Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths[*]

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Abstract

A multiparametric family of 2D Toda τ-functions of hypergeometric type is shown to provide generating functions for composite, signed Hurwitz numbers that enumerate certain classes of branched coverings of the Riemann sphere and paths in the Cayley graph of S_n. The coefficients $F^{c_1,\ldots,c_l}_{d_1,\ldots,d_m}(\mu,\nu)$ in their series expansion over products $P_\mu P'_\nu$ of power sum symmetric functions in the two sets of Toda flow parameters and powers of the $l + m$ auxiliary parameters are shown to enumerate $|\mu| = |\nu| = n$ fold branched covers of the Riemann sphere with specified ramification profiles μ and ν at a pair of points, and two sets of additional branch paints, satisfying certain additional conditions on their ramification profile lengths. The first group consists of l branch points, with ramification profile lengths fixed to be the numbers $(n - c_1, \ldots, n - c_l)$; the second consists of m further groups of "coloured" branch points, of variable number, for which the sums of the complements of the ramification profile lengths within the groups are fixed to equal the numbers $(d_1, \ldots, d_m)$. The latter are counted with signs determined by the parity of the total number of such branch points. The coefficients $F^{c_1,\ldots,c_l}_{d_1,\ldots,d_m}(\mu,\nu)$ are also shown to enumerate paths in the Cayley graph of the symmetric group S_n generated by transpositions, starting, as in the usual double Hurwitz case, at an element in the conjugacy class of cycle type μ and ending in the class of type ν, with the first l consecutive subsequences of $(c_1, \ldots, c_l)$ transpositions strictly monotonically increasing, and the subsequent subsequences of $(d_1, \ldots, d_m)$ transpositions weakly increasing.

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1 Introduction

In [12] a general method for interpreting $2D$ Toda τ-functions [32, 30, 31] of hypergeometric type [19, 29] as combinatorial generating functions for certain classes of paths in the Cayley graph of the symmetric group S_n was introduced. Examples included Okounkov's generating function for double Hurwitz numbers [26], which count the number of inequivalent n-fold branched covers of the Riemann sphere $\mathbf{P}^1$ having a pair of branch points at 0 and ∞ with specified ramification profile types μ and ν, and l additional branch points with simple ramification type. The equivalent combinatorial interpretation is the enumeration of k-step paths in the Cayley graph of S_n generated by transpositions, starting at an element in the conjugacy class of cycle type μ and ending in the class of cycle type ν.

Several similar examples of 2D Toda τ-functions of hypergeomatric type were studied in [9, 10, 12] and interpreted combinatorially in terms of counting paths in the Cayley graph generated by transpositions that are either strictly or weakly monotonically increasing, or some combination thereof. These included several cases that, by restriction of the flow variables to trace invariants of a pair of matrices, could be interpreted as matrix integrals of the Itzykson-Zuber-Harish-Chandra (HCIZ) type [14, 20, 9, 10], or variants thereof [16, Appendix A], [12]. In [33] a generating function was given for Grothendieck's *dessins d'enfants*, which is equivalent to the enumeration of branched covers of Riemann surfaces with three branch points, one of which has specified ramification profile, and the other two specified profile lengths. This was subsequently shown to be a KP τ-function, satisfying Virasoro constraints and topological recursion relations [18] and to have several equivalent representations as matrix integrals [2].

In [29] a large class of hypergeometric 2D Toda τ-functions was studied, including a family that, when the flow variables are restricted to the trace invariants of a pair of $N \times N$ matrices, can be interpreted as hypergeometric functions of matrix arguments [8]. (This was in fact the origin of the term "τ-function of hypergeometric type".) It follows moreover, from the results of [19, 27, 28] and [16, Appendix A] that these may all be represented as matrix integrals. In [1] a subclass of this family was noted to have the form of generating functions for Hurwitz numbers, but no general combinatorial or geometric interpretation was given. The combinatorial significance of the coefficients in the double power sum symmetric function expansions for these was indicated briefly in [12], as counting paths in the Cayley graph consisting of k strictly increasing subsequences of transpositions having given lengths.

In the present work, we view these as special cases of the more general class of hypergeometric 2 Toda τ-functions introduced in [29], and interpret them as generating functions for the enumeration of certain classes of branched covers of $\mathbf{P}^1$, and certain paths in the Cayley graph of S_n satisfying specified geometric and combinatorial constraints. The main results are stated in Theorems 2.1 and 2.2, which give both the geometric and combinatorial significance of the coefficients $F^{(c_1,...,c_l)}_{(d_1,...,d_m)}(\mu, \nu)$ for the expansions of these τ-functions in a basis of

products of power sum symmetric functions and monomials in the additional parameters.

Their enumerative geometrical significance, given in Theorem 2.1, is that they count, with signs, the n-sheeted branched covers of $\mathbf{P}^1$, having again a pair of branch points $(0, \infty)$, with ramification profiles given by a pair of partitions (μ, ν), plus two further families of branch points, satisfying specific conditions. The first family consists of l branch points whose ramification profiles have specified lengths $\{n - c_a\}_{a=1,\ldots k}$. The second consists of m "coloured" groups, each containing a variable number of branch points, but constrained so that the sums of the complements of the lengths of the ramification profiles of the points within each colour group are equal to the specified numbers $\{d_b\}_{b=1,\ldots m}$. The sign in the counting is $(-1)^{mn+\sum_b^m d_b}$ times the parity of the total number of coloured branch points.

The combinatorial meaning of the coefficients $F^{(c_1,\ldots,c_l)}_{(d_1,\ldots,d_m)}(\mu, \nu)$, given in Theorem 2.2, is the following: these again enumerate paths in the Cayley graph, starting at an element in the conjugacy class of cycle type μ and ending at one in the class of type ν, constrained so that the first k consecutive subsequences of transpositions of lengths $(c_1 \ldots, c_l)$ are each monotonically strictly increasing with respect to their larger elements, while those in the next successive subsequences of lengths $(d_1, \ldots, d_m)$ are weakly monotonically increasing.

All previously studied examples of generalized Hurwitz numbers can be recovered as special cases within this extended family of combinatorial/geometric generating functions.

2 The hypergeometric 2D Toda τ-functions $\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s})$

2.1 2D Toda τ-functions of hypergeometric type

A 2D Toda τ-function [32, 30, 31] consists of a lattice of functions $\tau^{2DT}(N, \mathbf{t}, \mathbf{s})$, labelled by the integers $N \in \mathbf{Z}$, depending differentiably on two infinite sequences of complex flow variables

$$\mathbf{t} = (t_1, t_2, \ldots), \quad \mathbf{s} = (s_1, s_2, \ldots), \tag{2.1}$$

and satisfying the infinite set of Hirota bilinear differential-difference equations, which are constant coefficient bilinear differential equations in the $(\mathbf{t}, \mathbf{s})$ variables, and finite difference equations in the lattice variable N. These can be concisely expressed through the following formal contour integral equality [32, 30, 31]

$$\oint_{z=\infty} z^{N'-N} e^{-\xi(\delta\mathbf{t},z)} \tau^{2DT}(N, \mathbf{t} + [z^{-1}], \mathbf{s}) \tau^{2DT}(N', \mathbf{t} + \delta\mathbf{t} - [z^{-1}], \mathbf{s} + \delta\mathbf{s}) =$$

$$\oint_{z=0} z^{N'-N} e^{-\xi(\delta\mathbf{s},z^{-1})} \tau^{2DT}(N - 1, \mathbf{t}, \mathbf{s} + [z]) \tau^{2DT}(N' + 1, \mathbf{t} + \delta\mathbf{t}, \mathbf{s} + \delta\mathbf{s} - [z]) \tag{2.2}$$

for any pair $N, N' \in \mathbf{Z}$, where

$$\xi(\mathbf{t}, z) := \sum_{i=1}^{\infty} t_i z^i, \quad [z]_i := \frac{1}{i} z^i, \tag{2.3}$$

understood as satisfied identically in the doubly infinite set of parameters

$$\delta\mathbf{t} = (\delta t_1, \delta t_2, \dots), \ \delta\mathbf{s} := (\delta s_1, \delta s_2, \dots). \tag{2.4}$$

These imply, in particular, the full set of KP (Kadomtsev-Petviashvili) Hirota bilinear relations [4] for either of the two sets of flow variables $\mathbf{t}$ and $\mathbf{s}$, for each N, as well as an infinite set of bilinear nearest neighbour difference equations linking the lattice sites N, N' to their neighbours. Such general 2D Toda τ-functions may be understood in terms of infinite abelian group actions on infinite flag manifolds [32, 30]. Using the Plücker embedding, they may be given a standard fermionic Fock space representation [31] as vacuum state expectation values. They may also be given an infinite series representation as sums over products $S_\lambda(\mathbf{t})S_\mu(\mathbf{s})$ of Schur functions, in which the coefficients are interpreted as Plücker coordinates of the associated infinite flag manifold [31, 16].

Remark 2.1. To clarify notational conventions, when Schur functions are expressed in this way, it is understood that their arguments $\mathbf{t}$ and $\mathbf{s}$ are related to the monomial sum symmetric functions as follows:

$$p_i = it_i, \quad p_i' = is_i, \tag{2.5}$$

and therefore

$$P_\mu(\mathbf{t}) = \prod_{i=1}^{\ell(\mu)} p_{\mu_i} = \prod_{i=1}^{\ell(\mu)} \mu_i t_{\mu_i}. \tag{2.6}$$

For the present work, only the special subfamily of τ-functions of *hypergeometric type* [19, 29] will be needed, for which this expansion reduces to a diagonal sum over products $S_\lambda(\mathbf{t})S_\lambda(\mathbf{s})$ of Schur functions of the same type. Further details and various applications of this subclass of 2D Toda τ-functions may be found in [19, 29, 28, 16, 17]. We give here only a brief summary of the essentials regarding the diagonal double Schur function series representation as needed in the present work.

For any map

$$\rho : \mathbf{Z} \to \mathbf{C}^\times$$
$$\rho : j \mapsto \rho_j, \tag{2.7}$$

we may define the following *content product* associated to the partition λ

$$r_\lambda(N) := r_0(N) \prod_{(ij)\in\lambda} r_{N+j-i}, \tag{2.8}$$

where

$$r_j := \frac{\rho_j}{\rho_{j-1}}, \quad r_0(N) := \prod_{j=0}^{N-1} \rho_j, \quad r_0(0) := 1, \quad r_0(-N) = \prod_{i=1}^{N} \rho_{-j}^{-1}, \quad N \in \mathbf{N}^+. \tag{2.9}$$

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As will be detailed further in Sections 4 and 5, this may be viewed as the eigenvalues of a certain family of operators acting either on the direct sum $\oplus_{n=1}^{\infty} \mathbf{Z}(\mathbf{C}[S_n])$ of the centers of the group algebras of S_n, $n \in \mathbf{N}$ or, equivalently, on a fermionic Fock space $\mathcal{F}$, and used thereby to define a 2D Toda τ-function of hypergeometric type. This may be expressed as a formal diagonal sum over products of Schur functions

$$\tau_r(N, \mathbf{t}, \mathbf{s}) := \sum_\lambda r_\lambda(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}), \tag{2.10}$$

where S_λ denotes the Schur function labelled by the integer partition $\lambda = \lambda_1 \geq \cdots \lambda_{\ell(\lambda)} > 0, 0 \ldots)$ of length $\ell(\lambda)$, and weight $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$. It follows from the fermionic representation (see [31, 29] and Section 5) that any such lattice of functions $\tau_r(N, \mathbf{t}, \mathbf{s})$ satisfies the Hirota bilinear relations (2.2).

In the following, we assume some familiarity with properties of the algebra Λ of symmetric functions in an arbitrary number of variables [23], the group algebra $\mathbf{C}[S_n]$, and irreducible characters $\chi_\lambda(\mu)$ of S_n [6]. The Frobenius character formula expresses the Schur functions S_λ linearly in terms of the power sum symmetric functions $\{P_\mu\}$,

$$S_\lambda = \sum_{\substack{\mu \\ |\mu|=|\lambda|}} \frac{\chi_\lambda(\mu) P_\mu}{Z_\mu}, \tag{2.11}$$

where $\chi_\lambda(\mu)$ is the irreducible character of S_n corresponding to the partition λ, evaluated on the conjugacy class of cycle type given by the partition μ, and

$$Z_\mu := \prod_{i=1}^{n} i^{j_i}(j_i)! = |\operatorname{stab}(\mu)|, \quad (j_i = \text{number of parts of } \mu \text{ equal to } i), \tag{2.12}$$

is the order of the stabilizer of any of the elements of the conjugacy class.

Substituting this into formula (2.10) gives an expansion over products of pairs of monomial sum symmetric functions

$$\tau_r(N, \mathbf{t}, \mathbf{s}) = \sum_{\substack{\mu, \nu \\ |\mu|=|\nu|}} G_r(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}) \tag{2.13}$$

where

$$G_r(\mu, \nu) = (Z_\mu Z_\nu)^{-1} \sum_{\substack{\lambda \\ |\lambda|=|\mu|=|\nu|}} r_\lambda(N) \chi_\lambda(\mu) \chi_\lambda(\nu). \tag{2.14}$$

We also use the notation

$$h_\lambda = \frac{|\lambda|!}{d_\lambda} \tag{2.15}$$

to denote the product of hook lengths [23], where

$$d_\lambda = \chi_\lambda(1^{|\lambda|}) = |\lambda|! \det \left(\frac{1}{(\lambda_i - i + j)!} \right) \tag{2.16}$$

is the dimension of the irreducible representation of S_n with character $\chi_\lambda(\mu)$.

2.2 The family of hypergeometric τ-functions $\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s})$

Choosing a set of $1 + l + m$ complex parameters $(q, \mathbf{w}, \mathbf{z})$,

$$\mathbf{w} := (w_1, \ldots, w_l), \quad \mathbf{z} := (z_1, \ldots, z_m), \tag{2.17}$$

Let $\rho_0^{(q,\mathbf{w},\mathbf{z})} = 1$ and for $j > 0$,

$$\rho_j^{(q,\mathbf{w},\mathbf{z})} := q^j \prod_{a=1}^{l}\prod_{b=1}^{m}\prod_{k=1}^{j} \left(\frac{1 + kw_a}{1 - kz_b}\right), \tag{2.18}$$

$$\rho_{-j}^{(q,\mathbf{w},\mathbf{z})} := q^{-j} \prod_{a=1}^{l}\prod_{b=1}^{m}\prod_{k=0}^{j-1} \left(\frac{1 + kz_b}{1 - kw_a}\right). \tag{2.19}$$

Then

$$r_j^{(q,\mathbf{w},\mathbf{z})} := q \prod_{a=1}^{l}\prod_{b=1}^{m} \left(\frac{1 + jw_a}{1 - jz_b}\right) \quad \text{for all } j \in \mathbf{Z} \tag{2.20}$$

and

$$r_\lambda^{(q,\mathbf{w},\mathbf{z})}(N) = r_0^{(q,\mathbf{w},\mathbf{z})}(N) \prod_{(i,j)\in\lambda} r_{N+j-i}^{(q,\mathbf{w},\mathbf{z})}, \tag{2.21}$$

where

$$r_0^{(q,\mathbf{w},\mathbf{z})}(N) = q^{\frac{1}{2}N(N-1)} \prod_{j=1}^{N-1} \left(\frac{1 + (N-j)w_a}{1 - (N-j)z_b}\right)^j, \quad N > 0, \quad r_0(0) = 1, \tag{2.22}$$

$$r_0^{(q,\mathbf{w},\mathbf{z})}(-N) = q^{\frac{1}{2}N(N+1)} \prod_{j=1}^{N} \left(\frac{1 - (N-j)w_a}{1 + (N-j)z_b}\right)^j, \quad N > 0. \tag{2.23}$$

The resulting hypergeometric 2D Toda τ-function is denoted

$$\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s}) = \sum_\lambda r_\lambda^{(q,\mathbf{w},\mathbf{z})}(N) S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s}) \tag{2.24}$$

$$= \sum_\lambda r_\lambda^{(q,\mathbf{w},\mathbf{z})}(N) \sum_{\mu,\nu,|\mu|=|\nu|=n} (Z_\mu Z_\nu)^{-1} \chi_\lambda(\mu)\chi_\lambda(\nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \tag{2.25}$$

Remark 2.2. Since $\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s})$ can be expressed in terms of $\tau^{(q,\mathbf{w},\mathbf{z})}(0, \mathbf{t}, \mathbf{s})$ by a simple transformation of parameters

$$\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s}) = T(q, N, \mathbf{w}, \mathbf{z})\tau^{(\tilde{q},\tilde{\mathbf{w}},\tilde{\mathbf{z}})}(0, \mathbf{t}, \mathbf{s}), \tag{2.26}$$

where

$$\tilde{w}_a := \frac{w_a}{(1 + Nw_a)}, \quad \tilde{z}_b := \frac{\tilde{z}_b}{(1 - Nz_b)}, \quad \tilde{q} = q\frac{\prod_{a=1}^{l}(1 + Nw_a)}{\prod_{b=1}^{m}(1 - Nz_b)}, \tag{2.27}$$

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and

$$T(q, N, \mathbf{w}, \mathbf{z}) = q^{\frac{1}{2}N(N-1)} \prod_{a=1}^{l} \prod_{b=1}^{m} \left(\frac{1 + N w_a}{1 - N z_b} \right)^{\frac{1}{2}N(N-1)} \tag{2.28}$$

we henceforth only consider the case $N = 0$, and simplify the notation to

$$\tau^{(q,\mathbf{w},\mathbf{z})}(0, \mathbf{t}, \mathbf{s}) =: \tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s}), \quad r_\lambda^{(q,\mathbf{w},\mathbf{z})}(0) =: r_\lambda^{(q,\mathbf{w},\mathbf{z})}. \tag{2.29}$$

Remark 2.3. If a positive integer $M > 0$ is chosen, and the flow variables $(\mathbf{t}, \mathbf{s})$ are restricted to be the trace invariants

$$t_i = \frac{1}{i} \operatorname{tr}(X^i) := [X]_i, \quad s_i = \frac{1}{i} \operatorname{tr}(Y^i) := [Y]_i \tag{2.30}$$

of a pair (X, Y) of $M \times M$ hermitian matrices whose eigenvalues are $\{x_i\}_{i=1,\ldots,M}$, $\{y_j\}_{j=1,\ldots,M}$, respectively, the series (2.24) for $N = 0$ only involves sums over Schur functions S_λ for which $\ell(\lambda) \le M$, and may be related to the so-called hypergeometric function of matrix arguments [8, 29] as follows. We may consistently choose $z_1 = -\frac{1}{M}$, since the restriction $\ell(\lambda) \le M$ implies that there is no value of j appearing in which the denominator factor in $r_j^{(q,\mathbf{w},\mathbf{z})}$ vanishes. Choosing the remaining parameters to be nonvanishing, and defining

$$u_a = \frac{1}{w_a}, \ a = 1, \ldots, l \ \text{ and } \ v_{b-1} := -\frac{1}{z_b}, \ b = 2, \ldots, m, \tag{2.31}$$

we have

$$\tau^{(q,\mathbf{w},\mathbf{z})}(0, [X], [Y]) =_l \Phi_{m-1}(u_1, \ldots, u_l; v_1, \ldots, v_{m-1} | qX, Y)$$

$$:= \frac{\det \left({}_l F_{m-1}(u_1, \ldots, u_l; v_1, \ldots, v_{m-1} | q x_i y_j) \right)_{i,j=1,\ldots,m}}{\Delta(\mathbf{x}) \Delta(\mathbf{y})}, \tag{2.32}$$

where ${}_l F_m$ is the usual general hypergeometric function

$$_l F_m(u_1, \ldots, u_l; v_1, \ldots, v_m | x) = \sum_{n=0}^{\infty} \frac{\prod_{a=1}^{l} (u_a)_n}{\prod_{b=1}^{m} (v_b)_n} \frac{x^n}{n!}. \tag{2.33}$$

Here ${}_l \Phi_m(a_1, \ldots, a_l; v_1, \ldots, v_m | X, Y)$ is what is known as the hypergeometric function of two matrix arguments [8].

Further expanding the coefficients in the formula (2.25) for $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s})$ as power series in the parameters $(q, \mathbf{w}, \mathbf{z})$, using multi-indices $\mathbf{c} = (c_1, \ldots, c_l) \in \mathbf{N}^l$, $\mathbf{d} = (d_1, \ldots, d_m) \in \mathbf{N}^m$, gives

$$\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s}) = \sum_{n=0}^{\infty} q^n \sum_{\substack{\mu, \nu \\ |\mu|=|\nu|=n}} \sum_{\mathbf{c} \in \mathbf{N}^l} \sum_{\mathbf{d} \in \mathbf{N}^m} \mathbf{w}^{\mathbf{c}} \mathbf{z}^{\mathbf{d}} F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu) P_\mu(\mathbf{t}) P_\nu(\mathbf{s}) \tag{2.34}$$

We are now ready to state the two main theorems:

Theorem 2.1. Geometric interpretation: generalized Hurwitz numbers. *The co-efficients $F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$ in the expansion (2.34) are equal to the number of n-sheeted inequivalent branched coverings of the Riemann sphere by a surface of genus g given by the Riemann-Hurwitz formula*

$$2g = 2 + \sum_{a=1}^{l} c_a + \sum_{b=1}^{m} d_b - \ell(\mu) - \ell(\nu), \tag{2.35}$$

counted with signs, as indicated below, such that the branch points consist of three classes:
i) A pair of branch points $(0, \infty)$, with ramification profiles (μ, ν).
ii) A set of l further branch points $\{q_a\}_{a=1,\ldots,l}$, with ramification profiles $\{\mu^{(a)}\}_{a=1,\ldots,l}$, the complement of whose lengths are

$$n - \ell(\mu^{(a)}) = c_a, \quad a = 1, \ldots, l \tag{2.36}$$

iii) A set of m further groups of branch points, $\{p_{b,i_b}\}_{\substack{b=1,\ldots m \\ i_b=1,\ldots j_b}}$, labeled by "colours" $b = 1, \ldots m$, with ramification profiles $\{\nu^{(b,i_b)}\}_{b=1,\ldots,m;i_b=1,\ldots j_b}$, where j_b is the number of points in the bth coloured group, such that the sum of the complements of the lengths of the ramification profiles at the points $\{p_{b,i_b}\}_{i_b=1,\ldots j_b}$ within the bth group is equal to d_b

$$\sum_{i_b=1}^{j_b} \left(n - \ell(\nu^{(b,i_b)})\right) = d_b, \quad b = 1, \ldots, m. \tag{2.37}$$

Each such covering is counted with a sign $(-1)^{mn+C+D}$, where

$$C := \sum_{b=1}^{m} j_b, \tag{2.38}$$

is the total number of coloured branched points and

$$D := \sum_{b=1}^{m} d_b = nC - \sum_{b=1}^{m} \sum_{i_b=1}^{j_b} \ell(\nu^{(b,i_b)}) \tag{2.39}$$

is the sum of the complements of the lengths of the ramification profiless of the coloured branch points.

Remark 2.4. Note that the number of branch points in each of the groups is variable, but the sum is only over those in group (ii) for which the lengths $\ell(\mu^{(a)})$ are fixed and those in group (iii) for which the sum of the complements of the lengths D are fixed. These therefore only involve signed sums over a finite number of individual Hurwitz numbers. It is also understood that, when applied to coverings by orientable surfaces, this interpretation of $F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$, is valid only if the genus g given by formula (2.35) is an integer. The case when it is a half integer is applicable to counting nonorientable covers.

Theorem 2.2. Combinatorial interpretation: multimonotonic paths in the Cayley graph. *The coefficients $F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$ in the expansion (2.34) are equal to the number of paths in the Cayley graph of S_n generated by transpositions $(a\,b)$, $a < b$, starting at an element in the conjugacy class with cycle type given by the partition μ and ending in the conjugacy class with cycle type given by partition ν, such that the paths consist of a sequence of*

$$k := \sum_{a=1}^{l} c_a + \sum_{b=1}^{m} d_b \tag{2.40}$$

transpositions $(a_1 b_1) \cdots (a_k b_k)$, divided into $l + m$ subsequences, the first l of which consist of $\{c_1, \ldots, c_l\}$ transpositions that are strictly monotonically increasing (i.e. $b_i < b_{i+1}$ for each neighbouring pair of transpositions within the subsequence), followed by $\{d_1, \ldots, d_m\}$ subsequences within each of which the transpositions are weakly monotonically increasing (i.e. $b_i \leq b_{i+1}$ for each neighbouring pair).

Evaluating at $\mathbf{t}_\infty := (1, 0, 0, \ldots)$ we have

$$S_\lambda(\mathbf{t}_\infty) = \frac{1}{h_\lambda}, \quad P_\nu(\mathbf{t}_\infty) = \delta_{\nu,(1)^{|\nu|}}. \tag{2.41}$$

Therefore setting $\mathbf{s} = \mathbf{t}_\infty$ in (2.24) and (2.34), $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s})$ restricts to the KP τ-function

$$\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{t}_\infty) = \sum_{\lambda} h_\lambda^{-1} r_\lambda^{(q,\mathbf{w},\mathbf{z})} S_\lambda(\mathbf{t}) = \sum_{n=0}^{\infty} q^n \sum_{\substack{\mu, \\ |\mu|=n}} \sum_{\mathbf{c} \in \mathbf{N}^l} \sum_{\mathbf{d} \in \mathbf{N}^m} \mathbf{w}^{\mathbf{c}} \mathbf{z}^{\mathbf{d}} F_{\mathbf{d}}^{\mathbf{c}}(\mu, (1)^n) P_\mu(\mathbf{t}), \tag{2.42}$$

where the coefficients $F_{\mathbf{d}}^{\mathbf{c}}(\mu, (1)^n)$ are the particular values corresponding to no branching at ∞. We therefore have the following corollary.

Corollary 2.3. Reduction to KP τ-function. *The KP τ-function $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{t}_\infty)$ is the generating function for the numbers $F_{\mathbf{d}}^{\mathbf{c}}(\mu, (1)^n)$ counting branched covers satisfying the same conditions as Theorem 2.1, but with no branch point at ∞. Equivalently, they are equal to the number of paths in the Cayley graph of S_n from an element in the conjugacy class with cycle type μ to the identity element (i.e. the number of factorizations of an element in the class μ as a product of transpositions) that satisfy the conditions of Theorem 2.2.*

As special cases, consider $(l, m) = (1, 0)$ and $(0, 1)$. For any positive integer $c \in \mathbf{N}_+$, and pair of partitions (μ, ν) with $|\mu| = |\nu| = n$, let $F_c^+(\mu, \nu)$ and $F_c^-(\mu, \nu)$ denote the number of n-sheeted branched covers of the Riemann sphere $\mathbf{P}^1$, up to automorphisms, with Euler characteristic

$$\chi = 2 - 2g = \ell(\mu) + \ell(\nu) - c, \tag{2.43}$$

having either an even ($F_c^+(\mu, \nu)$) or an odd number ($F_c^-(\mu, \nu)$) total number of branch points, including a pair $(0, \infty)$ with ramification profiles (μ, ν).

Let $F^c(\mu, \nu)$ and $F_c(\mu, \nu)$ be the composite Hurwitz number $F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$ with $(l, m) = (1, 0)$, $c_1 = c$, and $(l, m) = (0, 1)$, $d_1 = c$), respectively. According to Theorem 2.2, $F^c(\mu, \nu)$ is the number of strictly monotonically increasing products of c transpositions $(a_1 b_1) \cdots (a_c b_c)$ such that, if $g \in S_n$ is in the conjugacy class with cycle type μ, the product $(a_1 b_1) \cdots (a_c b_c) g$ is in the conjugacy class ν, while $F_c(\mu, \nu)$ is the number of products having the same property, but which are weakly monotonically increasing. The following is an immediate consequence of Theorems 2.1 and 2.2 for these two cases.

Corollary 2.4. The cases $(l, m) = (1, 0)$ and $(0, 1)$.

(i) $(l, m) = (1, 0)$:

In this case, there are at most three branch points, the ones at $(0, \infty)$ having ramification profiles (μ, ν) and a third one, whose profile λ has length

$$\ell(\lambda) = n - c. \tag{2.44}$$

These are therefore Belyi curves [33, 2, 18]. The combinatorial meaning of $F^c(\mu, \nu)$ is that it equals the number of paths in the Cayley graph of S_n consisting of sequences of c strictly monotonically increasing transpositions, starting at an element in the conjugacy class of cycle type μ and ending in the class of type ν.

(ii) $(l, m) = (0, 1)$:

$$F_c(\mu, \nu) = (-1)^{n+c} (F_c^+(\mu, \nu) - F_c^-(\mu, \nu)). \tag{2.45}$$

Thus, the number of weakly monotonically increasing paths of transpositions that lead from an element in the conjugacy class of type μ to the class of type ν is equal to the difference between the number of branched covers with an even or an odd number of branch points, branching profiles (μ, ν) at $(0, \infty)$ and Euler characteristic given by (2.43).

If ∞ is not a branch point; i.e., its profile is $\nu = (1)^n$, corresponding to the identity class in S_n, Corollaries 2.3 and 2.4 imply that the number of factorizations of any element $g \in S_n$ in the class μ as a product of c strictly monotonically increasing transpositions is equal to the number of Belyi curves with no branching at ∞ and c pre-images of the additional branch point. The number of factorizations into weakly increasing subsequences of transpositions is given by the difference (2.45) between the number of branched covers having an even or an odd number of branch points, for the Euler characteristic given by eq. (2.45), and $\nu = (1)^n$.

Remark 2.5. Further particular cases. The coefficients $F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$ with $m = 0$ (i.e. no branch points of type (iii)) and all $c_a = 1$ count those coverings that have simple ramification at all the points $\{q_a\}_{a=1,\dots,l}$ in group (ii), and hence these coincide with Okounkov's double Hurwitz number [26]. In the combinatorial interpretation, all the subsequences contain a single transposition,

therefore there is is no monotonicity imposed. Similarly, if $c_a = 0$ for all a's, and $d_b = 1$ for all b's, (or equivalently, if $l=0$), it follows that there can only be a single branch point in each colour group, and that it is simply ramified. This must therefore also equal Okounkov's double Hurwitz number, up to an overall sign. In the combinatorial interpretation there is again only one transposition in each sequence, so there is no condition of monotonicity. In fact, we may choose any subset of the c_a's and all the d_b's to equal 1, and the other c_a's to vanish, and the same result holds.

When $l = 0$ and $m = 1$ we have, combinatorially, a single sequence of d_1 weakly monotonically increasing transpositions. This was the case considered in [9, 10], and the τ-function shown to be identifiable with the Harish-Chandra-Itzykson-Zuber (HCIZ) matrix integral [20] when the expansion parameter is identified as $z = -1/N$ and the flow parameters equated to the trace invariants of a pair of Hermitian matrices. The case $l = 1, m = 1$ was explained combinatorially in [12], and given a matrix model representation. The case when l is arbitrary and $m = 0$ was considered in [1], and its combinatorial interpretation was given in [12], but no general enumerative geometric interpretation seems previously to have been provided, except for the case of Belyi curves where $l = 2$, $m = 0$ and μ is the trivial partition, which was studied in detail in [33, 2, 18]

A further special case can be obtained by choosing l arbitrary, $m = 0$, and summing over all coverings with l additional branch points and fixed genus. Letting

$$c := \sum_{a=1}^{l} c_a \qquad (2.46)$$

be the sum of the compliments of their ramification lengths (i.e. the number of pre-images), the Riemann-Hurwitz formula (2.43) holds. Let

$$F^{(c,l)}(\mu, \nu) = \sum_{\substack{c_1, \ldots, c_l \\ \sum_{a=1}^{l} c_a = c}} F^{(c_1, \ldots, c_l)}(\mu, \nu) \qquad (2.47)$$

be the total number of branched covers with up to $2 + l$ branch points, ramification profiles (μ, ν) at $(0, \infty)$ and genus given by (2.43). The specialization of the τ-function $\tau^{q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s})$ to the case $m = 0$, $w_1 = \cdots = w_l = w$ gives

$$\tau^{(q,(w)^{\otimes l})}(\mathbf{t}, \mathbf{s}) = \sum_{\lambda} q^{|\lambda|} (r_\lambda^{(1,w)})^l S_\lambda(\mathbf{t}) S_\lambda(\mathbf{s})$$

$$= \sum_{\lambda} q^{|\lambda|} \sum_{\substack{\mu, \nu \\ |\mu| = |\nu| = |\lambda|}} F^{(c,l)}(\mu, \nu) w^c P_\mu(\mathbf{t}) P_\nu(\mathbf{s}). \qquad (2.48)$$

The combinatorial interpretation of $F^{(c,l)}(\mu, \nu)$ is that it equals the number of c-step paths in the Cayley graph of S_n consisting of l sequential segments that are strictly monotonically increasing, with segment lengths anywhere between 1 and n.

The next two sections provide the proofs of Theorems 2.1 and 2.2.

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3 Geometric interpretation: generating function for generalized, signed Hurwitz numbers

3.1 Double power sum expansion of $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t},\mathbf{s})$

The Hurwitz number $H_{g_0}(\mu^{(1)},\ldots\mu^{(j)})$, where $|\mu^{(a)}| = n$, $a = 1,\ldots,j$ is the number of n sheeted branched covers, up to isomorphism, of a Riemann surface of genus g_0, with j branch points $\{q_1,\ldots,q_j\}$, whose ramification profiles are given by the partitions $\{\mu^{(a)}\}_{a=1,\ldots,j}$. The genus of the covering curve is given by the Riemann-Hurwitz formula

$$g = \frac{1}{2}\left(\sum_{a=1}^{j} \ell(\mu^{(a)}) - nj\right) + n(g_0 - 1) + 1 \tag{3.1}$$

Hurwitz numbers can be expressed as sums over products of the irreducible characters χ_λ of S_n evaluated at the conjugacy classes corresponding to the partitions using Frobenius' formula, [13], [22, Appendix A]:

$$H_{g_0}(\mu^{(1)},\ldots\mu^{(j)}) = \sum_{\lambda,\,|\lambda|=n} h_\lambda^{j+2g_0-2} \prod_{a=1}^{j} \frac{\chi_\lambda(\mu^{(a)})}{Z_{\mu^{(a)}}}. \tag{3.2}$$

We only consider the case where the base curve is the Riemann sphere, so $g_0 = 0$ and (3.2) becomes

$$H(\mu^{(1)},\ldots\mu^{(j)}) = \sum_{\lambda,\,|\lambda|=n} h_\lambda^{j-2} \prod_{a=1}^{j} \frac{\chi_\lambda(\mu^{(a)})}{Z_{\mu^{(a)}}}. \tag{3.3}$$

In order to apply this, we first express the content product formula (2.21) for $r_\lambda^{(q,\mathbf{w},\mathbf{z})}$ in terms of the extended Pochhammer symbols for partitions:

$$(u)_\lambda := \prod_{i=1}^{\ell(\lambda)} (u - i + 1)_{\lambda_i}, \quad (u)_i = \prod_{i=0}^{i-1} (u + i). \tag{3.4}$$

Let

$$u_a := \frac{1}{w_a}, \quad v_b := -\frac{1}{z_b}. \tag{3.5}$$

Then

$$\prod_{a=1}^{l} \prod_{(ij)\in\lambda} (1 + w_a(j - i)) = \prod_{a=1}^{l} \frac{(u_a)_\lambda}{u_a^{|\lambda|}}$$
$$\prod_{b=1}^{m} \prod_{(ij)\in\lambda} (1 - z_b(j - i)) = \prod_{b=1}^{m} \frac{(v_b)_\lambda}{(v_a)^{|\lambda|}}. \tag{3.6}$$

The Pochhammer symbols may be written in terms special evaluations of Schur functions [23, 29] as

$$(u)_\lambda = \frac{S_\lambda(\mathbf{t}(u))}{S_\lambda(\mathbf{t}_\infty)} = h_\lambda S_\lambda(\mathbf{t}(u)) \tag{3.7}$$

where

$$\mathbf{t}(u) := (u, u/2, \ldots, u/i, \ldots), \quad \mathbf{t}_\infty := (1, 0, 0, \ldots), \tag{3.8}$$

while the power sum symmetric functions at these values are given by

$$P_\mu(\mathbf{t}(u)) = u^{\ell(\mu)}, \quad P_\mu(\mathbf{t}_\infty) = \delta_{(\mu, 1^{|\mu|})} \tag{3.9}$$

Using the Frobenius character formula (2.11), this gives

$$(u)_\lambda = h_\lambda \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{Z_\mu} u^{\ell(\mu)}, \tag{3.10}$$

and hence

$$\prod_{a=1}^{l} \prod_{(i\,j)\in\lambda} (1 + w_a(j - i)) = \prod_{a=1}^{l} \left(h_\lambda \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{Z_\mu} w_a^{\ell^*(\mu)} \right), \tag{3.11}$$

$$\prod_{b=1}^{m} \prod_{(i\,j)\in\lambda} (1 - z_b(j - i)) = \prod_{b=1}^{m} \left(h_\lambda \sum_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)} \right), \tag{3.12}$$

where

$$\ell^*(\mu) := |\mu| - \ell(\mu), \quad \ell^*(\nu) := |\nu| - \ell(\nu) \tag{3.13}$$

are the complements of the lengths. Substituting into the content product formula gives

$$r_\lambda^{(q,\mathbf{w},\mathbf{z})} = q^{|\lambda|} \prod_{(i,j)\in\lambda} \left(\frac{\prod_{a=1}^{l}(1 + w_a(j - i))}{\prod_{b=1}^{m}(1 - z_b(j - i))} \right) = \frac{\prod_{a=1}^{l} h_\lambda \sum_{\mu, |\mu|=|\lambda|} \frac{\chi_\lambda(\mu)}{Z_\mu} w_a^{\ell^*(\mu)}}{\prod_{b=1}^{m} h_\lambda \sum_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)}}. \tag{3.14}$$

In order to expand as a power series in the parameters $\mathbf{w} = (w_1, \ldots w_l)$, $\mathbf{z} = (z_1, \ldots z_m)$, we express the factors appearing in the denominator as

$$h_\lambda \sum_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)} = 1 + h_\lambda {\sum}'_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)} \tag{3.15}$$

where $\sum'_{\nu, |\nu|=|\lambda|}$ means the sum with the identity class $\nu = 1^{|\lambda|}$ omitted. Expanding each denominator factor gives

$$\frac{1}{1 + h_\lambda {\sum}'_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)}} = \sum_{j_b=0}^{\infty} (-1)^{j_b} \left(h_\lambda {\sum}'_{\nu, |\nu|=|\lambda|} \frac{\chi_\lambda(\nu)}{Z_\nu} (-z_b)^{\ell^*(\nu)} \right)^{j_b}. \tag{3.16}$$

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3.2 Signed counting of constrained branched covers

We now combine the computations of the previous subsection for the content product formula expression $r_\lambda^{(q,\mathbf{w},\mathbf{z})}$ with the Frobenius character formula (2.11) to expand the τ-function (2.25) in a series consisting of products of pairs of power sum symmetric functions with coefficients that are Taylor series in the parameters $(q, w_1, \ldots, w_l, z_1, \ldots, z_m)$. Substituting the content product formulae (3.14) together with the denominator expansions (3.16) into (2.25), applying the Frobenius formula (3.3) for the Hurwitz numbers, and combining all monomial terms in like powers of w_a's and z_b's, we obtain

$$\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s}) = \sum_{n=1}^{\infty} q^n \sum_{\mu,\nu,|\mu|=|\nu|=n} \sum_{\mathbf{c}\in\mathbf{N}^l} \sum_{\mathbf{d}\in(\mathbf{N}^+)^m} F_{\mathbf{d}}^{\mathbf{c}}(\mu,\nu)\mathbf{w}^{\mathbf{c}}\mathbf{z}^{\mathbf{d}} P_\mu(\mathbf{t}) P_\nu(\mathbf{s}), \qquad (3.17)$$

where multi-index notation has been used:

$$\mathbf{w}^{\mathbf{c}} := \prod_{a=1}^{l} w_a^{c_a}, \quad \mathbf{z}^{\mathbf{d}} := \prod_{b=1}^{m} z_b^{c_b} \qquad (3.18)$$

and

$$F_{\mathbf{d}}^{\mathbf{c}}(\mu,\nu) = (-1)^{mn+D} \sum_{\substack{\mu^{(a)}, \; \ell^*(\mu^{(a)})=c_a \\ \nu^{(b,i_b)}, \; \sum_{i_b=1}^{j_b} \ell^*(\nu^{(b,i_b)})=d_b}} \sum_{j_1=1}^{d_1} \cdots \sum_{j_m=1}^{d_m} \sum_{i_1=1}^{j_1} \cdots \sum_{i_m=1}^{j_m} (-1)^C H(\mu,\nu,\{\mu^{(a)}\},\{\nu^{(b,i_b)}\}),$$

$$(3.19)$$

where

$$C = \sum_{b=1}^{m} j_b, \qquad (3.20)$$

is the total number of coloured branch points and

$$D = \sum_{b=1}^{m} d_b = \sum_{b=1}^{m} \sum_{i_b=1}^{j_b} \ell^*(\nu^{(b,i_b)}) \qquad (3.21)$$

is the sum of the complements of their ramification profile weights, which proves Theorem 2.1.

Remark 3.1. Note that because of the constraints

$$\ell^*(\mu^{(a)}) = c_a, \quad \sum_{i_b=1}^{j_b} \ell^*(\nu^{(b,i_b)}) = d_b \qquad (3.22)$$

and the fact that all partitions have the fixed weight

$$|\mu^{(a)}| = |\nu^{(b,i_b)}| = |\mu| = |\nu| = n, \qquad (3.23)$$

the number of terms in the sum (3.17) is finite.

4 Combinatorial interpretation: multimonotonic paths in the Cayley graph of S_n

4.1 The $\{C_\mu\}$ and $\{F_\lambda\}$ bases for the center $\mathbf{Z}(\mathbf{C}[S_n])$

The following is a brief version of the method developed in ref. [12], to which the reader is referred for further details. We recall two standard bases for the center $\mathbf{Z}(\mathbf{C}[S_n])$ of the group algebra, both labelled by partitions of weight n. The first consists of the cycle sums, defined by

$$C_\mu = \sum_{g \in cyc_\mu} g \text{ for } |\mu| = n, \tag{4.1}$$

where cyc_μ denotes the conjugacy class with cycle type μ. The second consists of the orthogonal idempotents, which may be defined as

$$F_\lambda := h_\lambda^{-1} \sum_{\mu, |\mu| = |\lambda|} \chi_\lambda(\mu) C_\mu. \tag{4.2}$$

These satisfy the relations

$$F_\lambda F_\mu = \delta_{\lambda\mu} F_\lambda. \tag{4.3}$$

from which it follows that these are eigenvectors under multiplication by any element of the center $\mathbf{Z}(\mathbf{C}[S_n])$. The *Jucys-Murphy elements* [21, 24, 3], defined as consecutive sums of transpositions

$$\mathcal{J}_b := \sum_{a=1}^{b-1} (ab), \quad b = 1, \ldots, n \tag{4.4}$$

generate a commutative subalgebra of $\mathbf{C}[S_n]$. Moreover, if $G \in \Lambda$ is a symmetric function, the substitution of the $\mathcal{J}_a$'s for the indeterminants gives an element of the center $\mathbf{Z}(\mathbf{C}[S_n])$

$$G(\mathcal{J}) := G(\mathcal{J}_1, \ldots, \mathcal{J}_n) \in \mathbf{Z}(\mathbf{C}[S_n]) \tag{4.5}$$

whose eigenvalues, under multiplication of F_λ are equal to the evaluation of G on the content of the Young diagram of the partition λ

$$G(\mathcal{J}) F_\lambda = G(\text{cont}(\lambda)) F_\lambda, \tag{4.6}$$

where $\text{cont}(\lambda)$ denotes the n element set consisting of the numbers $j - i$, where $(i, j) \in \lambda$ are the locations of the boxes in the Young diagram.

4.2 Multimonotonic paths in the Cayley Graph

In particular, if we choose G to be the generating functions of the elementary and complete symmetric functions

$$E(w, \mathcal{J}) = \prod_{a=1}^{n}(1 + w\mathcal{J}_a) = \sum_{j=0}^{n} e_j(\mathcal{J})w^j \qquad (4.7)$$

$$H(z, \mathcal{J}) = \prod_{a=1}^{n}(1 - z\mathcal{J}_a)^{-1} = \sum_{j=0}^{\infty} h_j(\mathcal{J})w^j, \qquad (4.8)$$

we obtain

$$E(w, \mathcal{J})F_\lambda = \prod_{(ij)\in\lambda}(1 + w(j - i))F_\lambda \qquad (4.9)$$

$$H(z, \mathcal{J})F_\lambda = \prod_{(ij)\in\lambda}(1 - z(j - i))^{-1}F_\lambda. \qquad (4.10)$$

Combining these by multiplication gives

$$q^{|\lambda|}\prod_{a=1}^{l}\prod_{b=1}^{m} E(w_a, \mathcal{J})H(z_b, \mathcal{J})F_\lambda = r_\lambda^{(q,\mathbf{w},\mathbf{z})}F_\lambda. \qquad (4.11)$$

On the other hand, the expansions (4.7), (4.8), together with the fact that

$$e_i(\mathcal{J}) = \sum_{\substack{(j_1,\ldots,j_i)\subset(1,\ldots,n) \\ j_1 < j_2 \cdots < j_i}} \mathcal{J}_{j_1}\cdots\mathcal{J}_{j_i} \qquad (4.12)$$

$$f_i(\mathcal{J}) = \sum_{\substack{(j_1,\ldots,j_i)\subset(1,\ldots,n) \\ j_1 \le j_2 \cdots \le j_i}} \mathcal{J}_{j_1}\cdots\mathcal{J}_{j_i} \qquad (4.13)$$

and the definition (4.4) of the Jucys-Murphy elements implies that if this same element is applied to the cycle sums, we obtain

$$q^{|\lambda|}\prod_{a=1}^{l}\prod_{b=1}^{m} E(w_a, \mathcal{J})H(z_b, \mathcal{J})C_\mu = q^{|\lambda|}\sum_{\nu,|\nu|=|\mu|} \tilde{F}_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)\mathbf{w}^{\mathbf{c}}\mathbf{z}^{\mathbf{d}}C_\nu, \qquad (4.14)$$

where $\tilde{F}_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu)$ is the number of products of k transpositions $(a_1b_1)\cdots(a_kb_k)$,

$$k = \sum_{a=1}^{l} c_a + \sum_{b=1}^{m} d_b \qquad (4.15)$$

satisfying

$$C_\nu = (a_1b_1)\cdots(a_kb_k)C_\mu \qquad (4.16)$$

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such that these may be grouped into successive sequences, corresponding to each of the factors in the product $\prod_{a=1}^{l} \prod_{b=1}^{m} E(w_a, \mathcal{J}) H(z_b, \mathcal{J})$, expanded in powers of w_a and z_b. These consist first of a sequence of l bands of transpositions having lengths c_a, $a = 1, \ldots, l$ that are strictly monotonically increasing in the second factors of $(a_i \ b_i)$, $b_i < b_{i+1}$, followed by m bands of lengths d_b, $b = 1, \ldots, m$, in which they are weakly monotonically increasing.

Substituting the change of basis formula (4.2) into (4.14) and equating the coefficients in the sums over the F_λ basis gives:

$$\chi_\lambda(\mu) r_\lambda^{(q,\mathbf{w},\mathbf{z})} = q^{|\lambda|} \sum_{\nu, |\nu|=|\mu|=|\lambda|} Z_\nu \chi_\lambda(\nu) \sum_{\mathbf{c} \in \mathbf{N}^l} \sum_{\mathbf{d} \in \mathbf{N}^m} \tilde{F}_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu) \mathbf{w}^{\mathbf{c}} \mathbf{z}^{\mathbf{d}}. \tag{4.17}$$

By the orthogonality of group characters

$$\sum_{\lambda, |\lambda|=|\mu|=|\nu|} \chi_\lambda(\mu) \chi_\lambda(\nu) = Z_\mu \delta_{\mu\nu} \tag{4.18}$$

this is equivalent to

$$\sum_{\lambda, |\lambda|=n} r_\lambda^{(q,\mathbf{w},\mathbf{z})} \chi_\lambda(\mu) \chi_\lambda(\nu) = q^n Z_\mu Z_\nu \sum_{\mathbf{d} \in \mathbf{N}^m} \tilde{F}_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu) \mathbf{w}^{\mathbf{c}} \mathbf{z}^{\mathbf{d}}. \tag{4.19}$$

Therefore, by eqs. (2.25), (2.34) we have

$$\tilde{F}_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu) = F_{\mathbf{d}}^{\mathbf{c}}(\mu, \nu), \tag{4.20}$$

which completes the proof of Theorem 2.2.

Remark 4.1. It follows from the results of [19] and [16, Appendix A] that all these generating functions have representations as matrix integrals. They therefore also satisfy Virasoro constraints, and their multitrace resolvent correlators may be computed through the methods of topological recursion [5].

5 Fermionic representation of $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s})$

Finally, we give the fermionic representation of the 2D Toda τ-functions $\tau^{(q,\mathbf{w},\mathbf{z})}(\mathbf{t}, \mathbf{s})$, following [4, 31, 29, 16, 17]. Any chain of 2D Toda τ-functions of hypergeometric type may be represented, in the fermionic Fock space approach, as vacuum state expectation values of the form

$$\tau_r(N, \mathbf{t}, \mathbf{s}) = \langle N | \hat{\gamma}_+(\mathbf{t}) \hat{C}_\rho \hat{\gamma}_-(\mathbf{s}) | N \rangle, \tag{5.1}$$

where $\langle N |$, $|N \rangle$ denote the left and right vacuum vectors in the Nth charge sector,

$$\hat{\gamma}_+(\mathbf{t}) := e^{\sum_{i=1}^{\infty} t_i J_i}, \quad \hat{\gamma}_-(\mathbf{s}); = e^{\sum_{i=1}^{\infty} s_i J_{-i}}, \quad \text{where } J_i := \sum_{j \in \mathbf{Z}} \psi_j \psi_{i+j}^\dagger, \quad i \in \mathbf{N}, \tag{5.2}$$

are the generators of the two infinite abelian Toda flows, $\{\psi_i, \psi_i^\dagger, i \in \mathbf{Z}\}$ are Fermi creation and annihilation operators that satisfy the usual anticomutation relations and vacuum annihilation conditions

$$[\psi_i, \psi_j^\dagger]_+ = \delta_{ij}, \quad \psi_i|N\rangle = 0 \text{ if } i < N, \quad \psi_i^\dagger|N\rangle = 0 \text{ if } i \geq N \tag{5.3}$$

and

$$\hat{C}_\rho = e^{\sum_{j \in \mathbf{Z}} T_j :\psi_j \psi_j^\dagger:} \tag{5.4}$$

is an element of the infinite abelian group of diagonal elements that generate *generalized convolution flows* [17]. The orthonormal Fermionic Fock basis states $|\lambda; N\rangle$ are labelled by pairs (λ, N) consisting of a partition λ and an integer N.

$$|\lambda; N\rangle := (-1)^{\sum_{i=1}^r b_i} \prod_{i=1}^r \psi_{a_i+N} \psi_{-b_i-1+N}^\dagger |N\rangle \tag{5.5}$$

where the partition λ, expressed in Frobenius notation [23], is $(a_1, \ldots, a_r | b_a, \ldots, b_r)$.

The double Schur function expansion (2.10) follows from the fact that Schur functions have the following fermionic matrix element expressions

$$S_\lambda(\mathbf{t}) = \langle \lambda; N | \hat{\gamma}_- \mathbf{t} | N \rangle = \langle N | \hat{\gamma}_- \mathbf{t} | \lambda; N \rangle \tag{5.6}$$

which, in turn, follow from Wick's theorem. Defining

$$\rho_j := e^{T_j}, \tag{5.7}$$

it follows that the basis vectors $|\lambda; N\rangle$ are eigenvectors of the convolution flow group elements

$$\hat{C}_\rho |\lambda; N\rangle = r_\lambda(N)|\lambda; N\rangle, \tag{5.8}$$

with eigenvalues $r_\lambda(N)$ given by the content product formula (2.8). Inserting a sum over a complete set of intermediate states, gives the double Schur function expansion (2.10) for $\tau_r(N, \mathbf{t}, \mathbf{s})$.

The particular case (2.24) corresponding to our family $\tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s})$ of generating functions is obtained by choosing the parameters ρ_j to be $\rho_j^{(q,\mathbf{w},\mathbf{z})}$, as defined in (2.19). The corresponding values of the convolution flow parameters are

$$T_j^{(q,\mathbf{w},\mathbf{z})} := j \ln q + \sum_{k=1}^j \sum_{a=1}^l \ln(1 + kw_a) - \sum_{k=1}^j \sum_{b=1}^m \ln(1 - kz_b) \quad \text{for } j > 0,$$

$$T_0^{(q,\mathbf{w},\mathbf{z})} = 0,$$

$$T_{-j}^{(q,\mathbf{w},\mathbf{z})} := -j \ln q + \sum_{k=0}^{j-1} \sum_{b=1}^m \ln(1 + kz_b) - \sum_{k=0}^{j-1} \sum_{a=1}^l \ln(1 - kw_a) \quad \text{for } j > 0, \tag{5.9}$$

which gives

$$\tau_r(N, \mathbf{t}, \mathbf{s}) = \tau^{(q,\mathbf{w},\mathbf{z})}(N, \mathbf{t}, \mathbf{s}). \tag{5.10}$$

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Remark 5.1. Besides the $N = 0$ sector of the fermionic Fock space, with orthonormal basis $\{|\lambda; 0\rangle\}$, on which the group of generalized convolution flows acts diagonally, and the direct sum $\oplus_{n=0}^{\infty} \mathbf{Z}(\mathbf{C}[S_n])$, where a corresponding infinite group acts diagonally, with the same eigenvalues, on the basis $\{F_\lambda\}$, as detailed in ref. ([12]), there is also the bosonic Fock space representation, in which the τ-function is viewed as a symmetric function of an infinite number of bosonic variables, with orthonormal basis given by the Schur functions $\{S_\lambda\}$. A corresponding infinite abelian group of operators acts diagonally in this space, with eigenvalues also given by a content product formula. Their infinitesimal generators are expressible as differential operators in the flow parameters $(t_1, t_2, \dots)$ with polynomial coefficients, referred to sometimes as *cut and join operators* [11, 1, 25]. These may be viewed as generating an abelian group through exponentiation (i.e. the solution of a diffusion-like equation), with a vacuum τ-function as initial condition. But these are not, in general, symmetries of the 2D Toda hierarchy and the solutions are not necessarily 2D Toda τ-functions, even though they admit series representations as diagonal sums over products of Schur functions. Only the smaller group, consisting of operators acting as generalized convolution flows [17], whose eigenvalues are of the content product form give rise to hypergeometric τ-functions.

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