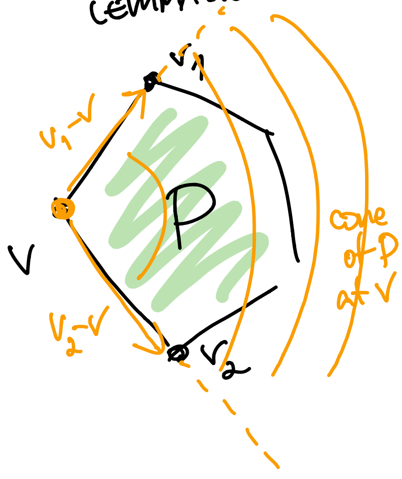
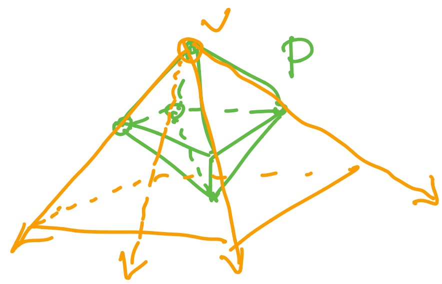
We'll need one further fact:

LEMMA: (see Ziegler's LEMMA 3.6) In a polytope P, if a vertex v has edge-neighbors $\{v_1, v_2, \ldots, v_s\}$, then

$$P \subset \underset{=}{v} + \underbrace{\mathbb{R}_{\geq 0}(v_1 - v) + \cdots + \mathbb{R}_{\geq 0}(v_s - v)}_{\text{called the vertex cone of } P \text{ at } v}$$

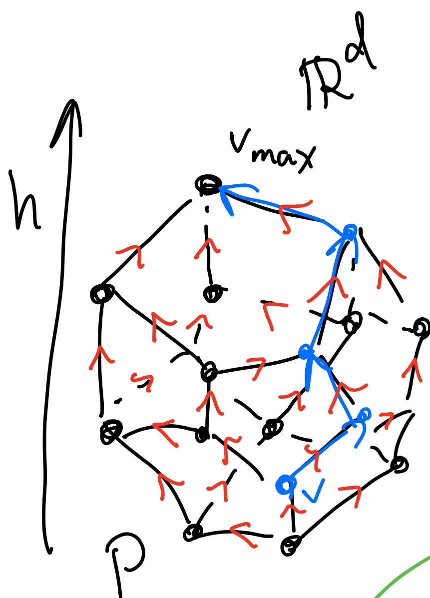




COROLLARY: If $h \in (\mathbb{R}^d)^*$ has $h(v) \neq h(v')$ $\forall$ vertices v of P, then $\exists$ a unique h-maximizing vertex v_{max} of P, and every other vertex v of P has a path

$$v_0 \to v_1 \to v_2 \to \cdots \to v_{\ell-1} \to v_\ell$$

with $v_0 = v$, $v_\ell = v_{max}$, $h(v_i) > h(v_{i-1})$ $\forall i \geq 1$



In other words, v_{max} is the unique sink if we direct the edges of P h-upward.

all edges directed into it, none out of it

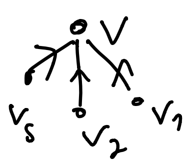
<u>proof</u>: Assuming $h(v) \neq h(v')$ $\forall$ vertices $v \neq v'$ in P, want to show that v_{max} is the only sink in the directed graph that directs the edges of P h-upward.

Suppose some vertex v was a sink. Name its neighbors $\{v_1, v_2, \ldots, v_s\}$ and then any $p \in P$ has, since

$$P \subset v + \mathbb{R}_{\geq 0}(v_1 - v) + \cdots + \mathbb{R}_{\geq 0}(v_s - v),$$

an expression

$$p = v + c_1(v_1 - v) + \cdots + c_s(v_s - v)$$



$$c_i \geq 0$$

so $$h(p) = h(v) + \underbrace{c_1}_{\geq 0}\underbrace{(h(v_1) - h(v))}_{<0} + \cdots + \underbrace{c_s}_{\geq 0}\underbrace{(h(v_s) - h(v))}_{<0}$$

$$h(p) \leq h(v) \quad \text{i.e. } v = v_{max} \quad \blacksquare$$

<u>REMARK</u>: We've shown in this setting that P also has a unique h-minimizing vertex v_{min}, and that v_{min} is the unique <u>source</u> in the directed graph.

THEOREM: Let $P \subset \mathbb{R}^d$ be a simple polytope with vertices $v_1, v_2, \dots, v_s$

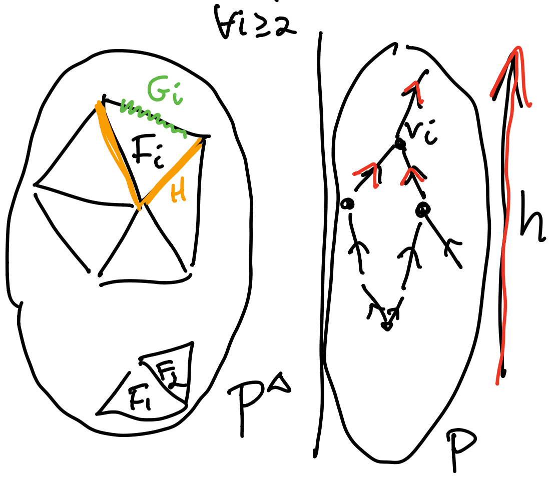
and $P^\Delta \subset (\mathbb{R}^d)^*$ its polar dual simplicial polytope

with corresponding facets $\underset{v_1^*}{\overset{}{F_1}} = v_1^*,\ F_2 = v_2^*,\ \dots,\ F_s = v_s^*$

and $h \in (\mathbb{R}^d)^*$ with $h(v_1) < h(v_2) < \dots < h(v_s)$.

Then $\Delta = \partial(P^\Delta)$ has $F_1, F_2, \dots, F_s$ as a shelling order,

in which $\overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}}) = \bigcup_{\substack{h\text{-downward} \\ \text{walls } H \text{ of } F_i}} \overline{H}$ $\quad \forall i \geq 2$

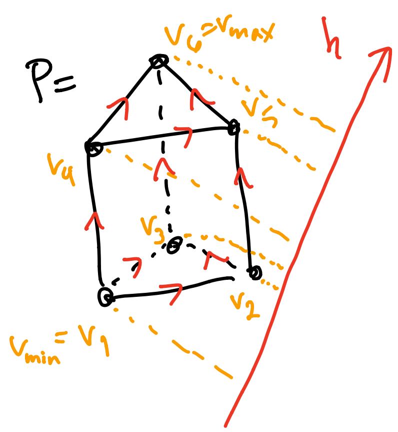


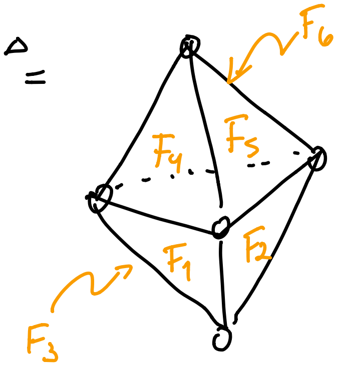
and hence the partitioning

$$\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$$

where $G_i = \bigcap_{\substack{h\text{-upward} \\ \text{walls } H' \text{ of } F_i}} H'$

so $\# G_i = h$-downdegree of v_i
$= $ indegree of v_i

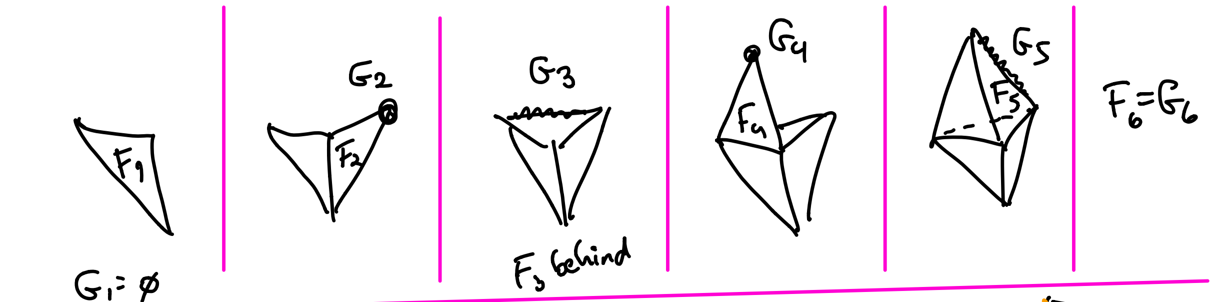




$\quad\ \ f_{-1}\ f_0\ f_1\ f_2$
$\underline{f} = (1, 5, 9, 6)$

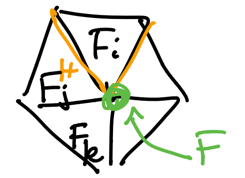
1
1 5
1 4 9
1 3 5 6

$\underline{h} = (1, 2, 2, 1)$
$\quad\ \ h_0\ \ h_1\ \ h_2\ \ h_3$

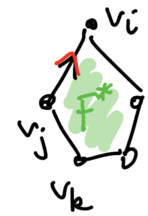


proof: We only need to show that $\forall i \geq 2$,

$$\overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}}) = \bigcup_{\substack{h\text{-downward} \\ \text{walls } H \text{ of } F_i}} \overline{H}$$



So given any F a face in $\overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}})$,



we need to show there is some wall H of F_i containing F; this means $H = F_i \cap F_j$ with $h(v_j) < h(v_i)$

Since $F \in \overline{F_i} \cap (\overline{F_1} \cup \dots \cup \overline{F_{i-1}})$, $\exists$ some F_k with $h(v_k) < h(v_i)$ and $F = F_i \cap F_k$.

But then look at F^* the dual face in P the simple polytope, which has v_i, v_k vertices on F^* since F_i, F_k are facets containing F.

We know v_i does not have the h-minimum value among vertices on F^* since $h(v_k) < h(v_i)$.

Hence inside F^*, v_i is not the unique source, it must have an h-downward edge to some v_j, that gives F_j with $H = F_i \cap F_j \supseteq F$ $\blacksquare$

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Where are we?

$\Delta = \partial\left(\begin{smallmatrix}\text{simplicial} \\ d\text{-dim'l} \\ \text{polytope}\end{smallmatrix}\right) \Longrightarrow \Delta$ shellable $\Longrightarrow$ Δ partitionable $\Longrightarrow$ $\underline{h}(\Delta) \geq 0$

Δ shellable $\Longrightarrow$ $K[\Delta]$ Cohen-Macaulayness $\Longrightarrow$ $\underline{h}(\Delta) \geq 0$

$\Delta = \partial(\ldots) \Downarrow$

$h_k(\Delta) = h_{d-k}(\Delta)$

Dehn-Sommerville equations

$\dim(\Delta) := d-1$, $\#\text{vertices} = n$ $\Longrightarrow$ $0 \leq h_k(\Delta) \leq \binom{n-d+k-1}{k}$

$\qquad /\!/$

$h_k\left(\partial C(n,d)\right)$, $0 \leq k \leq \lfloor d/2 \rfloor$

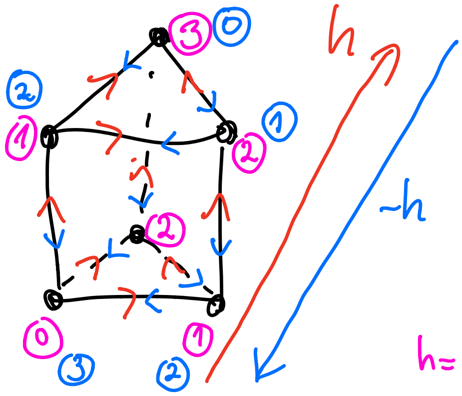
Immediate consequences of the shelling

(1) **THEOREM** (Dehn-Sommerville, $d \leq 5$: 1905, 1927) For a simplicial d-polytope P, then $h_k(P) = h_{d-k}(P)$ $\forall k$

proof: Insteading of choosing generic height function h to order vertices of P^Δ, choose $-h$ instead, and compare:

P^Δ

simple polytope



$h = \begin{pmatrix} h_0 & h_1 & h_2 & h_3 \\ 1, & 2, & 2, & 1 \end{pmatrix}$

$h_k(P) = \#$ vertices in P^Δ with $\underbrace{h\text{-downdegree}}_{=k}$

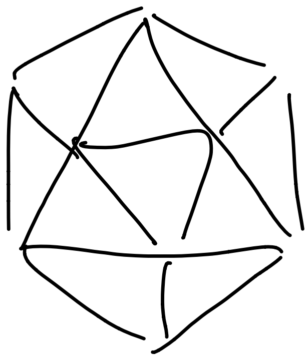
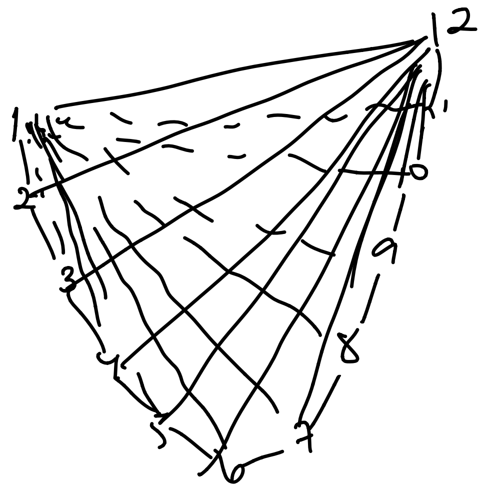
same as $(-h)$-updegree $= k$

or $(-h)$-downdegree $= d-k$

EXAMPLE: $d=3$

Simplicial 3-polytopes with $f_0 = n$ vertices all have the same $\underline{f}$-vector or $\underline{h}$-vector:

e.g. $n=12$

 or 

$$\begin{array}{ccccccc}
 & & & 1 = f_{-1} & & & \\
 & & 1 & & n = f_0 & & \\
 & 1 & & n-1 & & \boxed{3n-6} = f_1 & \\
1 & & n-2 & & 2n-5 & & \boxed{2n-4} = f_2 \\
\hline
(1, & & n-3, & & n-3, & & 1) \\
h_0 & & h_1 & & h_2 & & h_3
\end{array}$$

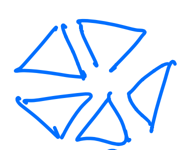
$$\Rightarrow \underline{f} = \overset{\begin{array}{cccc} f_{-1} & f_0 & f_1 & f_2 \end{array}}{(1, n, 3n-6, 2n-4)} \Big\} n=12 \quad = (1, 12, 30, 20)$$

$$\left.\begin{array}{l} h_3 = f_2 - f_1 - f_0 - f_{-1} \\ \quad = f - e + v - 1 \\ \quad = h_0 = 1 \end{array}\right\} \Rightarrow v - e + f = 2$$

Euler's formula

$$\left.\begin{array}{l}\text{Simplicial} \\ \text{(= all faces triangular)}\end{array}\right\} \Rightarrow 3f_2 = 2f_1$$

($\geq$ for all 3-polytopes)



② ~~CONJ~~ (Motzkin's U.B.C.) 1957
THEOREM (McMullen 1970)

All d-polytopes P with $f_0(P)=n$ vertices have $f_k(P) \le f_k(\underbrace{C(n,d)}_{\text{cyclic polytope}})$ $\forall k$.

proof: We saw using "vertex-pulling" that $\exists$ a simplicial d-polytope Q with $f_0(Q)=n=f_0(P)$ and $f_k(Q) \ge f_k(P)$ $\forall k$.

But then we showed

$$\dim_K(R_k) = h_k(Q) \le \binom{(n-d)+k-1}{k} \quad \forall k$$

$$\binom{(n-d)+k-1}{k} = h_k(C(n,d)) \text{ for } k=0,1,2,\ldots,\lfloor \tfrac{d}{2} \rfloor$$

$C(n,d)$ was $\lfloor \frac{d}{2} \rfloor$-neighborly, $f_k(C(n,d)) = \binom{n}{k}$

where
$R = K[\Delta] / (\Theta_1, \ldots, \Theta_d) \cong K[y_1, \ldots, y_{n-d}]$

where $\Theta_i \in R_1$ linear and $K[\Delta]$ is a finitely gen'd $K[\Theta_1, \ldots, \Theta_d]$-module

$\underline{\Theta} = (\Theta_1, \ldots, \Theta_d)$ is an l.s.o.p. linear system of parameters for $K[\Delta]$

So $h_k(Q) \le h_k(C(n,d))$ for $0 \le k \le \lfloor \frac{d}{2} \rfloor$

$$\begin{array}{ccc} h_k(Q) & \le & h_k(C(n,d)) \\ \text{D-S} \parallel & \Downarrow & \parallel \text{D-S} \\ h_{d-k}(Q) & \le & h_{d-k}(C(n,d)) \end{array}$$

i.e. $h_k(Q) \leq h_k(C(n,d))\ \forall k$

$$\Rightarrow f_{i-1} = \sum_{k=0}^{i} \underbrace{\binom{d-k}{d-i}}_{\geq 0} h_k \quad \text{for simplicial polytopes}$$

$$\Rightarrow f_{i-1}(Q) \leq f_{i-1}(C(n,d))\ \forall i \quad \blacksquare$$

③ DEF'N/COROLLARY: For $\Delta = \partial P$, P a simplicial d-polytope

and $R := K[\Delta]/(\underbrace{\theta_1, \dots, \theta_d}_{\text{any l.s.o.p. for } K[\Delta]})$

$$= \underbrace{R_0}_{\cong K} \oplus R_1 \oplus R_2 \oplus \dots \oplus \underbrace{R_d}_{\cong K}$$

with $R_d \cong K$ having $\{\underline{x}^F\}$ as K-basis
where F is any choice of a facet of Δ
$\quad\quad$ ‖ $(d-1)$-face

Any choice of an isomorphism
$R_d \xrightarrow[ev]{\sim} K$ is called an evaluation or degree map for R

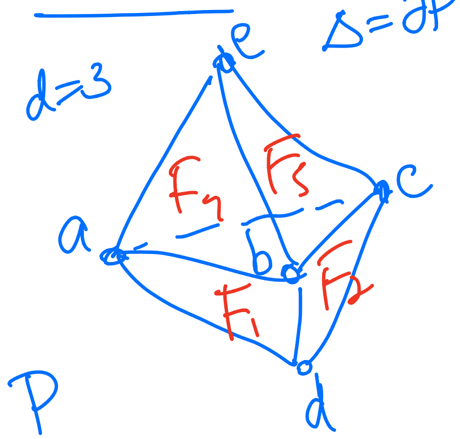
proof: We know $\dim_K(R_k) = h_k(P)$

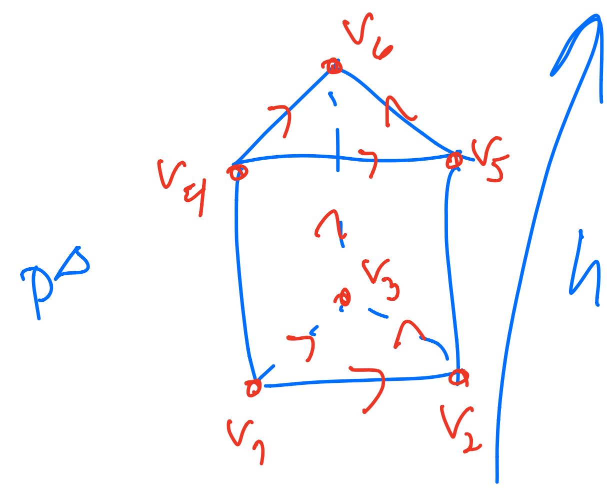
so $\dim_K(R_d) = h_d(P) = h_0(P) = 1.$

Interrupted proof for an...

EXAMPLE:

$d=3$ $\Delta = \partial P$





$$K[\Delta] = K[a,b,c,d,e] \big/ (de, abc)$$

$$\begin{array}{l} \theta_1 = d-e \\ \theta_2 = a-c \\ \theta_3 = b-c \end{array} \quad \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \end{array}$$

$\overset{\text{shelling}}{\Longrightarrow}$ $R = K[\Delta]/(\theta_1, \theta_2, \theta_3)$ has K-basis

$$\begin{array}{cccccc} G_1 & G_2 & G_3 & G_4 & G_5 & G_6 \\ \{1, & c, & ac, & e, & ce, & ace\} \end{array}$$

Back to proof...

We also we know $\{x^{G_i}\}_{i=1,2,\dots,s}$

in the shelling's partitioning $\Delta = \bigsqcup_{i=1}^{s} [G_i, F_i]$

give a K-basis for R.

And in the polytope shelling, it is only the last vertex v_s in P^Δ that has indegree $= d$, so only $F_s = G_s$.

Hence $R_d = K$-span of $\{\underline{x}^{F_s}\}$

But any vertex v of P^Δ can be made h-maximal by a choice of generic h,

so any facet F of P can be made the last facet $F_s = F$ in the shelling. $\boxtimes$

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Consequence (4):

THEOREM (Poincaré duality):

In the above setting where $\Delta = \partial P$

P a simplicial d-polytope and

$R := K[\Delta]/(\underbrace{\theta_1, \dots, \theta_d})$ any l.s.o.p. in $K[\Delta]_1$

then the bilinear form

$$\begin{aligned} R_k \times R_{d-k} &\longrightarrow K \\ (x\ ,\ y) &\longmapsto \mathrm{ev}(\underbrace{x \cdot y}_{\in R_d}) \end{aligned}$$

is a perfect/non degenerate bilinear form.

RECALL:

For fin. dim'l K-vector spaces V, W

a map

$$\begin{aligned} V \times W &\xrightarrow{\langle \cdot, \cdot \rangle} K \\ (v\ ,\ w) &\longmapsto \langle v, w \rangle \end{aligned}$$

is a bilinear form if
$$\begin{cases} \langle av, w \rangle = \langle v, aw \rangle = a \langle v, w \rangle & \forall a \in K \\ \langle v+v', w \rangle = \langle v, w \rangle + \langle v', w \rangle \\ \langle v, w+w' \rangle = \langle v, w \rangle + \langle v, w' \rangle \end{cases}$$

Call $\langle \cdot,\cdot \rangle$ a nondegenerate/perfect pairing if both
$$\begin{cases} \langle v,w\rangle = 0 \ \forall w\in W \Rightarrow v=0 \\ \text{and} \\ \langle v,w\rangle = 0 \ \forall v\in V \Rightarrow w=0 \end{cases}$$

EXERCISE

$\Longleftrightarrow$ the two K-linear maps
$$\begin{array}{ccc} V \to W^* & \text{and} & W \to V^* \\ v \mapsto \langle v, -\rangle & & w \mapsto \langle -, w\rangle \end{array}$$
are both isomorphisms of K-vector spaces

EXERCISE

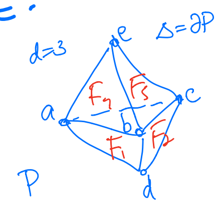
$\Longleftrightarrow$ $\dim_K V = \dim_K W =: n$

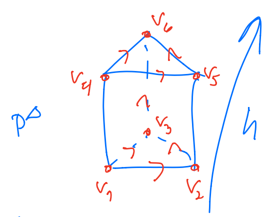
and if $\{v_i\}_{i=1,\dots,n}$, $\{w_i\}_{i=1,\dots,n}$ are K-bases for V, W

then $\big(\underbrace{\langle v_i, w_j\rangle}_{a_{ij}}\big)_{i,j=1,\dots,n}$ ← called the Gram matrix of $\langle \cdot,\cdot\rangle$

is a nonsingular/invertible matrix

EXAMPLE:





$K[\Delta] = K[a,b,c,d,e]/(de, abc)$

$$\begin{array}{l|c} & \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \theta_1 = d-e & \\ \theta_2 = a-c & \\ \theta_3 = b-c & \end{array} \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

$\overset{\text{shelling}}{\Longrightarrow}$ $R = K[\Delta]/(\theta_1,\theta_2,\theta_3)$ has K-basis

$$\{ \overset{G_1}{1}, \overset{G_2}{c}, \overset{G_3}{ac}, \overset{G_4}{e}, \overset{G_5}{ce}, \overset{G_6}{ace} \}$$

$$R = K[a,b,c,d,e]/(abc, de, d-e, a-c, b-c)$$
$\big\downarrow$ $b=c$, $a=c$, $d=e$

$$\cong K[c,e]/(c^3, e^2)$$

$$= K\text{-span of } \{\underbrace{1}_{R_0},\ \underbrace{c, e}_{R_1},\ \underbrace{c^2, ce}_{R_2},\ \underbrace{c^2e}_{R_3}\}$$

$R_0 \cong K$, $R_3 \cong K$, $ev(c^2e) := 1$

$$R_1 \times R_2 \longrightarrow K$$

has Gram matrix

$$\begin{array}{cc} & \overbrace{\begin{array}{cc} c^2 & ce \end{array}}^{R_2} \\ R_1 \left\{ \begin{array}{c} c \\ e \end{array} \right. & \begin{bmatrix} ev(c\cdot c^2) & ev(c\cdot ce) \\ ev(e\cdot c^2) & ev(e\cdot ce) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

nonsingular!

---

proof of Poincaré duality:

Consider the two K-bases $\{\underline{x}^{G_i}\}_{i=1,2,\dots,s}$ for R

$\{\underline{x}^{G_i'}\}_{i=1,2,\dots,s}$ for R

that come from a generic $h \in (\mathbb{R}^d)^*$ and $-h$.

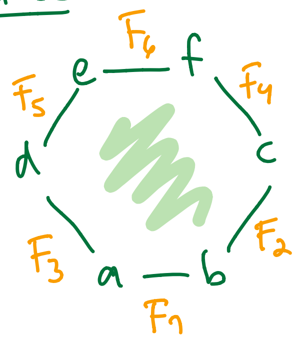
Note that we can re-write them as $\{\underline{x}^{G_i}\}_{i=1,2,\dots,s}$

and $\{\underline{x}^{F_i \setminus G_i}\}_{i=1,2,\dots,s}$ (i.e. $F_i \setminus G_i = G_i'$)

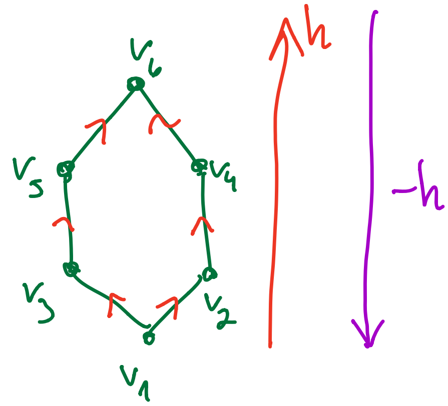
We CLAIM that their Gram matrix $(\langle \underline{x}^{G_i}, \underline{x}^{F_j \setminus G_j} \rangle)$

will be invertible upper-triangular.

EXAMPLE:

$P =$ 

simplicial

P^Δ simple 

K-bases for $R = K[\Delta]/(\theta_1, \theta_2)$

from shellings...

	F_1	F_2	F_3	F_4	F_5	F_6
	ab	bc	ad	cf	de	ef
from h:	$\{1,$	$c,$	$d,$	$f,$	$e,$	$ef\}$
from $-h$:	$\{ab,$	$b,$	$a,$	$c,$	$d,$	$1\}$

(multiplication precursor) Gram matrix for $\underset{k}{R_1} \times \underset{d-k}{R_1} \xrightarrow{\langle\cdot,\cdot\rangle} K$ pairing

(just in $K[\Delta]$)

R_1 using $-h$ basis

R_1 using h basis

$$\begin{array}{c} \\ c \\ d \\ f \\ e \end{array}
\begin{array}{c}
\begin{array}{cccc} b & a & c & d \end{array} \\
\left[\begin{array}{cccc}
bc & ac & c^2 & cd \\
\cancel{bd}^{\,0} & ad & cd & d^2 \\
\cancel{bf}^{\,0} & \cancel{af}^{\,0} & cf & df \\
\cancel{be}^{\,0} & \cancel{ae}^{\,0} & \cancel{ce}^{\,0} & de
\end{array}\right]
\end{array}$$

below diagonal, these all vanish already in $K[\Delta]$. (*)

diagonal has all facet monomials F_i, with $ev(\underline{x}^{F_i}) \neq 0$.

proof: To prove assertion $(*)$,

note that if $(\underline{x}^{G_i})(\underline{x}^{F_j \setminus G_j}) \neq 0$ in $K[\Delta]$,

we need some facet $\begin{cases} F_k \supseteq G_i \\ \text{and } F_k \supseteq F_j \setminus G_j \end{cases}$

But the h-shelling then implies $\boxed{h(v_k) \geq h(v_i)}$

while the $(-h)$-shelling implies $-h(v_k) \geq -h(v_j)$

$$\Downarrow$$

$$\boxed{h(v_k) \leq h(v_j)}$$

$$h(v_i) \leq h(v_j)$$

$\blacksquare$