Math 8680 Apr. 21, 2021

Re-interpreting $(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{r-1})$

This combinatorial interpretation is key to the A-H-K proof.

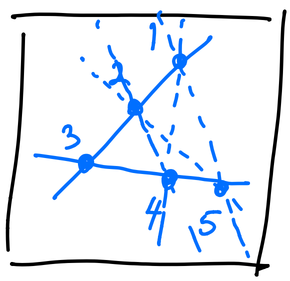
DEF'N: In matroid M on $E=\{1,2,\ldots,n\}$, call a chain/flag of flats $(\emptyset \neq) F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \,(\neq E)$

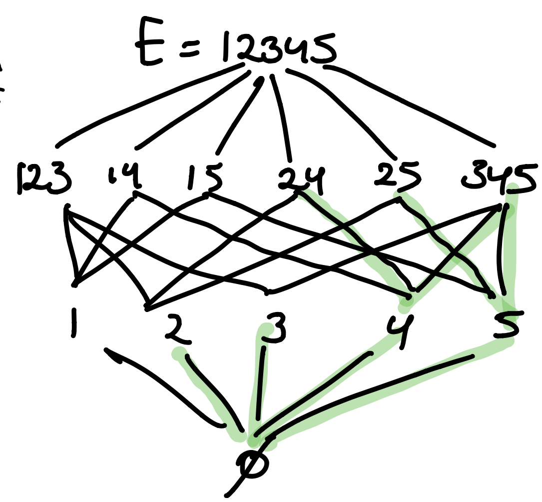
- **descending** if $\min(F_1) > \min(F_2) > \cdots > \min(F_k) > \underset{\min(E)}{\overset{1}{\shortparallel}}$
- **initial** if $r(F_i) = i$ for $i=1,2,\ldots,k$

THEOREM: $(\bar{w}_0, \bar{w}_1, \ldots, \bar{w}_{r-1})$ has $\underset{1=}{\bar{w}_0}$ for $k=1,2,\ldots,r-1$

$\bar{w}_k$ = #descending initial flags $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of flats in M

EXAMPLE:

$M =$ 

poset of flats L_M 

had $(w_0, w_1, w_2, w_3) = (1,5,8,4)$

$(\bar{w}_0, \bar{w}_1, \bar{w}_2) = (1,4,4)$

counts $F_1 =$ 2
3
4
5

counts $F_1 \subset F_2$ = 4 — 24
4 — 345
5 — 25
5 — 345

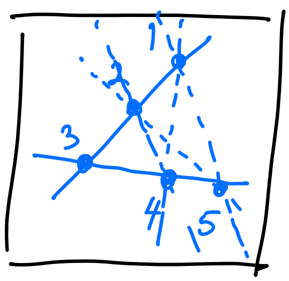
Before proving it, it helps to learn one last matroid construction:

Truncation: $\mathrm{Trunc}(M)$ has same ground set E as M, but whose flats are

$$\mathcal{F}(\mathrm{Trunc}(M)) := \{\text{flats } F \in \mathcal{F}(M) : r(F) \neq r(M)-1\}$$

It models $(v_i)_{i \in E} \rightsquigarrow (\pi(v_i))_{i \in E}$

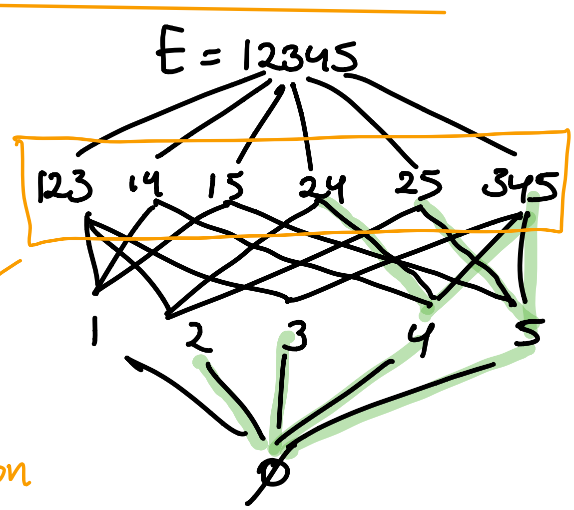
$$\underset{K^r}{\cap} \xrightarrow{\ \pi\ } K^{r-1}$$ a generic K-linear projection (which might require a field extension $K \subset \overline{K}$)

$M =$ 

$\{v_1, \dots, v_5\} \subset \mathbb{R}^3$

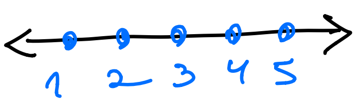
$\downarrow \pi$

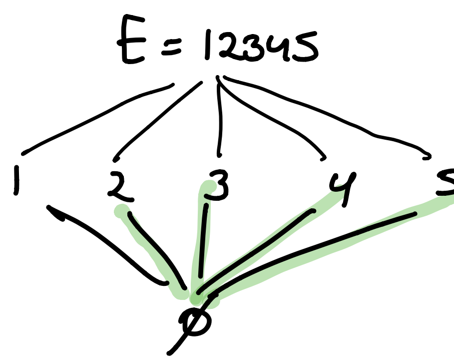
$\{\pi(v_1), \dots, \pi(v_5)\} \subset \mathbb{R}^2$

$\mathcal{F}(M)$ 

removed in truncation

$\mathrm{Trunc}(M)$



$\mathcal{F}(\mathrm{Trunc}(M))$ 

proof of THEOREM: It suffices to prove only the assertion

$$\overline{\omega}_{r-1} = \#\text{ descending (initial) flags } F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r-1}$$

since for all other $k = 1, 2, \ldots, r-2$, the notion of descending initial flags $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ is same for M and $\mathrm{Trunc}(M)$, while $(\omega_0, \omega_1, \ldots, \omega_{r-1})$, $(\overline{\omega}_0, \overline{\omega}_1, \ldots, \overline{\omega}_{r-2})$ are also same for M, $\mathrm{Trunc}(M)$.

Note that $\overline{\omega}_{r-1} = \omega_r = (-1)^{r(E)} \mu(\emptyset, E)$, and so it suffices to show for every flat F of M that

$$m(F) := (-1)^{r(F)} \cdot \#\text{ descending flags in } F$$

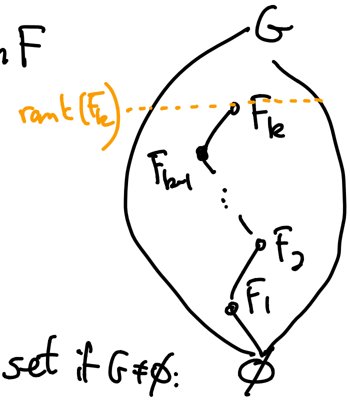
$$F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{r(F)-1} \subsetneq F_{r(F)} = F$$

meaning $\min(F_i) > \min(F_{i+1})$

satisfies the Möbius function recursion $\displaystyle\sum_{F:\, \emptyset \subseteq F \subseteq G} m(F) = \delta_{\emptyset, G}$:

$$\sum_{F:\, \emptyset \subseteq F \subseteq G} m(F) = \sum_{F:\, \emptyset \subseteq F \subseteq G} (-1)^{r(F)} \cdot \#\text{descending flags in } F$$

$$= \sum_{\text{initial flags } \emptyset \neq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subseteq G} (-1)^{\mathrm{rank}(F_k)}$$



Do a sign-reversing involution on the summation set if $G \neq \emptyset$:

if $\min F_k = \min G$ then remove F_k, giving $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k-1} (\subset G)$

if $\min F_k > \min G$ then add $F_{k+1} = F_k \vee \min G$ giving $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_{k-1} \subsetneq F_k (\subseteq G)$ ∎

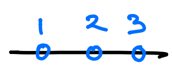
Math 8680 Apr. 23, 2021

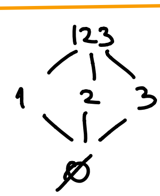
Chow rings and Bergman fans for matroids

A-H-K re-interpret $(\bar\mu_0, \bar\mu_1, \dots, \bar\mu_{r-1})$ for a matroid M inside Feichtner & Yuzvinsky's (2003) Chow ring $A(M)$

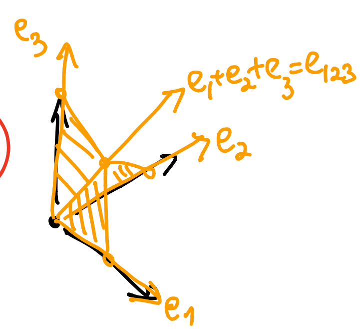
DEF'N: The pre-Bergman fan $\hat{\Sigma}(M)$ for a matroid M on $E=\{1,2,\dots,n\}$ lives in $\mathbb{R}^n$ with basis $e_1,\dots,e_n$ and has rays spanned by $e_F := \sum_{i\in F} e_i$ for all non-empty flats $F \in \mathcal{F}(M)-\{\varnothing\}$ with cones spanned by $\{e_{F_1},\dots,e_{F_k}\}$ for chains $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$

The Bergman fan $\Sigma(M)$ lives in $\mathbb{R}^n / \mathbb{R}(\underbrace{e_1+\dots+e_n}_{e_E})$ with rays e_F for non-empty, proper flats $F \in \mathcal{F}(M)-\{\varnothing, E\}$ cones again spanned by $\{e_{F_1},\dots,e_{F_k}\}$ for chains $F_1 \subsetneq \cdots \subsetneq F_k$

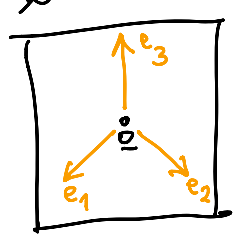
EXAMPLE: $M =$ 

has $L_M =$ 

$\hat{\Sigma}(M) \subset \mathbb{R}^3$

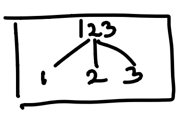


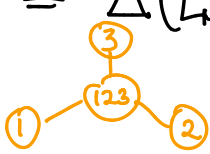
$\Sigma(M) \subset \mathbb{R}^3 / \mathbb{R}\begin{bmatrix}1\\1\\1\end{bmatrix}$



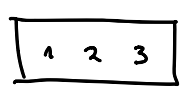
Note their associated simplicial complexes are order complexes:

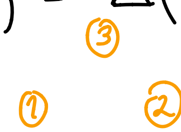
$\Delta_{\hat{\Sigma}(M)} \cong \Delta(L_M - \{\varnothing\})$





$\Delta_{\Sigma(M)} \cong \Delta(L_M - \{\varnothing, E\})$





PROP/DEF'N: The fans $\Sigma = \hat{\Sigma}(M), \Sigma(M)$ have isomorphic "cohomology" rings $H(\Sigma) := R_\Sigma / A_+ R_\Sigma \cong \mathbb{R}[\Delta_\Sigma]/(\underline{\Theta}_\Sigma)$,
Fleming-Karu
called the Chow ring $A(M)$, which has two presentations:

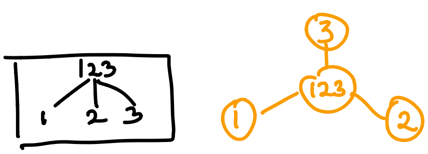
$$A(M) = \underbrace{\mathbb{R}[x_F]_{F\in\mathcal{F}(M)-\{\emptyset\}} \Big/ \left(x_F x_G \,:\, \begin{matrix} F,G \text{ incomparable} \\ F\not\subseteq G,\ G \not\subseteq F \end{matrix} \right)}_{K[\Delta_{\hat{\Sigma}(M)}]} \Big/ \underbrace{\Big(\sum_{F:\, i\in F} x_F \Big)_{i=1,2,\dots,n}}_{\substack{(\theta_1,\dots,\theta_n) \\ = (\underline{\theta}_{\hat{\Sigma}(M)})}}$$

Feichtner-Yuzvinsky 2003

$$= \underbrace{\mathbb{R}[x_F]_{F\in\mathcal{F}(M)-\{\emptyset,E\}} \Big/ \left(x_F x_G \,:\, F,G \text{ incomparable} \right)}_{K[\Delta_{\Sigma(M)}]} \Big/ \underbrace{\Big(\sum_{F:\, i\in F\neq E} x_F - \sum_{F:\, j\in F\neq E} x_F \Big)_{i\neq j}}_{= (\theta_{ij})_{i\neq j} = (\underline{\theta}_{\Sigma(M)})}$$

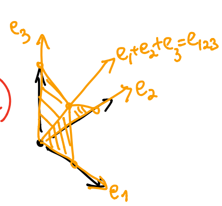
A-H-K 2015

EXAMPLE

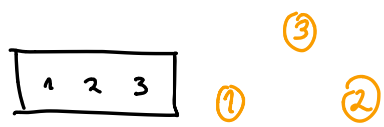


$$A(M) = K[x_1, x_2, x_3, x_{123}] \Big/ \left(\begin{matrix} x_1x_2, \\ x_1x_3, \\ x_2x_3, \\ \theta_1 = x_{123} + x_1, \\ \theta_2 = x_{123} + x_2, \\ \theta_3 = x_{123} + x_3 \end{matrix} \right)$$

$\hat{\Sigma}(M)$
$\cap$
$\mathbb{R}^3$

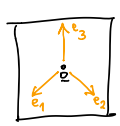


$$\cong K[x_1]/(x_1^2)$$



$$A(M) = K[x_1, x_2, x_3] \Big/ \left(\begin{matrix} x_1x_2, \\ x_1x_3, \\ x_2x_3, \\ \theta_{12} = x_1 - x_2, \\ \theta_{13} = x_1 - x_3, \\ \theta_{23} = x_2 - x_3 \end{matrix} \right)$$

$\Sigma(M)$
$\cap$
$\mathbb{R}^3 / \mathbb{R}\left[\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right]$



$$\cong K[x_1]/(x_1^2)$$

proof: The fact that those two presentations agree with $\mathbb{K}[\Delta_\Sigma]/(\underline{\Theta}_\Sigma)$ for $\Sigma = \hat{\Sigma}(M), \Sigma(M)$ is straightforward:

- use $x_1, \ldots, x_n$ as $\mathbb{R}$-basis for $(\mathbb{R}^n)^*$, giving $\theta_i = \sum_{F : i \in F} x_F$

- use $x_i - x_j$ as $\mathbb{R}$-spanning set for $\left(\mathbb{R}^n / \mathbb{R}(e_1 + \cdots + e_n)\right)^*$, giving $\theta_{ij} = \sum_{i \in F \neq E} x_F - \sum_{j \in F \neq E} x_F$

(EXER 3(a) in 2nd part HW)

Can check an isomorphism between the rings is induced from these back-and-forth maps:

$$
\begin{array}{ccccc}
x_F & \mathbb{R}[x_F]_{F \in \mathcal{F}(M) - \{\emptyset\}} & & \text{if } F \neq E, & \\
 & & & x_F & x_E \\
g \uparrow & g \uparrow \; \downarrow f & & f \downarrow & f \downarrow \\
x_F & \mathbb{R}[x_F]_{F \in \mathcal{F}(M) - \{\emptyset, E\}} & & x_F & -\alpha
\end{array}
$$

where $\alpha := \alpha_i := \sum_{F : i \in F \neq E} x_F$

for any $i = 1, 2, \ldots, n$

$\blacksquare$

From what we've proven generally about $H(\Sigma)$ for fans Σ,

$A(M) = H(\Sigma_M)$ is $\mathbb{R}$-spanned by square-free monomials

having support in $\Delta_{\Sigma_M} = \Delta(L_M - \{\phi, E\})$,

i.e. monomials $x_{F_1} x_{F_2} \cdots x_{F_k}$ with $(\phi \neq) F_1 \subsetneq \cdots \subsetneq F_k (\neq E)$

so $A(M) = A^0 \oplus A^1 \oplus A^2 \oplus \cdots \oplus A^{r-1}$ if $r = r(M)$

$A^0 = \mathbb{R}$

It turns out that $A^{r-1} \cong \mathbb{R}$ with $\mathbb{R}$-basis $\{x_E^{r-1}\}$ in the F-Y presentation

(not obvious!)

or $\{\alpha^{r-1}\}$ in the A-H-K presentation

so that we can define an evaluation/degree isomorphism

$$\begin{array}{ccc} A^{r-1} & \xrightarrow[\sim]{ev} & \mathbb{R} \\ f & \longmapsto & \langle f \rangle \end{array}$$

with $\alpha^{r-1} \longmapsto +1$

F-Y actually give an $\mathbb{R}$-basis for $A(M)$, via Gröbner theory.

DEF'N: Given a monomial order $\prec$ on $k[x_1, \ldots, x_n] = k[\underline{x}]$

a total order $m \prec m'$ which is a well-ordering compatible with multiplication: $m \prec m' \Rightarrow m \cdot m'' \prec m' \cdot m''$

one can talk about the initial/leading term $in_\prec(f) = m_0$

for $f = c_0 m_0 + c_1 m_1 + \cdots + c_t m_t$, where $m_0 \succ m_1, m_2, \ldots, m_t$

One then calls $\{g_1, \ldots, g_s\}$ a Gröbner basis (w.r.t. $\prec$) for the ideal $I = (g_1, \ldots, g_s)$ they generate if every $f \in I$ has $\mathrm{in}_\prec(f)$ divisible by at least one of $\mathrm{in}_\prec(g_i)$.

FACT: Every ideal $I \subset K[\underline{x}]$ has a Gröbner basis, which can be found using Buchberger's algorithm and checked using Buchberger's criterion.

(easy) PROP: If $\{g_1, \ldots, g_s\}$ are a Gröbner basis for $I = (g_1, \ldots, g_s)$ w.r.t. $\prec$, then $K[\underline{x}]/I$ has a K-basis given by the standard monomials

$$:= \left\{ \begin{array}{l} \text{monomials in } K[\underline{x}] \\ \text{divisible by none of } \mathrm{in}_\prec(g_i) \end{array} \right\}$$

Feichtner & Yuzvinsky applied Buchberger's Criterion to prove the following result.

THEOREM The following is a Gröbner basis presentation using lex order with $x_G > x_F$ if $G \supsetneq F$:

$$A(M) \cong \mathbb{R}[x_F]_{F \in \mathcal{F}(M) - \{\phi\}} \Big/ \Big(\underbrace{x_F x_G}_{\substack{F,G \\ \text{incomparable}}}, \underbrace{x_F \Big(\sum_{H: H \supseteq G} x_H\Big)^{r(G)-r(F)}}_{\text{if } F < G}, \underbrace{\Big(\sum_{H: H \supseteq G} x_H\Big)^{r(G)}}_{\forall G} \Big)$$

lex-leading terms: $x_F x_G$ if F, G incomparable; $x_F x_G^{r(G)-r(F)}$ if $F < G$; $x_G^{r(G)}$

As a consequence, the quotient $A(M)$ has $\mathbb{R}$-basis given by these standard monomials, that is, the monomials divisible by none of the lex-leading terms in the GB:

$$\left\{ x_{F_1}^{a_1} x_{F_2}^{a_2} \cdots x_{F_k}^{a_k} : (\phi \neq) F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subseteq E, \ a_i \leq r(F_i) - r(F_{i-1}) - 1 \right\}$$

In particular, the only standard monomial of degree $r-1$ is x_E^{r-1}, so it gives a basis for $A^{r-1}(M)$.

EXAMPLE It's worth comparing two $\mathbb{R}$-bases for $A(M) = H(\Sigma_M)$ when M is the Boolean matroid $(v_i)_{i=1,\dots,n}$ that has $\{v_1, \dots, v_n\}$ lin. independent, so L_M is the Boolean algebra of all subsets of $\{1, 2, \dots, n\}$

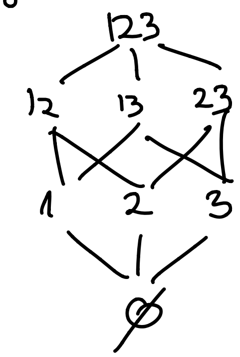
This is the unique situation where

$\Sigma_M = \mathcal{N}(P) = \mathcal{F}(P^\Delta)$ for a simple polytope,

that therefore also has a shelling basis for $H(\Sigma_M) \cong A(M)$

e.g. $n=3$

$L_M =$



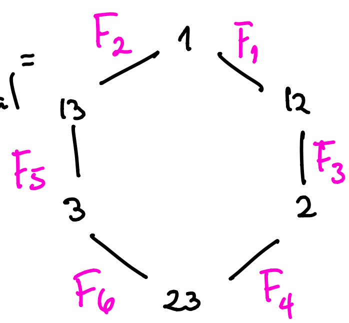
F-Y basis for $A(M)$ is

$$y_E^{r-1}$$

$$\Big\{ \underbrace{1,}_{A^0} \;\Big|\; \underbrace{x^1_{12},\, x^1_{13},\, x^1_{23},\, x^1_{123},}_{A^1} \;\Big|\; \underbrace{x^2_{123}}_{A^2} \Big\}$$

Not a basis known before F-Y.

$P^\Delta_{\text{simplicial}} =$



A standard shelling order $F_1, \dots, F_6$ shown

Shelling basis $\{x^{G_i}\}_{i=1,\dots,6}$ for $A(M)$ is

$$\Big\{ \underbrace{1,}_{A^0} \;\Big|\; \underbrace{x_{13},\, x_2,\, x_{23},\, x_3,}_{A^1} \;\Big|\; \underbrace{x_3 x_{23}}_{A^2} \Big\}$$

this generalizes to the descent monomial basis of A. Garsia, having a monomial for each permutation $w = (w_1, w_2, \dots, w_n)$ in S_n of form $\prod_{i: w_i > w_{i+1}} x_{w_1 w_2 \cdots w_i}$

e.g. $w = (3,1,5,4,2) \in S_5$ has descent monomial

$$x_3\, x_{135}\, x_{1345}$$

This choice of evaluation/degree isomorphism $A^{r-1}(M) \xrightarrow[\sim]{\langle\cdot\rangle} \mathbb{R}$

sending $\alpha^{r-1} \longmapsto +1$

where $\alpha := \alpha_i := -x_E = \sum_{\substack{\emptyset \neq F \neq E: \\ i \in F}} x_F$ for any $i = 1, 2, \ldots, n$

turns out to be very natural ...

PROP: (A-H-K PROP 5.8) For any flag $(\emptyset \neq) F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k (\neq E)$ of proper flats, one has in $A^r(M)$ that

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = \begin{cases} \alpha^{r-1} & \text{if } F_1 \subsetneq \cdots \subsetneq F_k \text{ is initial, i.e. } r(F_i) = i \ \forall i \\ 0 & \text{otherwise.} \end{cases}$$

In particular, when $k=0$, any two maximal flags $(F_i)_{i=1}^{r-1}$, $(F_i')_{i=1}^{r-1}$ have

$$x_{F_1} \cdots x_{F_{r-1}} = x_{F_1'} \cdots x_{F_{r-1}'} = \alpha^{r-1}$$

$$\Big\downarrow \langle\cdot\rangle = \text{evaluation/degree map}$$

$$+1$$

proof: A key observation: if one picks any $i \notin F$, then

$$x_F \cdot \alpha = x_F \sum_{\substack{\emptyset \neq G \neq E: \\ i \in G}} x_G = x_F \sum_{\substack{\emptyset \neq G \neq E \\ F \cup \{i\} \subseteq G}} x_G$$

In particular, if $r(F) = r-1$ then $x_F \alpha = 0$.

To show $x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = 0$ if (F_i) is not initial, use descending induction on k.

If $k = r-2$, $x_{F_1} x_{F_2} \cdots x_{F_{r-2}} \alpha = 0$ since (F_i) not initial implies $r(F_{r-2}) = r-1$ and $x_{F_{r-2}} \alpha = 0$ by KEY OBSERVATION.

If $k < r-2$, choose any $i \notin F_k$, so

$$x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} \underset{\text{KEY OBSERVATION}}{=} x_{F_1} x_{F_2} \cdots x_{F_k} \Big(\sum_{\substack{G: \\ G \supseteq F_k \cup \{i\}}} x_G \Big) \alpha^{r-2-k} \underset{\text{induction}}{=} 0$$

To show $x_{F_1} x_{F_2} \cdots x_{F_k} \alpha^{r-1-k} = \alpha^{r-1}$ if (F_i) is initial, use ascending induction on k.

If $k=1$, pick $i \in F_1$ (so F_1 = the rank 1 flat spanned by $\{i\}$) and

$$\alpha^{r-1} = \alpha \cdot \alpha^{r-2} = \Big(\sum_{\substack{F: \\ F \ni i}} x_F \Big) \alpha^{r-2} = x_{F_1} \cdot \alpha^{r-2}$$

by the non-initial case already proven

If $k > 1$, pick $i \in F_k \setminus F_{k-1}$ (so F_k = the rank k flat spanned by $F_{k-1} \cup \{i\}$) and use induction to write

$$\alpha^{r-1} = x_{F_1} \cdots x_{F_{k-1}} \alpha^{r-1-(k-1)} = x_{F_1} \cdots x_{F_{k-1}} \alpha \cdot \alpha^{r-1-k}$$

KEY OBSERVATION

$$= x_{F_1} \cdots x_{F_{k-1}} \Big(\sum_{\substack{G: \\ G \supseteq F \cup \{i\}}} \Big) \alpha^{r-1-k}$$

by the non-initial case already proven

$$= x_{F_1} \cdots x_{F_{k-1}} x_{F_k} \alpha^{r-1-k} \quad \blacksquare$$

This lets A-H-K reinterpret $(\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{r-1})$ using

$$\left.\begin{aligned} \alpha &= \alpha_i = \sum_{\substack{\emptyset \neq F \neq E:\\ i \in F}} x_F \\ \beta &= \sum_{\emptyset \neq F \neq E} x_F - \alpha = \sum_{\substack{\emptyset \neq F \neq E:\\ i \notin F}} x_F \end{aligned}\right\} \text{for any } i = 1, 2, \ldots, n$$

PROP: In $A(M)$, one has

$$\beta^k = \sum_{\substack{\text{descending flags } (\emptyset \neq) F_1 \subsetneq \cdots \subsetneq F_k (\neq E = F_{k+1}) \\ \min(F_i) > \min(F_{i+1})}} x_{F_1} x_{F_2} \cdots x_{F_k}$$

and hence one can reinterpret $\overline{w}_k$ as

$$\langle \alpha^{r-1-k} \beta^k \rangle = \# \text{ descending initial flags } (\emptyset \neq) F_1 \subsetneq \cdots \subsetneq F_k (\neq E) = \overline{\omega}_k$$

proof: Induct on k.

BASE CASE $k=1$ has $\beta^1 = \beta = \sum_{\substack{\emptyset \neq F \neq E \\ 1 \notin F}} x_F = \sum_{\substack{(\emptyset \neq) F_1 (\neq E): \\ \min F_1 > \min E}} x_F$ ✓

INDUCTIVE STEP:

$$\beta^k = \beta \cdot \beta^{k-1} = \sum_{\substack{(\emptyset \neq) F_1 \subsetneq \cdots \subsetneq F_k (\neq E) \\ \min F_i > \min F_{i+1}}} \beta \cdot x_{F_1} x_{F_2} \cdots x_{F_k} = \sum_{F_1 \subsetneq \cdots \subsetneq F_k} \sum_{\substack{F: \\ \min F_1 \notin F}} \underbrace{x_F \cdot x_{F_1} x_{F_2} \cdots x_{F_k}}$$

vanishes unless $F \subseteq F_1$

but then $\min F_1 \notin F \Rightarrow \min F > \min F_1$

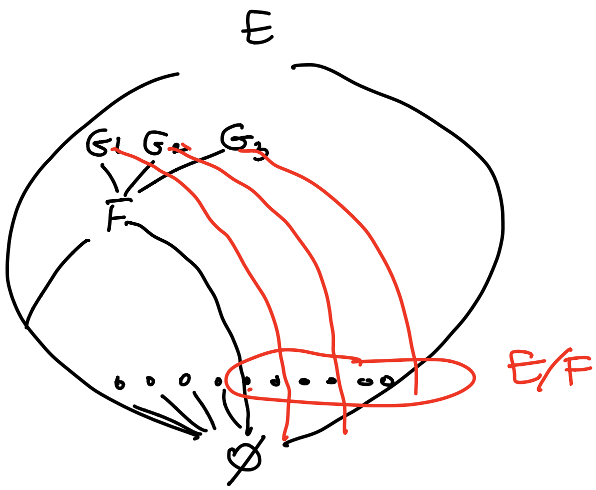
i.e. $F \subsetneq F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ is descending ∎

Flat axioms are important in the proofs interpreting $\langle \alpha^k \beta^{r-1-k} \rangle = \overline{w}_k$

(F1)

(F2)

(F3) $\forall$ flats F the flats G covering F partition $E \setminus F$



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Now the Kähler package for $A(M)$ becomes relevant.

THEOREM (A-H-K 2015) For any matroid M of rank r, the Chow ring $A(M) = \underset{\overset{\|}{\mathbb{R}}}{A^0} \oplus A^1 \oplus \cdots \oplus \underset{\underset{\mathbb{R}}{\downarrow \wr}}{A^{r-1}} \underset{\underset{\langle f \rangle}{\downarrow}}{\ni} \underset{\underset{+1}{\downarrow}}{\alpha^{r-1}}$

satisfy the Kähler package PD, HL, HRM

with Lefschetz elements $\ell = \sum_{\emptyset \neq F \neq E} c_F x_F \in A^1(M)$

coming from any coefficients $\{c_F\}_{F \in \mathcal{F}(M)}$ that are the restriction of a strictly submodular function $2^E \to \mathbb{R}$, $A \mapsto c_A$

$\left(c_{A \cup B} + c_{A \cap B} < c_A + c_B \text{ for } A \neq B \right)$

Let's see how this would imply the log-concavity THM for $(\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{r-3}, \overline{w}_{r-2}, \overline{w}_{r-1})$ i.e. $\overline{w}_k^2 \geq \overline{w}_{k-1} \cdot \overline{w}_{k+1}$.

1. Note one only needs to prove the last inequality

$$\overline{w}_{r-2}^{\,2} \geq \overline{w}_{r-3} \cdot \overline{w}_{r-1}$$

because the rest follow upon repeatedly replacing

$$\underset{(\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{r-3}, \overline{w}_{r-2}, \overline{w}_{r-1})}{M} \rightsquigarrow \underset{(\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{r-3}, \overline{w}_{r-2})}{\mathrm{Trunc}(M)}$$

2. The last inequality $\bar{\omega}_{r-2}^{2} \geq \bar{\omega}_{r-3} \cdot \bar{\omega}_{r-1}$ can be rephrased

$$\begin{array}{ccc} & \alpha & \beta \\ \begin{array}{c}\alpha \\ \\ \beta\end{array} & \multicolumn{2}{c}{\begin{bmatrix} \underbrace{\langle \alpha \cdot \alpha \cdot \beta^{r-3} \rangle}_{=\bar{\omega}_{r-3}} & \underbrace{\langle \alpha \cdot \beta \cdot \beta^{r-3} \rangle}_{=\bar{\omega}_{r-2}} \\ \underbrace{\langle \beta \cdot \alpha \cdot \beta^{r-3} \rangle}_{=\bar{\omega}_{r-2}} & \underbrace{\langle \beta \cdot \beta \cdot \beta^{r-3} \rangle}_{=\bar{\omega}_{r-1}} \end{bmatrix}} \end{array}$$

has its determinant

$$\bar{\omega}_{r-3}\bar{\omega}_{r-1} - \bar{\omega}_{r-2}^{2} \leq 0$$

nonpositive

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3. Replacing β by ℓ for any Lefschetz element $\ell \in A^1(M)$, the desired inequality would indeed hold:

$$\det \begin{array}{ccc} & a & \ell \\ \begin{array}{c}\alpha \\ \\ \ell\end{array} & \multicolumn{2}{c}{\begin{bmatrix} \langle \alpha \cdot \alpha \cdot \ell^{r-3} \rangle & \langle \alpha \cdot \ell \cdot \ell^{r-3} \rangle \\ \langle \ell \cdot \alpha \cdot \ell^{r-3} \rangle & \langle \ell \cdot \ell \cdot \ell^{r-3} \rangle \end{bmatrix}} \end{array} \leq 0$$

holds because this symmetric matrix expresses the quadratic form $Q_\ell(-)$ on the 2-plane $\mathbb{R}\ell \oplus \mathbb{R}\alpha$ inside $A^1(M)$, which has orthogonal decomposition from HRM:

$$A^1(M) = \ell \cdot A^0(M) \oplus PA^1(M)$$

$$A^0(M) = \mathbb{R} \quad (\uparrow \cdot \ell)$$

$Q_\ell(-)$ pos. def. | $Q_\ell(-)$ neg. def.

$\Rightarrow$ signature of $Q_\ell(-)$ on $\mathbb{R}\ell \oplus \mathbb{R}\alpha$ is either $(+,-)$ or $(+,0)$

$\Rightarrow$ $\det \leq 0$ either way

4. The element $\beta = \sum_{\substack{\emptyset \neq F \neq E: \\ i \notin F}} x_F = \sum_F b_F \cdot x_F$ where $b_A := \begin{cases} 0 \text{ if } i \in A \\ 1 \text{ if } i \notin A \end{cases}$

is not necessarily a Lefschetz element, since $A \mapsto b_A$ is only weakly submodular, not strictly.

$b_{A\cup B} + b_{A\cap B} \leq b_A + b_B$

But once we check that there exist some strictly submodular functions, e.g. $L_A := \#A(n - \#A)$

Very easy — HW EXERCISE 1(a) from 2nd part

then any $b_\epsilon := b + \epsilon \cdot L$ with $\epsilon > 0$ is strictly submodular,

so $\beta = \lim_{\epsilon \to 0} \beta_\epsilon$ where $\beta_\epsilon = \sum_F b_\epsilon(F) \cdot x_F$

for continuously varying Lefschetz elements $\{\beta_\epsilon\}_{\epsilon > 0}$.

Since each β_ϵ has its 2x2 matrix $\det \leq 0$, in the limit, the matrix for β also has $\det \leq 0$.

$\blacksquare$