in \widetilde{X}$ we get $\widetilde{x}H = \phi(\widetilde{x})H$, thus $\rho\phi = \rho$. Hence $H \leq Cov(\widetilde{X}|\widetilde{X}/H)$.

Now take $\psi \in Cov(\widetilde{X}|\widetilde{X}/H)$. If $\widetilde{x} \in \widetilde{X}$ then $\widetilde{x}H = \psi(\widetilde{x})H$, thus there is a $\phi \in H$ with $\phi\psi(\widetilde{x}) = \widetilde{x}$. Since $H \leq Cov(\widetilde{X}|\widetilde{X}/H)$, it follows that we must have $\phi\psi \in Cov(\widetilde{X}|\widetilde{X}/H)$. Since covering transformations are determined by what they do to one point, we have that $\phi\psi$ is the identity map, hence $\phi = \psi^{-1} \in H$. It now follows that $H = Cov(\widetilde{X}|\widetilde{X}/H)$.

Finally, consider $\Psi\Phi$. This gives a composition $\widetilde{X}/H \mapsto Cov(\widetilde{X}|\widetilde{X}/H) = H \mapsto \widetilde{X}/H$ where the middle equality was proven above. Thus we have that Φ and Ψ are inverses of one another. Therefore there is a set-theoretic bijection between $\mathcal{C}$ and $\mathcal{G}$ as defined above.

8.5 Corollary. *Let $\widetilde{X} \overset{\rho}{\to} X$ be a universal covering. Then for any $H \leq Cov(E|X)$ we have $\pi_1(\widetilde{X}/H) \cong H$.*

Proof. With the above conditions, and using the maps of *Theorem 8.4*, we have $H = \Phi\Psi(H)$ and $\Phi\Psi(H) = Cov(\widetilde{X}|\widetilde{X}/H)$. Furthermore, $Cov(\widetilde{X}|\widetilde{X}/H) \cong \pi_1(\widetilde{X}/H)$ by *Corollary 7.5*. Therefore $\pi_1(\widetilde{X}/H) \cong H$.

The last theorem gives a way to find all the total spaces over a given base space X. Moreover, since $Cov(\widetilde{X}|X) \cong \pi_1(X, x_0)$ by *Theorem 7.5*, we have actually shown that the total spaces over X are determined (up to homeomorphism) by the subgroups of the fundamental group of X. Let's put this to use by classifying all the covering spaces of the circle.

8.6 Example. Classify all the total spaces over S^1. Recall that $\mathbb{R}$ was shown to be a universal covering space over S^1. Also, *example 6.1* showed that $Cov(\mathbb{R}|S^1) \cong \mathbb{Z}$. Apply *Theorem 8.4* to get that any total space is equivalent to $\mathbb{R}/H$ where H is a subgroup of $\mathbb{Z}$ thus $H = \mathbb{Z}/n\mathbb{Z}$ for some $n \in \mathbb{Z}$. Note $\mathbb{R}/\mathbb{Z}/n\mathbb{Z} \cong S^1$. Thus $\mathbb{R}$ and S^1 are the only total spaces over S^1. Moreover, this allows us to classify all the covering spaces over S^1 as being $(\mathbb{R}, \exp)$ and $(\mathbb{R}/\mathbb{Z}/n\mathbb{Z}, \rho_n)$ for all nonzero $n \in \mathbb{Z}$ as was developed in *exercise 2.5*. Furthermore, *Corollary 8.5* shows that $\pi_1\big(\mathbb{R}/\mathbb{Z}/n\mathbb{Z}\big) \cong \mathbb{Z}/n\mathbb{Z}$.

9 Branched Covering Spaces

Take any finite group G which acts on E. What can be said about the orbit space E/G? In particular, does this fit in with covering spaces? It turns out that $\rho : E \to E/G$ is not necessarily a covering space. Take the following as a counter example.

9.1 Example. let $G = \mathbb{Z}/n\mathbb{Z}$, for nonzero $n \in \mathbb{Z}$, act on the complex plane $\mathbb{C}$ via counter-clockwise rotation about the origin by a degree of $2\pi/n$. Now consider the projection map $\rho_n : \mathbb{C} \to \mathbb{C}/G$, if this is a covering space then for any two $z_1, z_2 \in \mathbb{C}/G$ we would have $|\rho^{-1}(z_1)| = |\rho^{-1}(z_2)|$ (recall *Proposition 4.3*). But if we let $z_1 = 0$ and $z_2 \neq 0$, then we see that $|\rho^{-1}(z_1)| = 1$ and $|\rho^{-1}(z_2)| = n$. Thus ρ_n fails to induce a covering space.

In the above example, if we remove the point $z = 0$ from the domain, then what we end up with is $\rho_n : \mathbb{C}\backslash 0 \to \mathbb{C}\backslash 0$. Since S^1 is a strong deformation retract of $\mathbb{C}\backslash 0$ we can utilizing *Theorem 3.2* to show that these covering spaces can be described by $S^1 \overset{\rho_n}{\to} S^1$, now appeal to *example 8.6* to get the classification of these covering spaces.

Moreover, each point of $\mathbb{C}$ has an orbit under this action of order $[G : G_z]$, where G_z is isotropy subgroup of z. It happens to be that the point $z = 0$ has $|G_0| = n$ but for all other $z \in \mathbb{C}$ we have $|G_z| = 1$. A final note on this example, in order to find that the point $z = 0$ was a "bad" point we had to look at its image's pullback. This leads us to define two terms.

Definition. Let G act on a space E. If $e \in E$ and $|G_e| > 1$, then e is called a *singular point*. The image of e, $\rho(e)$, is called a *branch point*.

If G is any group acting on a space E ,then E/G has the quotient topology induced by the projection map $\rho : E \to E/G$. Also E/G will be nicely connected since E is nicely connected and ρ is continuous, thus E/G is a space (as we have defined them). We follow with another definition, a definition which generalizes that of a covering space.

Definition. Let E and X be two spaces. We call $\rho : E \to X$ a *branched covering* if there exists a co-dimension 2 subset $B \subset X$ such that $E\backslash B^{-1} \overset{\rho}{\to} X\backslash B$ is a covering space. We call B the *branch set* and $B^{-1} := \rho^{-1}(B)$ the *singular set*.

If the resulting covering space from above is regular, then we have a *regular branched covering*. Notice, a covering space is just a special case of a branched covering. To see this note, given a covering space $E \overset{\rho}{\to} X$, that $B = \emptyset$ which is vacuously a co-dimension 2 subset of X and that the singular set is also empty. By removing these sets from their respected spaces, nothing is topologically changed, thus we are left with a covering space. The concept of branched coverings becomes very useful if we require one additional property for these groups, as will be seen in the next theorem due to [4, 5, 13], the theorem will follow after we define this property.

Definition. A group G acts *properly discontinuously* on a space E if for each $e \in E$ there is an open neighborhood U of e such that $U \cap Ug = \emptyset$ for all $g \in G$ $(g \neq 1)$.

9.2 Theorem. *Given a group G that acts on E such that $\rho : E \to E/G$ is a branched covering. If G acts properly discontinuously on $E\backslash B^{-1}$, where B^{-1} is the singular set, then $\rho : E \to E/G$ is a regular branched covering.*

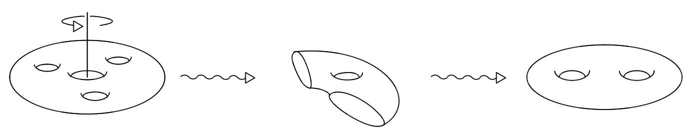
Proof. The group G can be viewed as a subgroup of covering transformations since each $g \in G$ can be regarded as a homeomorphism on $E\backslash B^{-1}$ with the property $\rho g = \rho$. Now take any $eG \in \big(E/G\big)\backslash B$ with B the branch set, then $\rho^{-1}(eG) = \{e_1 \in E | e_1 g = e \text{ for some } g \in G\}$ is a discrete set on which G clearly acts transitively. Hence $Cov(E|E/G)$ acts transitively on the fiber. Therefore, $\rho : E \to E/G$ is a regular branched covering.

Let us now consider some examples which demonstrates how branched coverings of surfaces can be used to study the symmetries of a surface. But first let us clarify what we mean by a surface.

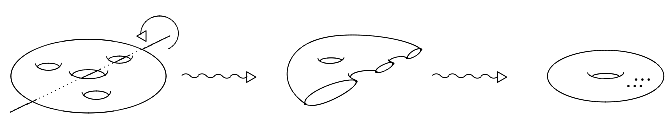
Definition. A *surface* will be defined as a 2-dimensional compact, orientable manifold without boundary. Denote a surface by Σ_g where $g \geq 0$ is the genus. If G is a group that acts on Σ_g, then we will denote the orbit space by $\Sigma_{h,b}$ where h is the genus and $b = |B|$, the order of the branch set. It is understood that $\Sigma_{h,0} = \Sigma_h$.

The following, adapted from [5, 6], shows how branched coverings can appear when you consider looking at a symmetry group acting on a given surface.

9.3 Example. Consider $G = \mathbb{Z}/3\mathbb{Z}$ acting on Σ_4 via rotation by an angle of $2\pi/3$ about the indicated axis as pictured on the left. We identify a fundamental domain under this action (center) and then identify the corresponding boundaries together, thus gluing together the ends. Doing this yields Σ_2 (right). This is a branched covering, $\rho : \Sigma_4 \to \Sigma_2$ which has a branch set of order 0, clearly no points are fixed under this action. Hence $\Sigma_4 \xrightarrow{\rho} \Sigma_2$ is a covering space.



9.4 Example. We can also let $G = \mathbb{Z}/2\mathbb{Z}$ act on Σ_4 via rotation by an angle of π (as pictured on the left). We identify a fundamental domain (center) and then identify the corresponding boundaries together, thus closing up the openings. Doing this yields $\Sigma_{1,6}$ (right). In this case we have the branched covering $\rho : \Sigma_4 \to \Sigma_{1,6}$, notice that the order of the branch set is 6.



10 The Riemann-Hurwitz Relation

There is a well known topological invariant of a topological space called the Euler Characteristic. This invariant has an important connection to branched coverings as will be shown.

Definition. The *Euler Characteristic* of a space X is given by $\chi(X) := \sum_n (-1)^n \text{rank} H_n(X; \mathbb{Z})$ where $H_n(X; \mathbb{Z})$ is the n^{th} homology group of X.

If we let $X = \Sigma_g$ with $g \geq 0$ (*i.e.* X is a surface) then the Euler Characteristic is found using a triangulation of the surface. In this case we see that $\chi(\Sigma_g) = F_{aces} - E_{dges} + V_{erticies}$, or equivalently $\chi(\Sigma_g) = 2 - 2g$. We are able to relate the Euler Characteristic to branched coverings of surfaces by noting that this *is* a topological invariant since its definition is in terms of homology [4], thus is independent of the choice of triangulation made on a surface. This allows one the ability to prove the following well known relationship between two surfaces where one is a branched covering of the other.

Riemann-Hurwitz Relation. *Given a branched covering $\rho : \Sigma_g \to \Sigma_{h,b}$. We have the relation,*

$$\chi(\Sigma_g) = |G|\chi(\Sigma_{h,b}) - D_{h,b},$$

where $D_{h,b} = |G| \sum_{x \in B}(1 - \frac{1}{n_x})$ with $n_x = |G_y|$ for $y \in \rho^{-1}(x)$ and $x \in B$.

Proof. Take a sufficiently small triangulation of $\Sigma_{h,b}$ so that each triangle is contained in an evenly covered neighborhood and such that if $x \in B$ then x is a vertex of this triangulation. Denote $\chi(\Sigma_{h,b}) = F - E + V$ where F, E, and V are the number of faces, edges, and verticies (respectively) of the chosen triangulation. Since ρ is surjective, the pullback of this triangulation is clearly a triangulation of Σ_g, thus we just need to count the number of faces, edges, and verticies to calculate the Euler Characteristic of Σ_g.

Each evenly covered neighborhoods pullback will contain $|G|$ copies of the triangle contained within it. Thus we have $|G|$ faces and $|G|$ edges. Ideally we would expect there to be $|G|$ verticies, but at each branch point, the pullback is missing $|G| - |yG|$ vericies from this expectation. Thus, accounting for this defect we yield the equation,

$$\chi(\Sigma_g) = |G|F - |G|E + |G|V - \sum_{x \in \Sigma_{h,b}} (|G| - |yG|).$$

G is finite so $|yG| = |G|/|G_y|$. Define $n_x := |G_y|$ for $y \in \rho^{-1}(x)$ and $x \in B$, note that n_x is well-defined by *Proposition 1.3*, also notice that $n_x \neq 1$ if and only if $x \in B$. Thus, with a slight rearranging of terms and re-indexing the summation, we can define the following,

$$D_{h,b} := |G| \sum_{x \in B} (1 - \frac{1}{n_x}).$$

Therefore we have what was needed to be shown,

$$\chi(\Sigma_g) = |G|\chi(\Sigma_{h,b}) - D_{h,b}.$$

The term $D_{h,b}$ in the *Riemann-Hurwitz Relation* is called the *branched defect* of the branched covering and will now be used to help create an upper bound on the order of a group which can act on a given surface of genus $g \geq 2$. This upper bound is well known from *Hurwitz's Automorphisms Theorem* [16], we developed it as follows.

Hurwitz's Inequality. *Given a finite group G which acts on Σ_g for $g \geq 2$, then the order of G can be bounded above as follows,*

$$|G| \leq 84(g - 1).$$

Proof. Let G act on Σ_g for $g \geq 2$. It is clear that $\rho : \Sigma_g \to \Sigma_{h,b}$ is a branched covering, let us look at the branched defect $D_{h,b}$ and considering some cases:

Case 1: $D_{h,b} = 0$. Then it must be that $b = 0$, so the *Riemann-Hurwitz Relation* becomes $2g - 2 = |G|(2h - 2)$. Since $g \geq 2$ it is clear from the last equality that $h \geq 2$. Thus $(g - 1) \geq |G|$.

Case 2: $D_{h,b} \neq 0$. Then it is necessary for $D_{h,b} \geq \frac{b}{2}|G|$. Thus the *Riemann-Hurwitz Relation* becomes $2g - 2 \geq |G|(2h - 4 + \frac{b}{2})$. With some rearranging we get,

$$\frac{4(g - 1)}{(4h - 4 + b)} \geq |G|,$$

as long as $4h-4+b>0$, which happens if $h\geq 1$ or if $h=0$ and $b\geq 5$. In both of these cases we get $4(g-1)\geq |G|$.

Now consider $h=0$ and $b\in\{1,2,3,4\}$. If $b\in\{1,2\}$ then the *Riemann-Hurwitz Relation* does not hold, so these values of b cannot happen. If $b=4$ then *Riemann-Hurwitz Relation* gives $2(g-1)=|G|\big(2-(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4})\big)$. To get an upper bound we must make $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}+\frac{1}{n_4}$ as large as possible and strictly less that 2. Setting $n_i=2$ for $i=1,2,3$ and $n_4=3$ works, and shows $12(g-1)\geq |G|$.

This leaves us with the case where $h=0$ and $b=3$. Plugging these into the *Riemann-Hurwitz Relation* we get,

$$2(g-1)=|G|\big(1-(\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3})\big).$$

For this equation to give an upper bound we must make $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}$ as large as possible and strictly less than 1. It turns out that setting $n_1=2$, $n_2=3$, and $n_3=7$ works. Thus our equation becomes $2(g-1)\geq |G|\big(1-\frac{41}{42}\big)$, which simplifies to $84(g-1)\geq |G|$.

Gathering all the inequalities from above yields the following,

$$|G|\leq\begin{cases} g-1, & b=0\\ 4(g-1), & h\geq 1 \text{ or } h=0 \text{ and } b\geq 5\\ 12(g-1), & (h,b)=(0,4)\\ 84(g-1), & (h,b)=(0,3)\end{cases}. \tag{4}$$

Clearly, $84(g-1)$ is the largest value, thus will yield the most general upper bound. Therefore we have shown that $|G|\leq 84(g-1)$ for $g\geq 2$.

As is pointed out by [16] the bound in *Hurwitz's Inequality*, $|G|=84(g-1)$, is obtained. One such group is the symmetry group of the surface created by the $(2,3,7)$ triangle tiling of the hyperbolic plane once one identifies the boundary of a $2g$-gon under a particular relation, which will form a surface of genus g. These groups are called *Hurwitz groups* and the related surface a *Hurwitz surface*.

10.1 Example. Let $G=\mathbb{Z}/3\mathbb{Z}$ acts on Σ_4 (as in *example 9.3*), then the *Riemann-Hurwitz Relation* shows us that $-6=(3)\chi(\Sigma_{h,b})-D_{h,b}$. This action clearly has no fixed points, thus $D_{h,b}=0$, so we now have $-6=3\chi(\Sigma_{h,0})$. It must be that $\chi(\Sigma_{h,0})=-2$, hence $2-2h=-2$ which gives $h=2$. Therefore $\Sigma_4/G=\Sigma_{2,0}$ which agrees with *example 9.3*. This also illustrates the first bound in *equation 4* is obtained, $|G|=g-1$.

10.2 Example. Let $G=\mathbb{Z}/2\mathbb{Z}$ act on Σ_4 (as in *example 9.4*), then the *Riemann-Hurwitz Relation* shows us that $-6=2\chi(\Sigma_{h,b})-D_{h,b}$. This action was shown to induced 6 branch points in a base space of genus 1, so we now have $-6=2\chi(\Sigma_{1,6})-D_{1,6}$. Thus we find that the branched defect is $D_{1,6}=6$, this means G fixes each singular point.

11 Kulkarni's Congruence

If G is a group that acts on a surface Σ_g, $g\geq 0$, then the natural projection map $\rho:\Sigma_g\to\Sigma_{h,b}$ will induce a branched covering. Moreover, we have shown that these spaces and the group G are linked by the *Riemann-Hurwitz Relation*. Recall G is assumed to be finite, so *Sylow's Theorem* [3, 14] says there is a collection of p-subgroups in G of maximum order for each prime p dividing the order of G.

Definition. A group G is said to be a *p-group* if $|G| = p^{n_p}$ for some prime p with $n_p \in \mathbb{Z}_{>0}$. Define the *exponent* of G to be the natural number e_p such that $g^{p^{e_p}} = g$ for all $g \in G$, and such that p^{e_p} is minimal. Note $0 < e_p \leq n_p$. We also define the *cyclic p-deficiency* of G to be the non-negative integer $n_p - e_p$.

Now consider this setup: G induces an orientation preserving action on Σ_g thus yielding the orbit space $\Sigma_{h,b}$. By considering the *Riemann-Hurwitz Relation* we are able to follow [8] and develop a relation between the genus of this surface with the p-subgroups found within the group G and their cyclic p-deficiencies.

Kulkarni's Congruence. *Let G be a finite group inducing an orientation preserving action on Σ_g for $g \geq 2$. There exists a natural number N depending only on the cyclic p-deficiencies of G such that $g \equiv 1(N)$.*

Proof. Let us first consider G as a p-group with $|G| = p^{n_p}$. Assume G acts on Σ_g with $g \geq 2$, then we have the branched covering space $\rho : \Sigma_g \to \Sigma_{h,b}$. We can use the *Riemann-Hurwitz Relation* to yield,

$$\begin{aligned}1 - g =& \frac{1}{2}\big(|G|(2-2h) - D_{h,b}\big)\\ =& |G|(1-h) - \frac{1}{2}|G|\sum_{x\in B}(1 - \frac{1}{n_x})\\ =& p^{n_p}(1-h) - \frac{1}{2}p^{n_p}\sum_{x\in B}(1-\frac{1}{n_x}).\end{aligned}$$

We note that $n_x = p^j$ for some $1 \leq j \leq n_p$ since $n_x \big| |G|$. Now, multiplying the above equation by -1, then re-arranging the summation yields,

$$\begin{aligned}g - 1 =& p^{n_p}(h-1) - \frac{1}{2}p^{n_p}\sum_{j=1}^{e_p} a_j(\frac{1}{p^j} - 1)\\ =& p^{n_p}(h-1) - \frac{1}{2}p^{n_p - j}\sum_{j=1}^{e_p} a_j(1 - p^j).\end{aligned}$$

Where a_j is the number of branch points whose isotropy subgroups are of order p^j. Notice the summation runs up to e_p, the exponent of G. The following claim gives justification for this.

Claim. *Let $x \in B$ then for any $y \in \rho^{-1}(x)$, G_y is cyclic of order p^j for $1 \leq j \leq e_p$.*

Proof. Suppose $g_1 \in G_y$ and $|g_1| = p^j$ such that j is maximal. We have $yg_1 = y$ and since this action preserves orientation it must be that G acts on a disk centered at y via rotation with angle $2\pi k/p^j$ with $(k,p) = 1$. Without loss of generality let $k = 1$. If $g_2 \in G_y$, then g_2 gives a rotation of $2\pi k'/p^{j'}$ where $j' \leq j$. Thus $g_2 = g_1^{k'(p^{j-j'})}$, hence G_y is cyclic. Furthermore, since G_y is cyclic it is generated by a singleton and thus the order of the group is the order of this singleton, which must be less that or equal to p^{e_p}.

Continuing with the above equation, and with a little factoring, we yield,

$$g - 1 = p^{n_p - e_p}\big(p^{e_p}(h-1) - \frac{1}{2}p^{e_p - j}\sum_{j=1}^{e_p} a_j(1-p^j)\big). \tag{5}$$

We now consider a few cases for p:

Case 1: p is odd. *Equation 5* gives,

$$\begin{aligned} g-1 =& p^{n_p-e_p}\big(p^{e_p}(h-1)-\frac{1}{2}p^{e_p-j}\sum_{j=1}^{e_p}a_j(1-p^j)\big) \\ =& p^{n_p-e_p}\big(p^{e_p}(h-1)-p^{e_p-j}\sum_{j=1}^{e_p}a_j\frac{(1-p^j)}{2}\big). \end{aligned}$$

Note that since p is odd, $\frac{(1-p^j)}{2}\in\mathbb{Z}$ for all j. Thus we have shown that $g\equiv 1(p^{n_p-e_p})$.

Case 2: $p=2$. If G is cyclic ($i.e\ n_2=e_2$), then *equation 5* shows that $g\equiv 1(2^{n_2-e_2})$ since $2^{n_2-e_2}=1$ and $g-1\in\mathbb{Z}$. Now consider when $p=2$ and $e_2<n_2$. *Equation 5* then becomes,

$$\begin{aligned} g-1 =& 2^{n_2-e_2}\big(2^{e_2}(h-1)-2^{e_2-j-1}\sum_{j=1}^{e_2}a_j(1-2^j)\big) \\ =& 2^{n_2-e_2-1}\big(2^{e_2+1}(h-1)-2^{e_2-j}\sum_{j=1}^{e_2}a_j(1-2^j)\big). \end{aligned}$$

Notice that the value in the parenthesis above is an integer, thus $g\equiv 1(2^{n_2-e_2-1})$.

Putting it all together gives, for any p-group G acting on Σ_g, a number p^{f_p} with,

$$f_p=\begin{cases} n_p-e_p, & \text{if } p \text{ is odd or } p=2 \text{ and } n_2=e_2 \\ n_2-e_2-1, & \text{if } p=2 \text{ and } n_2>e_2 \end{cases}.$$

This number f_p is a positive integer and is clearly dependent only on the cyclic p-deficiency of G such that $g\equiv 1(p^{f_p})$. To extend this to any finite group G we note the existence of G's Sylow p-subgroups [3, 14] all of which act on Σ_g since G does. Thus we have a set of congruences: $g\equiv 1(p_i^{f_{p_i}})$ for all $p_i\big||G|$ where the p_i's are distinct primes for all i. Now apply the *Chinese Remainder Theorem* [3, 14] to get $g\equiv 1\big(\Pi_{p\big||G|}p^{f_p}\big)$. Note that $\Pi_{p\big||G|}p^{f_p}\in\mathbb{N}$ and this value is also only dependent on the cyclic p-deficiencies of G. Letting $N=\Pi_{p\big||G|}p^{f_p}$ finishes the proof.

So, for any finite group G acting on a surface of genus g, there is a natural number N dependent on G's algebraic structure, in which $g\equiv 1(N)$. One might wonder if this congruence works both ways, In other words if $g\equiv 1(N)$ does G act on Σ_g? It is not necessarily true, but a close converse can be created and is as follows.

Kulkarni's Congruence (Converse)**.** *Let G be a finite group. Let N be the natural number, as developed prior, which is dependent only on the cyclic p-deficiencies of G. If $g\equiv 1(N)$, then G acts on Σ_g except for finitely many exceptional values of g.*

The proof comes from looking at the non-negative integer solutions to a diophantine equation which is related to the *Riemann-Hurwitz Relation*, the full proof is beyond the scope of this paper, but may be found in [8].

12 Further Applications

Consider, a surface Σ_g of genus g and a finite group G that acts on Σ_g such that $\rho : \Sigma_g \to \Sigma_{h,b}$ is a branched covering with b branch points. We now show that a lower bound can be put on g if this branched covering has only one branch point, *i.e.* $b = 1$.

12.1 Lemma. *There is no group, G, which acts on the sphere, Σ_0, such that $\rho : \Sigma_0 \to \Sigma_0/G$ is a branched covering with only one branch point.*

Proof. Assume there is such a G, then we have the branched covering $\rho : \Sigma_0 \to \Sigma_{0,1}$. By the *Riemann-Hurwitz Relation* we get,

$$\chi(\Sigma_0) = |G|\chi(\Sigma_{0,1}) - D_{0,1},$$

where $D_{0,1} = |G|(1 - \frac{1}{n_x})$ with $B = \{x\}$. Using the fact that $\chi(\Sigma_0) = 2$, we can simplify this equation to get,

$$2 = |G|\big(2 - (1 - \frac{1}{n_x})\big) = |G| + \frac{|G|}{n_x}.$$

Clearly $|G| \geq 1$, but we also have that $\frac{|G|}{n_x} \geq 1$. This *must* mean that $|G| = 1$, hence G is the trivial group. It follows that all points in Σ_0 have isotropy subgroups of order 1, including $x \in B$. This contradicts our assumption that x is a branch point, $G_x \not> 1$. Therefore, no such group G exists.

Thus we have a lower bound on the genus of the total surface, $g \geq 1$, if given a branched covering with one branch point. Next we will form a lower bound on the genus of the base surface, h, in this branched covering.

12.2 Proposition. *The sphere Σ_0 has no a branched coverings with only one branch point.*

Proof. Let G act on Σ_g, $g \geq 0$, such that $\rho : \Sigma_g \to \Sigma_{h,1}$ is a branched covering. Assume $h = 0$, then by the *Riemann-Hurwitz Relation* we have,

$$\chi(\Sigma_g) = |G|\chi(\Sigma_{0,1}) - |G|\sum_{x \in B}(1 - \frac{1}{n_x}),$$

where $B = \{x\}$. Using the facts that $\chi(\Sigma_g) = 2-2g$ and $\chi(\Sigma_{0,1}) = 2$, gives us the following equation,

$$2 - 2g = 2|G| - |G| + \frac{|G|}{n_x} = |G| + \frac{|G|}{n_x}.$$

Noting that $|G| > 1$ and that $\frac{|G|}{n_x} \geq 1$ gives us: $2 - 2g > 2$, thus $1 > g$, hence $g = 0$. Now we appeal to *Lemma 12.1* to show that no such branched covering exists. It follows that $h \neq 0$, hence $\Sigma_{h,1}$ is not the sphere.

Now we have, given a branched covering with one branch point, that the genus of the total surface must be greater or equal to 1 and the same must go for the genus of the base surface. We can now strengthen the lower bound on the genus g of the total surface.

12.3 Proposition. *If G acts on Σ_g such that $\rho : \Sigma_g \to \Sigma_{h,1}$ is a branched covering with one branch point, then $g \geq 2$.*

Proof. By *Proposition 12.2* we know that g and h are both greater than 0. Consider $g = 1$, since $g \geq h$ we must have $h = 1$. Assume $\rho : \Sigma_1 \to \Sigma_{1,1}$ is a branched covering. Then the *Riemann-Hurwitz Relation* yields,

$$\chi(\Sigma_g) = |G|\chi(\Sigma_{1,1}) - D_{h,1}.$$

Plugging in the knowns, we simplify the above equation to $0 = -|G| + \frac{|G|}{n_x}$, hence $n_x = 1$ which contradicts the assumption that x is a branch point. So we must have that $g \geq 2$.

It has been shown that a branched covering, $\rho : \Sigma_g \to \Sigma_{h,b}$ with $b = 1$, must have a lower bound on g of 2 and a lower bound on h of 1. We will finish with a proof which shows that if the sphere has a branched covering with two branch points, then the total surface must itself be a sphere.

12.4 Proposition. *If a finite group G acts on the surface Σ_g such that $\rho : \Sigma_g \to \Sigma_{0,2}$ is a branched covering with two branch points, then $g = 0$. Moreover, for all odd primes $p \big| |G|$ the Sylow p-subgroup of G are cyclic and the Sylow 2-subgroup is cyclic, dihedral, generalized quaternion, or semi-dihedral.*

Proof. Consider the branched covering $\Sigma_g \xrightarrow{\rho} \Sigma_{0,2}$. We see by the *Riemann-Hurwitz Relation* that,

$$\chi(\Sigma_g) = |G|\chi(\Sigma_{0,2}) - |G|(2 - \frac{1}{n_{x_1}} - \frac{1}{n_{x_2}}),$$

for $B = \{x_1, x_2\}$. Simplification of this equation yields $2(g-1) = \frac{|G|}{n_{x_1}} + \frac{|G|}{n_{x_2}}$. Since $\frac{|G|}{n_{x_i}} \geq 1$ for $i = 1, 2$, the right-hand side must be greater that 2, thus $g - 1 \geq 1$ so $g = 0$. But this also means $2 = \frac{|G|}{n_{x_1}} + \frac{|G|}{n_{x_2}}$, which implies that both isotropy subgroups are of order $|G|$ (*i.e.* G fixes both of the singular points attributed to these two branch points).

Furthermore, using *Kulkarni's Congruence* we have the relation $g \equiv 1(N)$, we can do this since [7] extends *Kulkarni's Congruence* to all $g \geq 0$. So $g - 1 = kN$ for some $k \in \mathbb{Z}$, but $g = 0$ hence $-1 = kN$. Since N is the product of powers it is positive, so it must be that $N = 1$, the only way for this to happens is if $f_p = 0$ for all primes $p \big| |G|$. Hence, if p is odd then the cyclic deficiency of all the p-subgroups of G is 0, thus cyclic. Moreover, if $p = 2$ then $f_2 \leq 1$, it follows from [8] that the Sylow 2-subgroup of G is either cyclic, dihedral, generalized quaternion, or semi-dihedral.

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