Math 8680 Feb 17, 2021

What remains to be done from our mini-overview?

The green things below.

$\Delta = \partial\left(\begin{smallmatrix}\text{simplicial}\\ d\text{-dim'l}\\ \text{polytope}\end{smallmatrix}\right) \Longrightarrow \Delta$ shellable

Δ shellable $\Longrightarrow$ Δ partitionable $\Longrightarrow$ $\underline{h}(\Delta) \geq 0$

Δ shellable $\Longrightarrow$ $K[\Delta]$ Cohen-Macaulayness $\Longrightarrow$ $\underline{h}(\Delta) \geq 0$

$\Delta = \partial(\ldots) \Downarrow h_k(\Delta) = h_{d-k}(\Delta)$

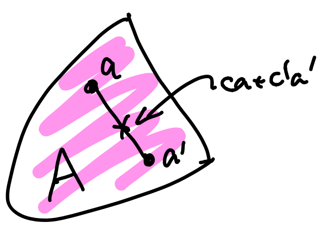
$\dim(\Delta) := d-1$, $\#\text{vertices} = n$ $\Rightarrow$ $0 \leq h_k(\Delta) \leq \binom{n-d+k-1}{k}$

$\binom{n-d+k-1}{k} = h_k(\partial C(n,d))$, $0 \leq k \leq \lfloor d/2 \rfloor$

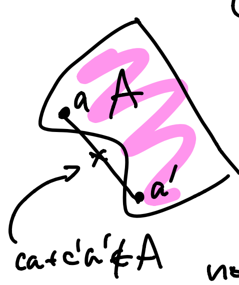
Polytopes!!

[See Ziegler's "Lectures on polytopes"]

DEF'NS: $A \subseteq \mathbb{R}^d$ is <u>convex</u> if $\forall a, a' \in A$ the points $\{ca + c'a' : c, c' \geq 0,\ c + c' = 1\} \subseteq A$



convex



not convex

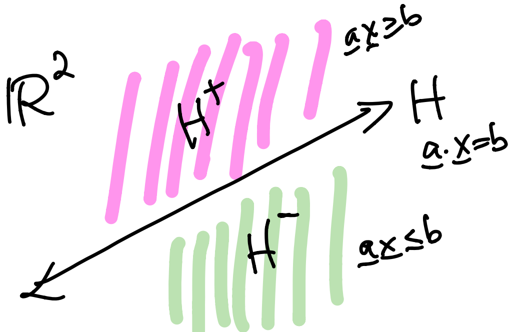
The <u>convex hull</u> $\operatorname{conv}(A)$ = "smallest" convex set $C \supseteq A$

$$:= \bigcap_{\substack{\text{convex } C \subset \mathbb{R}^d:\\ C \supseteq A}} C = \left\{ \sum_{i=1}^{s} c_i a_i : \begin{array}{l} a_i \in A \\ c_i \geq 0 \\ \sum_{i=1}^{s} c_i = 1 \end{array} \right\}$$

A hyperplane (affine) $H = \{ \underline{x} \in \mathbb{R}^d : \underline{a} \cdot \underline{x} = b \}$ for some $\underline{a} \neq 0$ in $\mathbb{R}^d$, $b \in \mathbb{R}$

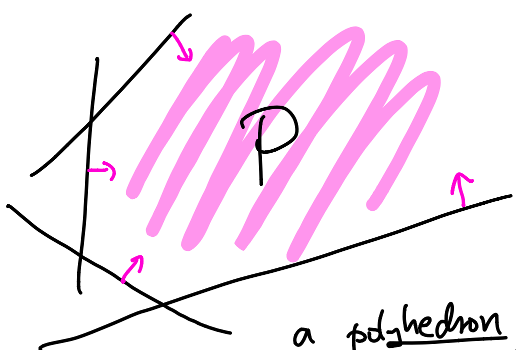
and has two half-spaces $H^+ = \{ \underline{x} \in \mathbb{R}^d : \underline{a} \cdot \underline{x} \geq b \}$

$H^- = \{ \underline{x} \in \mathbb{R}^d : \underline{a} \cdot \underline{x} \leq b \}$

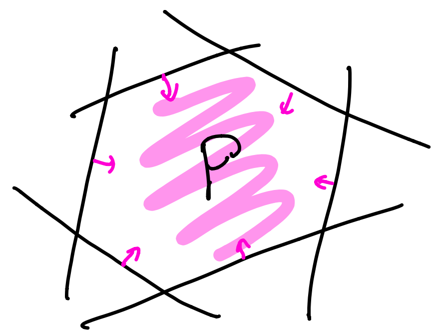


A polyhedron $P \subset \mathbb{R}^d$ is a finite intersection $P = \bigcap_{i=1}^{s} H_i^+$ of half-spaces

and if it is bounded, it is called a (convex) polytope.



a polyhedron, not a polytope



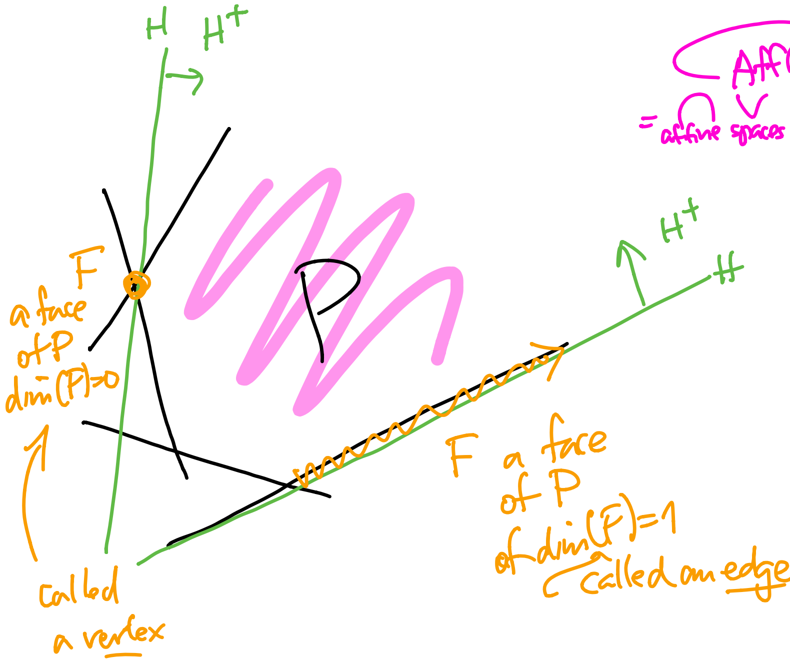
a polytope

A face $F \subset P$ a polyhedron is an intersection $F = H \cap P$ where $H^+ \supseteq P$

say H^+ supports P

Faces F have a <u>dimension</u> $\dim(F) := \dim(\underbrace{\mathrm{Aff}(F)})$

$\mathrm{Aff}(A) :=$ <u>affine hull</u> of $A \subset \mathbb{R}^d$
$=$ smallest affine space containing A
$= \bigcap_{\text{affine spaces } V \supseteq A} V = \left\{ \sum_{i=1} c_i a_i : a_i \in A,\ \sum_{i=1} c_i = 1 \right\}$



$\emptyset$ empty set is a face

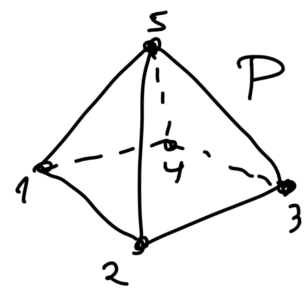
$\dim(\emptyset) = -1$

CONVENTION:
$F = P$ is the (improper) face of P

∂P (boundary of P) $= \left\{ \text{proper faces } F \neq P \right\}$

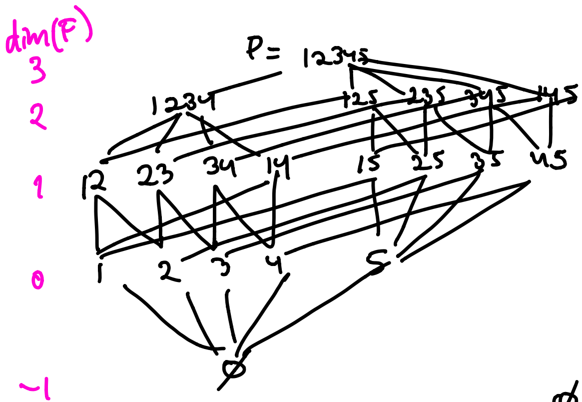
POLYTOPE FACTS (see Ziegler)

- A <u>polytope</u> $P = \mathrm{conv}(\{\text{vertices of } P\})$



$P = \mathrm{conv}\{1,2,3,4,5\}$

- The <u>poset</u> of faces of P ordered by inclusion $\mathrm{Faces}(P)$ has ...

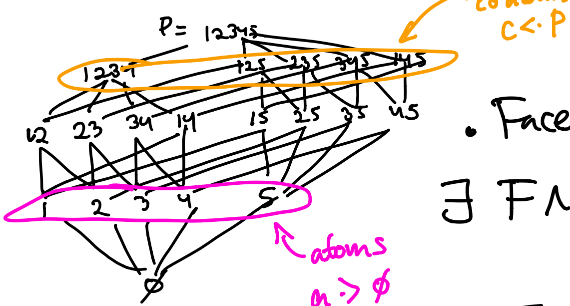


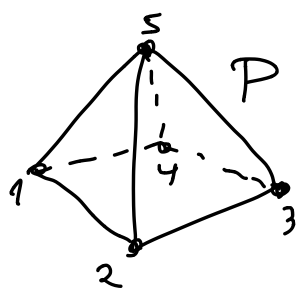
- $\mathrm{Faces}(P)$ is <u>finite</u>
- Has every <u>maximal</u> (under inclusion) <u>chain</u> (totally ordered subset) of the same length

$$\underset{\substack{\| \\ F_{-1}}}{\emptyset} \subset F_0 \subset F_1 \subset F_2 \subset \dots \subset \underbrace{F_{\dim(P)-1}}_{\text{called a facet}} \subset \underset{\substack{\| \\ F_{\dim P}}}{P}$$

- Faces(P) is a <u>graded/ranked</u> poset, ranked by $\dim(F)$, i.e. if $F \lessdot F'$ then $\dim(F) = \dim(F') - 1$.

  "is covered by"





- Faces(P) is a <u>lattice</u>, meaning $\forall$ faces F, G

  $\exists\ F \wedge G :=$ greatest lower bound for F, G
  $= F \cap G$

  $F \vee G :=$ smallest upper bound for F, G
  $= \bigwedge_{\substack{H \in \text{Faces}(P) \\ H \supseteq F, G}} H = \bigcap_{\text{same } H} H$

- Faces(P) is <u>coatomic</u>, i.e. $F = \bigwedge_{\substack{\text{coatoms} \\ c \geq F}} c$

  i.e. face $F = \bigcap_{\substack{\text{facets } G: \\ G \supseteq F}} G$

- Faces(P) is <u>atomic</u>,

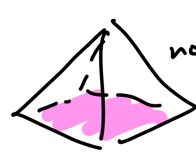
  i.e. $F = \bigvee_{\substack{\text{atoms} \\ a \leq F}} a$

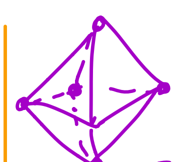
  i.e. a face F = smallest face containing its vertices $\left(= \text{conv}\begin{pmatrix}\text{its} \\ \text{vertices}\end{pmatrix}\right)$

- When P is <u>simplicial</u>, meaning all (proper) faces/facets are <u>simplices</u>, then $\Delta := \partial P$ = boundary faces of P is an (abstract) <u>simplicial</u> <u>complex</u> on vertex set $V :=$ vertices(P)

  d-dim'l convex hull of $(d+1)$-points

$d=1$ $\quad d=2$ $\quad d=3$





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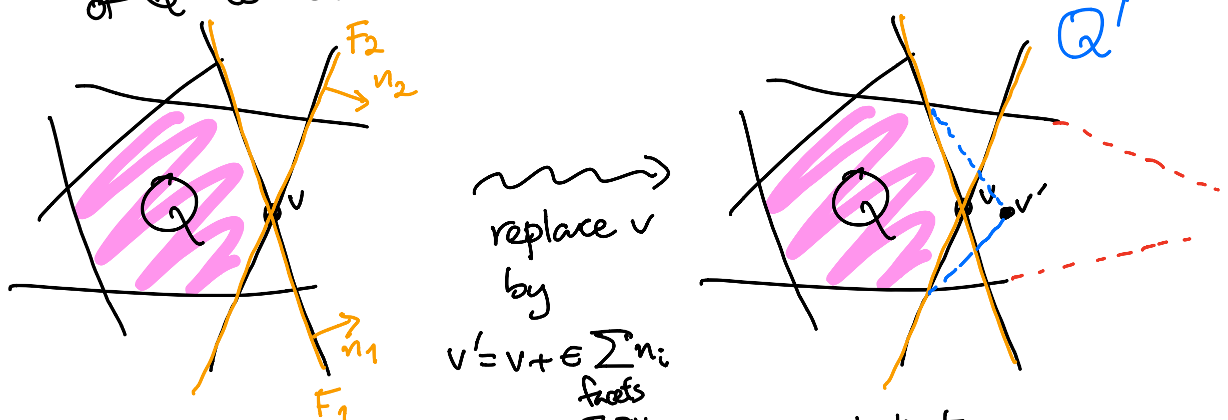
The upper bound conjecture

Q: Fixing d, n, how large can $f_k(P)$ be for a d-dim'l polytope with $f_0(P) = n$ vertices?

PROP: One can restrict attention to simplicial polytopes, since for every d-polytope Q, $\exists$ a simplicial polytope P having $f_0(P) = f_0(Q)$ and $f_k(P) \geq f_k(Q)$ $\forall k$.

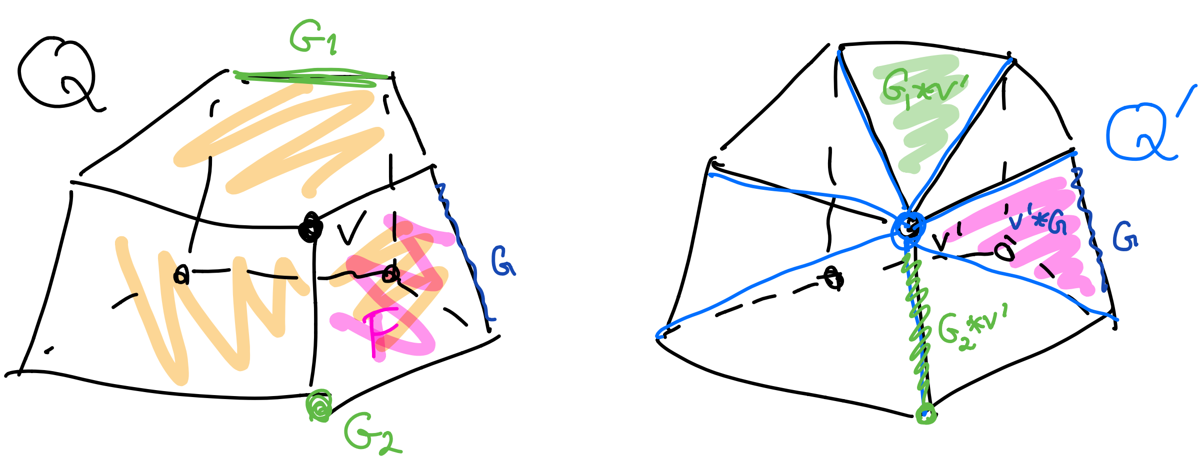
(see Grünbaum's "Convex polytopes" §5.2)

proof sketch: Apply the following vertex-pulling process at each vertex of Q to obtain P:



replace v by $v' = v + \epsilon \sum_{\substack{\text{facets} \\ F_i \ni v}} n_i$

where $\epsilon > 0$ is small enough that v' crosses no facet hyperplane for facets $F \not\ni v$ of Q

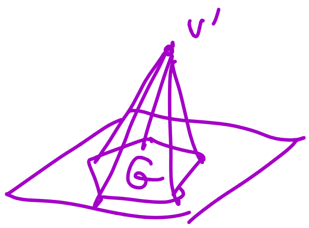


One can show that Q' has exactly these faces:

(a) faces F of Q with $v \notin F$

(b) faces of form $G * v'$ for faces G of Q not containing v with $G \subseteq F$ a face of Q containing v

cone/pyramid with base G and apex v'



Note we get an injection

$$\{k\text{-faces of } Q\} \hookrightarrow \{k\text{-faces of } Q'\}$$

$$F \longmapsto \begin{cases} F & \text{if } v \notin F \\ G * v' & \text{for any facet } G \text{ of } F,\ G \not\ni v \text{ if } v \in F \end{cases}$$

Now pull every vertex of Q to get P, and P is simplicial $\boxtimes$

Note that in a simplicial polytope P,

$$f_{k-1}(P) \leq \binom{n}{k} \quad \text{where } n = f_0(P)$$

since $\partial P = \Delta$ is a simplicial complex with n vertices

Can we have equality here?

Yes, the $(n-1)$-simplex does

$n=4$

$\underline{f} = (f_{-1}, f_0, f_1, f_2, f_3)$
$= (1, 4, 6, 4, 1)$

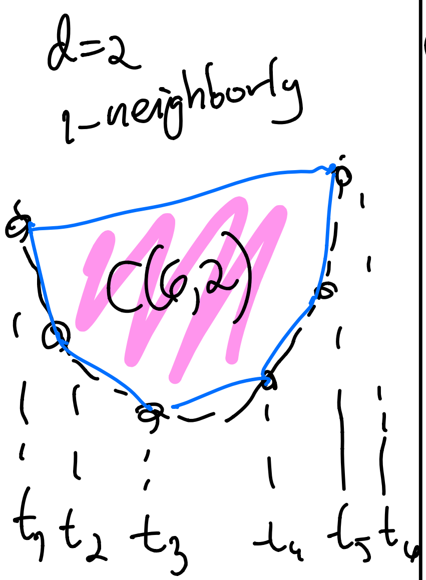
It can happen for non-simplices that
$f_{k-1}(P) = \binom{n}{k}$, in which case we call P
a <u>k-neighborly polytopes</u>

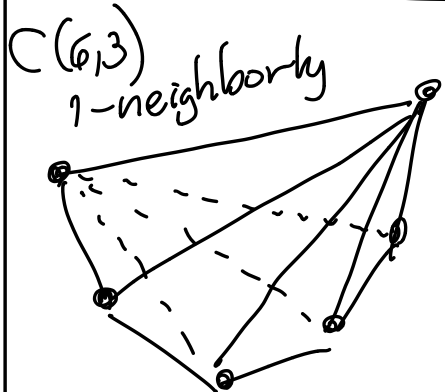
<u>PROP</u>: Any d-dim'l <u>cyclic polytope</u> with
n vertices

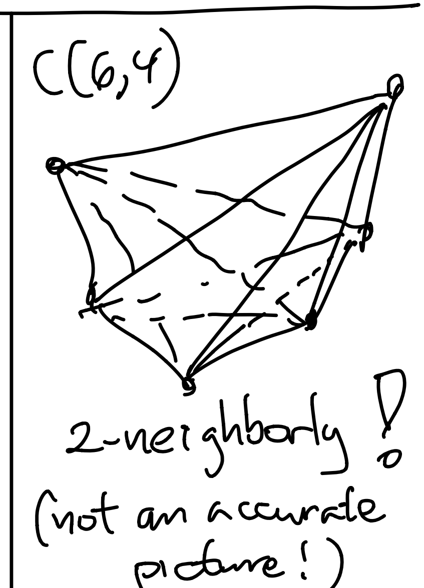
$$C(n,d) := \operatorname{conv}\{\underline{x}(t_1), \underline{x}(t_2), \dots, \underline{x}(t_n)\} \subset \mathbb{R}^d$$

where
$$\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ t^3 \\ \vdots \\ t^d \end{bmatrix} \quad \text{and } t_1 < t_2 < \dots < t_n$$

is <u>simplicial</u> and $\lfloor d/2 \rfloor$-neighborly







REMARKS:

① A d-polytope cannot be $\left(\lfloor \frac{d}{2} \rfloor + 1\right)$-neighborly without being a simplex, i.e. $d = n-1$ (maybe an EXERCISE for HW?)

② Motzkin thought maybe all $\lfloor \frac{d}{2} \rfloor$-neighborly polytopes have same face poset as $C(n,d)$, but that's vastly false.

③ HW Exercise 5 is telling us the facial structure of $C(n,d)$ (Gale's evenness criterion)

④ Points on the curve $\left\{ \begin{bmatrix} t \\ t^2 \\ t^3 \\ \vdots \\ t^d \end{bmatrix} \right\}$ also come up in real algebraic geometry (Shapiro–Shapiro Conj, e.g.)

proof: If $C(n,d)$ were not simplicial, then some facet has $\geq d+1$ vertices $\underline{x}(t_1), \underline{x}(t_2), \ldots, \underline{x}(t_{d+1})$ lying on some affine hyperplane $c_0 + c_1x_1 + c_2x_2 + \cdots + c_dx_d = 0$ in $\mathbb{R}^d$

(i.e. $\underline{c} \cdot \underline{x} = -c_0$)

giving a nontrivial solution to

$$\underbrace{\begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ \vdots & \vdots & & & \vdots \\ 1 & t_{d+1} & t_{d+1}^2 & \cdots & t_{d+1}^d \end{bmatrix}}_{} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

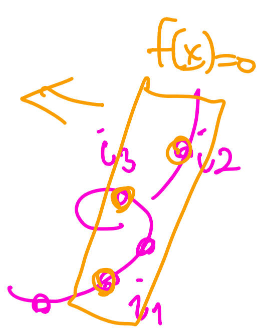
invertible, Vandermonde matrix with $t_1 < t_2 < \cdots < t_{d+1}$

$$\det = \prod_{1 \leq i < j \leq d+1} (t_j - t_i) \neq 0 \quad . \quad \text{Contradiction}$$

To show $C(n,d)$ is $\lfloor d/2 \rfloor$-neighborly, given any $F = \{i_1, i_2, \ldots, i_k\} \subset \{1,2,\ldots,n\}$ with $2k \leq d$ we'll find an affine function $f: \mathbb{R}^d \to \mathbb{R}$

with $f(\underline{x}(t_i)) = 0$ for $i \in F$

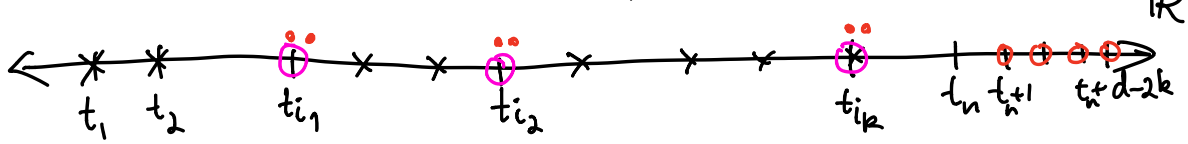
$f(\underline{x}(t_i)) \begin{matrix} >0 \\ <0 \end{matrix} \Big\}$ same sign for $i \notin F$



We CLAIM: This $f(\underline{x})$ does it:

$$f(\underline{x}) \overset{\text{def}}{=} \det \begin{bmatrix} 1 & 1 & 1 & & 1 & 1 & 1 & \cdots & 1 \\ x_1 & & & & & & & & \\ x_2 & \underline{x}(t_{i_1}) & \underline{x}(t_{i_1}+\epsilon) & \cdots & \underline{x}(t_{i_k}) & \underline{x}(t_{i_k}+\epsilon) & \underline{x}(t_n+1) & \cdots & \underline{x}(t_n+d-2k) \\ \vdots & & & & & & & & \\ x_d & & & & & & & & \end{bmatrix}$$

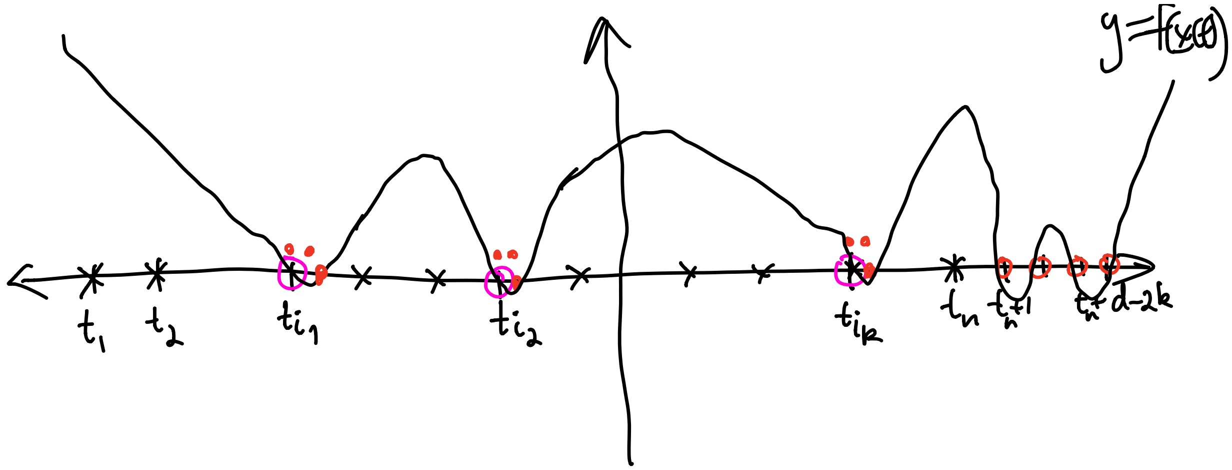
$\underbrace{\hspace{6em}}_{k \text{ pairs}}$ $\qquad$ $\underbrace{\hspace{4em}}_{d-2k \text{ more columns}}$



Note • $f(\underline{x})$ is affine-linear $\mathbb{R}^d \to \mathbb{R}$

• $f(\underline{x}(t))$ is a polynomial in t, of degree of degree d because the t^d coefficient is a nonvanishing Vandermonde determinant

• We've exhibited d roots already due to due to repeated matrix columns, namely $t = t_{i_1}, t_{i_1}+\epsilon, \ldots, t_{i_k}, t_{i_k}+\epsilon$, $t_n+1, t_n+1, \ldots, t_n+d-2k$



i.e. $f(\underline{x}(t_i)) > 0$ for $i \notin F = \{i_1, \dots, i_k\}$

$\blacksquare$

Hence

Motzkin conjectured:

(UBC) $\quad f_k(P) \leq f_k(C(n,d))$

$\forall$ d-polytopes with n vertices.

COROLLARY: Any cyclic polytope $C(n,d)$
(or any $\lfloor d/2 \rfloor$-neighborly simplicial polytope)
has $f_{k-1}(C(n,d)) = \binom{n}{k}$ for $0 \le k \le \lfloor d/2 \rfloor$
by definition of $\lfloor d/2 \rfloor$-neighborly
and $h_k(C(n,d)) = \binom{(n-d)+k-1}{k}$ for $0 \le k \le \lfloor d/2 \rfloor$

proof: We saw that when we have two sequences, with d fixed,

$$\underline{f} = (f_{-1}, f_0, f_1, \ldots)$$

$$\underline{h} = (h_0, h_1, h_2, \ldots)$$

related by

$$\sum_{k=0}^{\infty} f_{k-1} \left(\frac{t}{1-t}\right)^k = \frac{\sum_{k=0}^{\infty} h_k t^k}{(1-t)^d}$$

then $\underline{f}$ and $\underline{h}$ have the unitriangular relation for $\underline{h}(\Delta), \underline{f}(\Delta)$ of $(d-1)$-dim'l simplicial complex for all j.
$(h_0, h_1, \ldots, h_j)$ vs. $(f_{-1}, f_0, \ldots, f_{j-1})$

Take $f_k = \binom{n}{k}$ for $k = 0, 1, 2, \ldots, n$ (so this is $f_k(C(n,d))$ only for $k \le \lfloor d/2 \rfloor$)

and then

$$\sum_{k=0}^{\infty} \binom{n}{k} \left(\frac{t}{1-t}\right)^k = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{t}{1-t}\right)^k = \left(1 + \frac{t}{1-t}\right)^n$$

$$= \frac{1}{(1-t)^n} = \frac{1}{(1-t)^d} \frac{1}{(1-t)^{n-d}}$$

$$= \frac{1}{(1-t)^d} (1-t)^{-(n-d)}$$

$$= \frac{1}{(1-t)^d} \sum_{k=0}^{\infty} (-t)^k \binom{-(n-d)}{k}$$

$$= \frac{1}{(1-t)^d} \sum_{k=0}^{\infty} t^k \cdot \binom{(n-d)+k-1}{k}$$

Hence $h_k(C(n,d)) = \binom{(n-d)+k-1}{k}$ for $0 \le k \le \lfloor \frac{d}{2} \rfloor$. $\blacksquare$

Where are we?

$$\begin{tikzcd}
 & & \Delta \text{ partitionable} \arrow[dr, Rightarrow] & \\
\Delta = \partial\left(\substack{\text{simplicial}\\ d\text{-dim'l}\\ \text{polytope}}\right) \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \Delta \text{ shellable} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \underline{h}(\Delta) \ge 0 \\
h_k(\Delta) = h_{d-k}(\Delta) & & K[\Delta] \text{ Cohen-Macaulayness} \arrow[ur, Rightarrow] &
\end{tikzcd}$$

$\dim(\Delta) := d-1$, $\#\text{vertices} = n$ $\Rightarrow$ $0 \le h_k(\Delta) \le \binom{n-d+k-1}{k} = h_k(\partial C(n,d))$, $0 \le k \le \lfloor d/2 \rfloor$

Fans, normal fans, face fans & polar duals

To shell a simplicial polytope's boundary, we'll order its facets by a linear functional on vertices of its polar dual polytope!

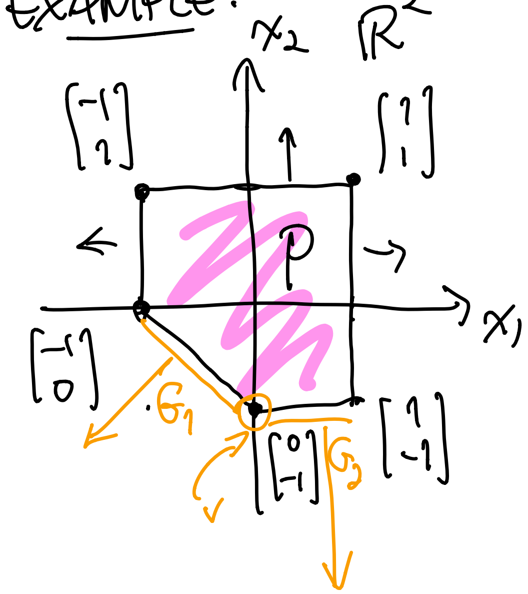
<u>DEF'N:</u> Given a polytope $P \subset \mathbb{R}^d$ and a (proper) boundary face $G \subsetneq P$, let $N_P(G) := \{$ linear functionals $f \in (\mathbb{R}^d)^*$: f achieves its max value on P at all $p \in G \}$

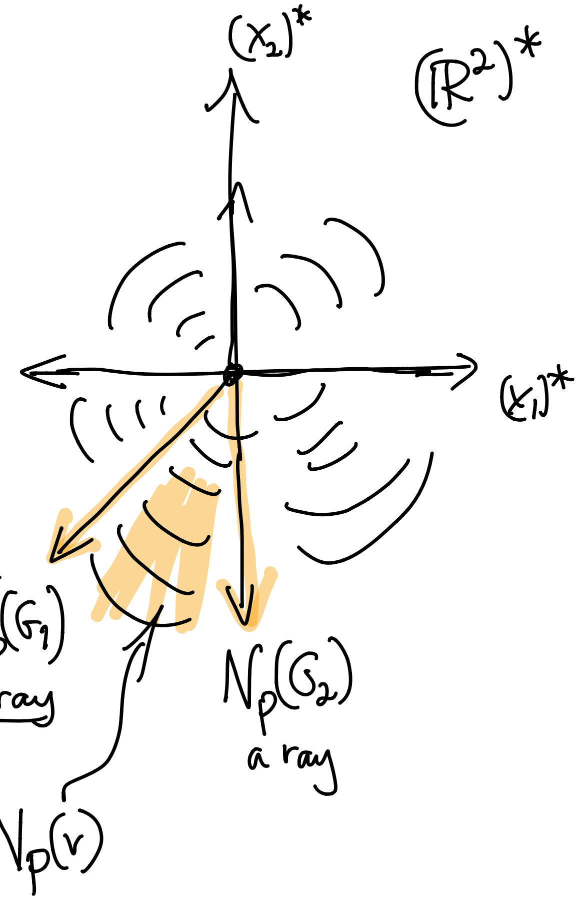
$N_P(G)$: (closed) normal cone of P at G

$N_P^{\text{open}}(G) := \{$ same except not achieved on any $p \in F \supsetneq G$ in $P \}$

(open) normal cone of P at G (relatively open)

EXAMPLE:





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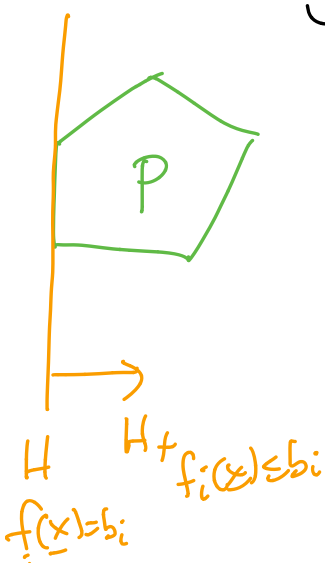
<u>PROP</u>: If G a face of P has facets containing G being $\{F_1, \dots, F_s\}$ with

$$F_i = \underbrace{\{\underline{x} \in \mathbb{R}^d : f_i(\underline{x}) = b_i\}}_{H_i} \cap P, \quad f_i \in (\mathbb{R}^d)^*, \quad H_i^+ = \{\underline{x} \in \mathbb{R}^d : f_i(\underline{x}) \leq b_i\}$$



then $N_P(G) = \mathbb{R}_{\geq 0} \cdot f_1 + \dots + \mathbb{R}_{\geq 0} \cdot f_s$

$=$ cone spanned by $\{f_1, \dots, f_s\}$ in $(\mathbb{R}^d)^*$

$N_P^{open}(G) = \mathbb{R}_{>0} \cdot f_1 + \dots + \mathbb{R}_{>0} \cdot f_s$

<u>(half) proof</u>: At least both inclusions $\supseteq$ are not hard:

If $f = \sum_{i=1}^{s} c_i f_i$ with $c_i > 0$, then then any $p \in P$, has

$$f(p) = \sum_{i=1}^{s} \underbrace{c_i}_{>0} \underbrace{f_i(p)}_{\leq b_i}$$ with equality $\Leftrightarrow p \in F_i$

$$\leq \sum_{i=1}^{s} c_i b_i \text{ with equality } \Leftrightarrow p \in F_1, \dots, F_s$$

$$\Leftrightarrow p \in G \qquad \blacksquare$$

DEF'N: A <u>polyhedral cone</u> $C \subset \mathbb{R}^d$ (or $(\mathbb{R}^d)^*$) is a finite intersection of <u>linear</u> halfspaces

$$C = \bigcap_{i=1}^{s} H_i^+ \quad \text{where } H_i^+ = \{x \in \mathbb{R}^d : f_i(x) \geq 0\}$$

for some $f_i \in (\mathbb{R}^d)^*$

(so $\underline{0} \in C$ always)

It's called <u>pointed</u> if it contains no lines through $\underline{0}$.

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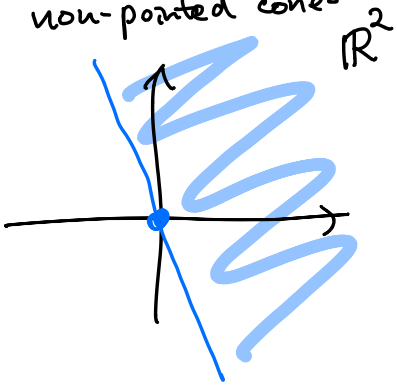
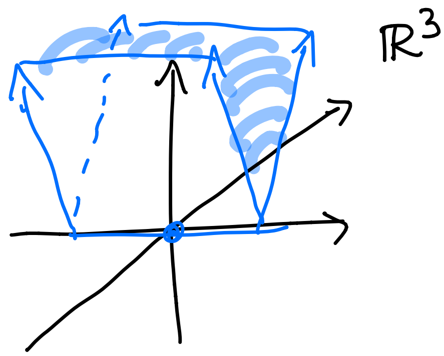
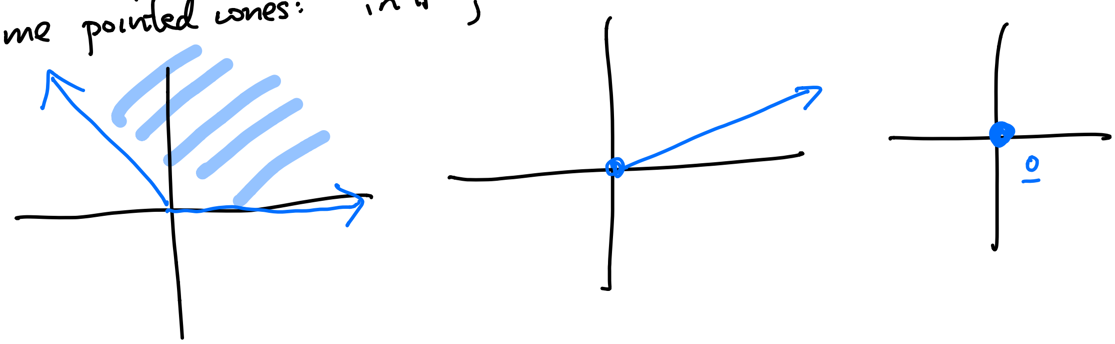
two non-pointed cones:

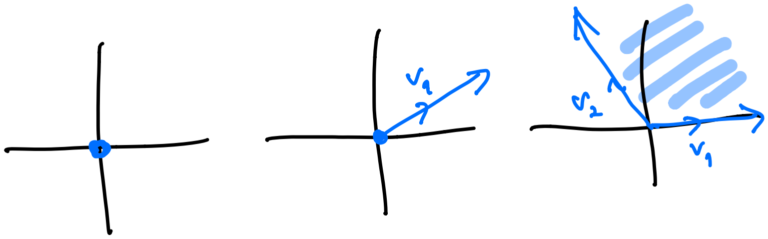




Some pointed cones: in $\mathbb{R}^2$,

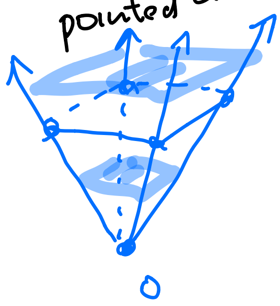


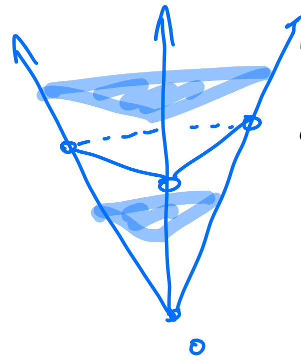
A pointed cone C is called a <u>simplicial cone</u> if $\exists \{v_1, \dots, v_s\}$ s.t. $C = \mathbb{R}_{\geq 0} \cdot v_1 + \mathbb{R}_{\geq 0} \cdot v_2 + \dots + \mathbb{R}_{\geq 0} \cdot v_s$ for some $\{v_1, \dots, v_s\} \subset \mathbb{R}^d$ <u>linear independent</u>!



all pointed cones in $\mathbb{R}^2$ are simplicial

In $\mathbb{R}^3$, here's a non simplicial pointed cone





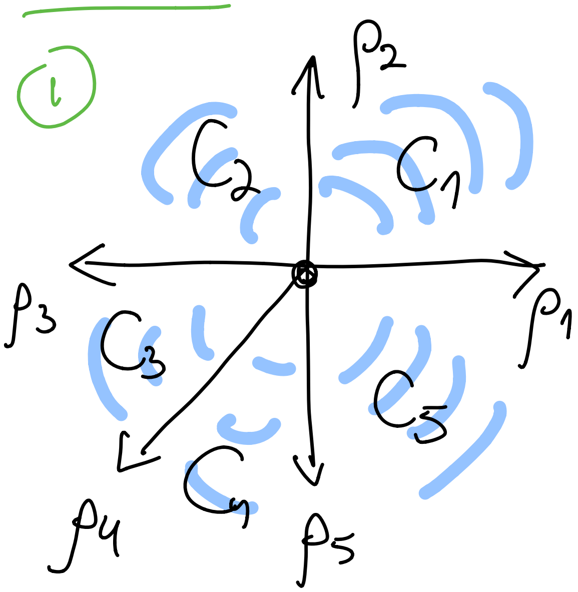
a simplicial cone in $\mathbb{R}^3$

A <u>fan</u> $\Sigma = \{C_i\}$ in $\mathbb{R}^d$ is a collection of cones C_i which is closed under taking faces/subcones and has $C_i \cap C_j$ is a face of each C_i, C_j.

Say Σ is a <u>complete fan</u> (in $\mathbb{R}^d$) if $\bigcup_i C_i = \mathbb{R}^d$

EXAMPLES

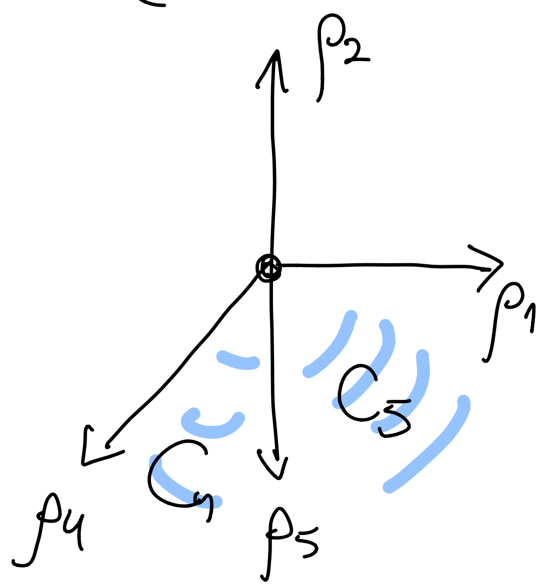
(1)



$\Sigma = \{\{\underline{0}\}, \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, C_1, C_2, C_3, C_4, C_5\}$

is a complete fan in $\mathbb{R}^2$

(2) A non-complete (sub-) fan in $\mathbb{R}^2$



$\Sigma' = \{\{\underline{0}\}, \rho_1, \rho_2, \rho_4, \rho_5, C_4, C_5\}$

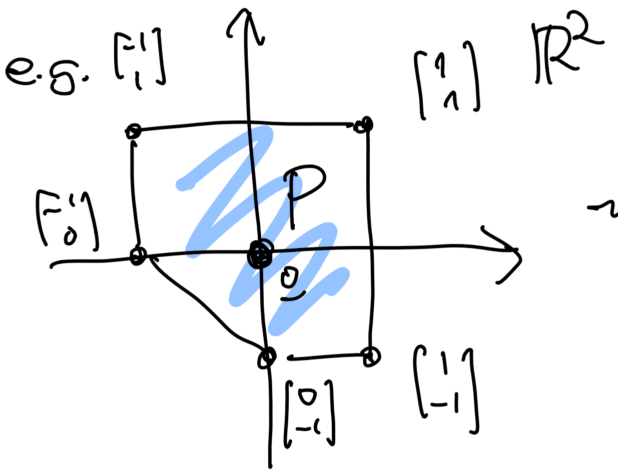
(3) Every d-dim'l polytope $P \subset \mathbb{R}^d$, containing the origin in its interior, has three related objects ...

- **face fan** $\mathcal{F}(P) \subset \mathbb{R}^d$ $:= \{ \text{cones } C_F \text{ through each proper face } F \subsetneq P \}$

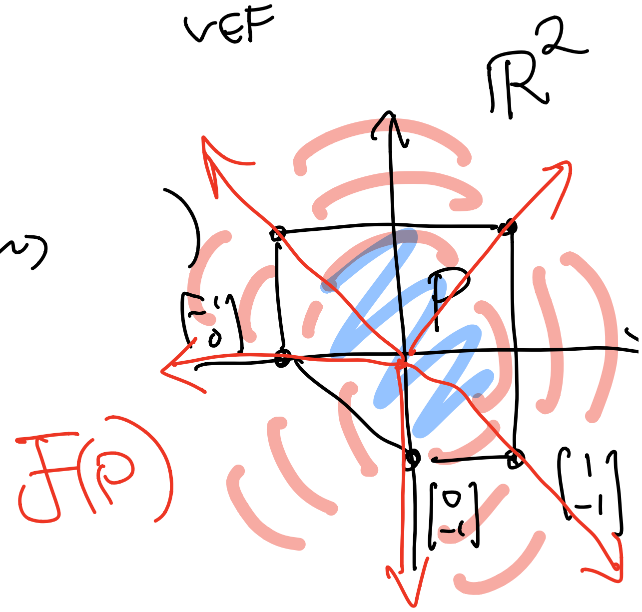
a complete fan

i.e. $C_F := \sum_{\text{vertices } v \in F} \mathbb{R}_{\geq 0} \cdot v$

e.g.



⇝



- **normal fan** $\mathcal{N}(P) := \{ N_P(G) : \text{non-empty faces } G \neq \emptyset \text{ of } P \}$ $\subseteq (\mathbb{R}^d)^*$

- **polar polytope**

$$P^{\Delta} := \{ f \in (\mathbb{R}^d)^* : f(p) \leq 1 \quad \forall p \in P \} \subset (\mathbb{R}^d)^*$$

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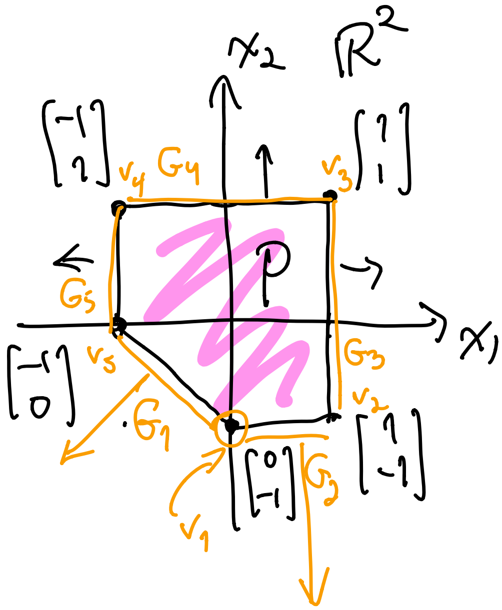
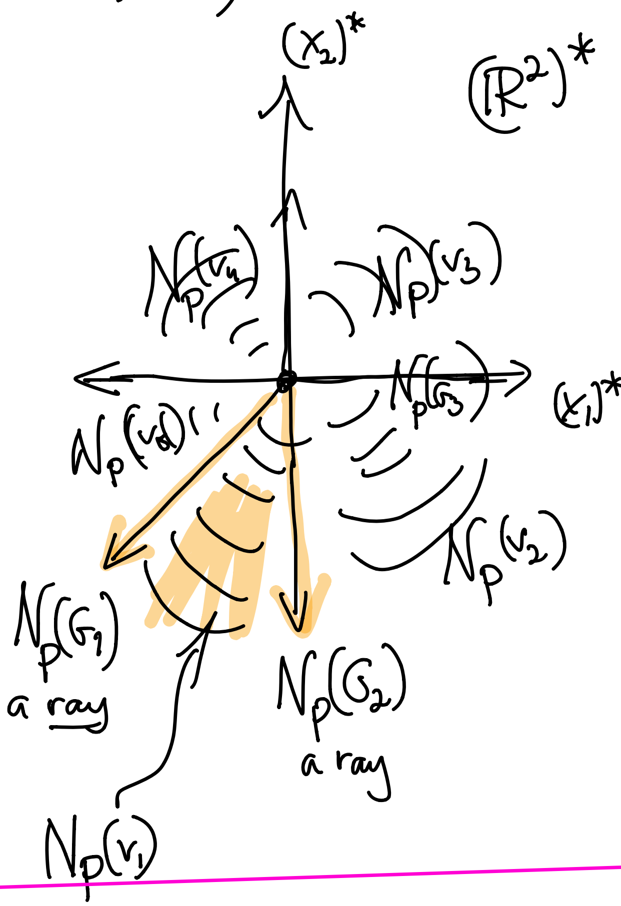
P $\qquad\qquad$ $\mathcal{N}(P)$ a complete fan in $(\mathbb{R}^d)^*$

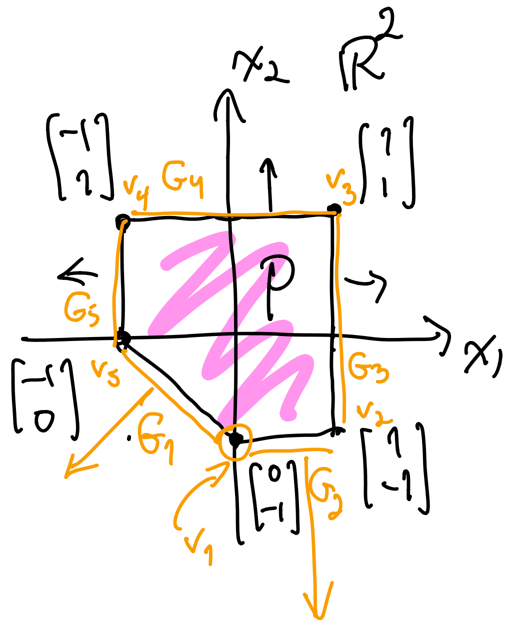




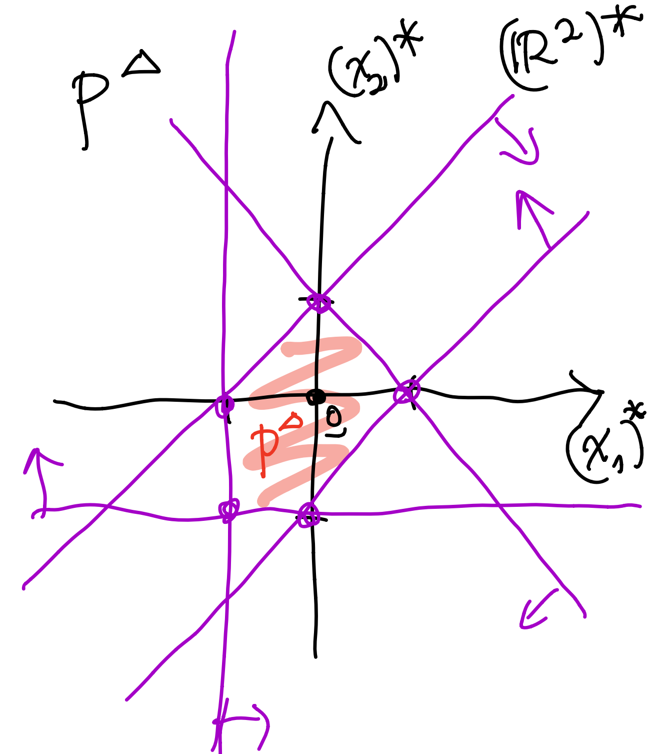
polar polytope

$$P^{\Delta} := \{ f \in (\mathbb{R}^d)^* : f(p) \leq 1 \ \forall p \in P \}$$

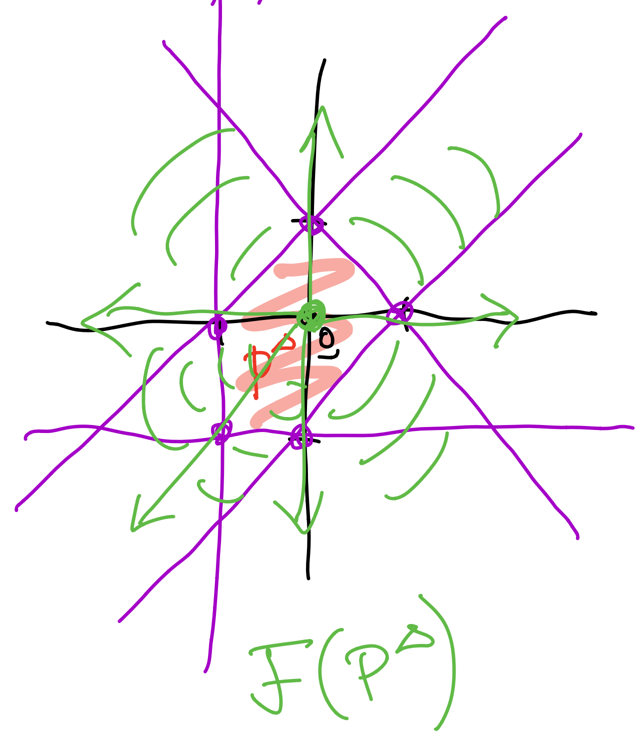
$$\subset (\mathbb{R}^d)^* \quad = \{ f \in (\mathbb{R}^d)^* : f(v) \leq 1 \ \forall \text{ vertices } v \in P \}$$



x_2
$\mathbb{R}^2$
$\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
v_4
G_4
v_3
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
P
G_5
x_1
$\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
v_5
G_3
v_2
G_1
$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\begin{bmatrix} 0 \\ -1 \end{bmatrix}$
G_2
v_1



P^Δ
$(x_2)^*$
$(\mathbb{R}^2)^*$
$(x_1)^*$
$\underline{0}$
P^Δ



$\underline{0}$
P^Δ
$\mathcal{F}(P^\Delta)$

FACTS:

- $(P^\Delta)^\Delta = P$
- The face posets $\text{Faces}(P) \xrightarrow{\sim} \text{Faces}(P^\Delta)$ are in bijection

$$F \longmapsto \{f \in (\mathbb{R}^d)^* : f(p) = 1 \ \forall p \in F, \ f(p) < 1 \ \forall p \notin F\}$$

with $\underset{\text{in } P}{F \subseteq G} \iff \underset{\text{in } P^\Delta}{F^* \supseteq G^*}$

i.e. $\text{Faces}(P^\Delta)$ is the opposite/dual poset of $\text{Faces}(P)$

EXAMPLE:

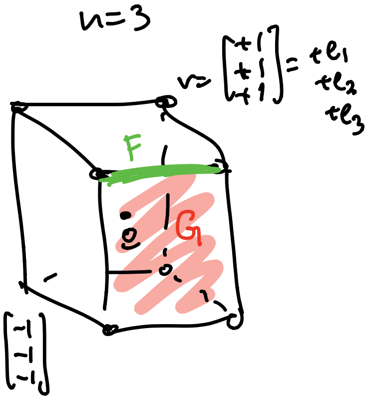
$P = n\text{-cube in } \mathbb{R}^n$

$Q_n = [-1,+1]^n$

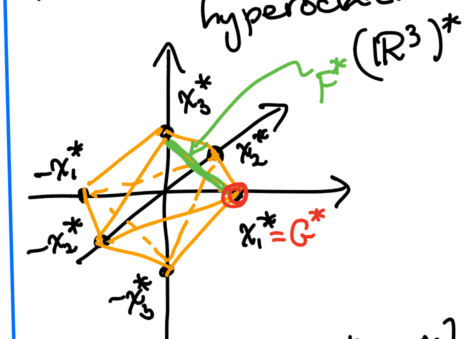
$= \text{conv}\{\pm e_1 \pm e_2 \pm \dots \pm e_n\}$

$= \bigcap_{i=1}^n \{x_i \le 1\} \wedge \bigcap_{i=1}^n \{x_i \ge -1\}$

$n=3$



P^Δ = cross-polytope/hyperoctahedron



$= \text{conv}\{\pm x_1^*, \pm x_2^*, \dots, \pm x_n^*\}$

$= \bigcap \{\pm x_1 \pm x_2 \pm \dots \pm x_n \le 1\}$

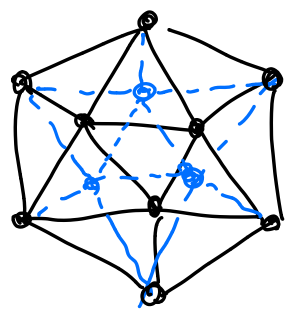
$\dim P = d$

- P is simplicial $\iff$ P^Δ is simple

all facets/faces are simplices; so every facet has exactly d vertices

every vertex v lies on exactly d edges (or, equivalently, on exactly d facets)

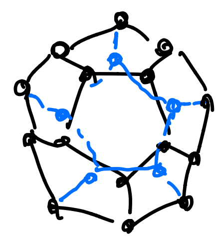
e.g.

$P =$ 

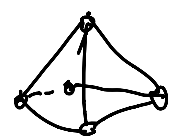
$\cap$

$\mathbb{R}^3$

simplicial

P^Δ 

simple



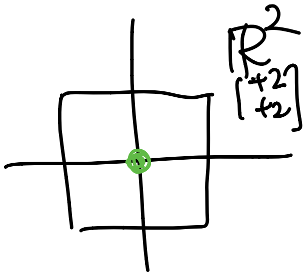
neither simplicial nor simple

REMARK: Changing the choice of $\underline{0}$ in interior of P does change P^Δ in $(\mathbb{R}^d)^*$, but doesn't affect the comb. structure of P^Δ or $\mathcal{F}(P)$ or $\mathcal{F}(P^\Delta) = \mathcal{N}(P)$

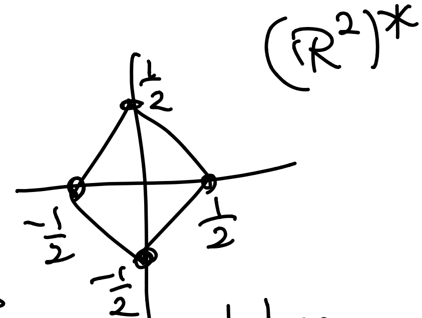
i.e. $\text{Faces}(P) = \text{Faces}(P^\Delta)^{opp}$ are all unchanged

poset

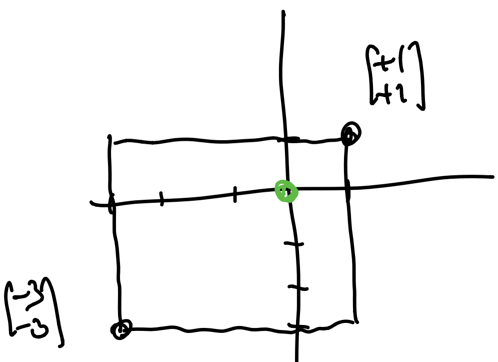
e.g.



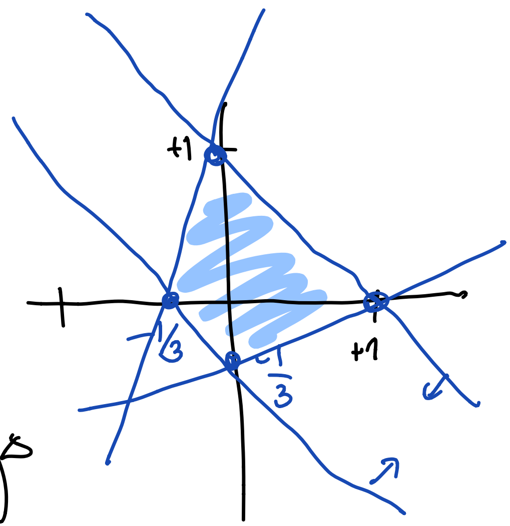
$Q_2 = [-2,+2]^2$



Q_2^Δ = cross polytope

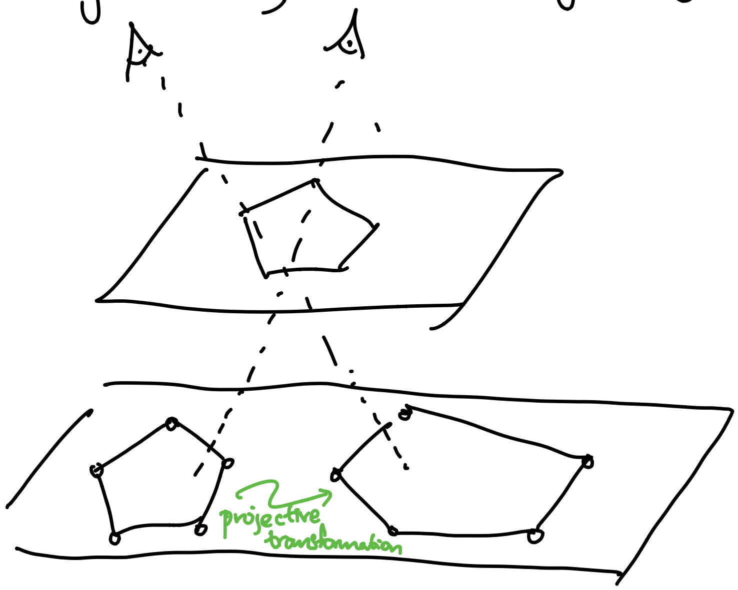


$Q_2' = [-3,+1]^2$



$(Q_2')^\Delta$

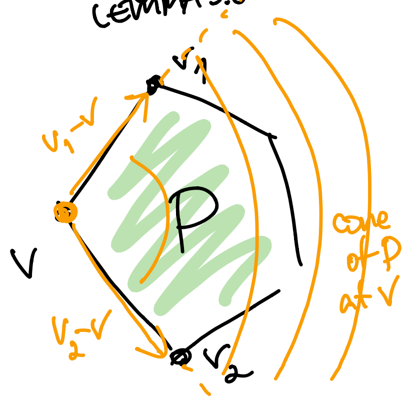
In general, P^Δ changes by a projective transformation in $(\mathbb{R}^d)^*$



We'll need one further fact:

LEMMA: (see Ziegler's LEMMA 3.6) In a polytope P, if a vertex v has edge-neighbors $\{v_1, v_2, \ldots, v_s\}$, then

$$P \subset \underset{=}{v} + \underbrace{\mathbb{R}_{\geq 0}(v_1 - v) + \ldots + \mathbb{R}_{\geq 0}(v_s - v)}_{\text{called the vertex cone of } P \text{ at } v}$$



COROLLARY: If $h \in (\mathbb{R}^d)^*$ has $h(v) \neq h(v') \ \forall$ vertices v of P, then $\exists$ a <u>unique</u> h-maximizing vertex v_{max} of P, and every other vertex v of P has a path

$$\underset{\underset{v}{\shortparallel}}{v_0} \to v_1 \to v_2 \to \ldots \to v_{\ell-1} \to \underset{\underset{v_{max}}{\shortparallel}}{v_\ell}$$

with $h(v_i) > h(v_{i-1}) \ \forall i \geq 1$