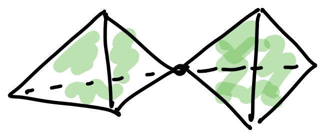
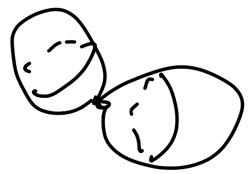
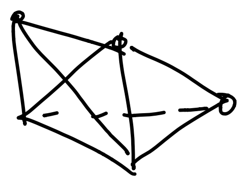
Math 8680 March 19, 2021



$\cong$ homeo.



neither is Cohen-Macaulay



$\simeq$ homotopy equiv. to

Cohen-Macaulay

<u>Reisner's Thm:</u> $K[\Delta]$ is Cohen-Macaulay (as a ring)

$\iff$ $\tilde{H}_i(\mathrm{lk}_\Delta(F); K) = 0$ if $i < \dim(\mathrm{lk}_\Delta(F))$

<u>Klee:</u> If Δ is an <u>Euler complex</u> $\left\{ \begin{array}{l} \text{pure} \\ + \\ \tilde{\chi}(\mathrm{lk}_\Delta(F)) = (-1)^{\dim(\mathrm{lk}_\Delta(F))} \end{array} \right.$

then $h_k(\Delta) = h_{d-k}(\Delta)$ $\forall k$
(Dehn-Sommerville eqns)

<u>DEF'N:</u> An $\mathbb{N}$-graded (comm.) K-algebra R is <u>Gorenstein</u> if it is Cohen-Macaulay <u>and</u> for some (or <u>any</u>) h.s.o.p. $(\underline{\theta}) = (\theta_1, \ldots, \theta_d)$

the quotient $A = R/(\underline{\Theta}) = A_0 \oplus A_1 \oplus \dots \oplus A_t$

satisfies Poincaré duality meaning $A_t \cong K$

and picking $A_t \xrightarrow[\sim]{ev} K$

and $\forall\, k \le \frac{t}{2}$, the pairing $A_k \times A_{t-k} \xrightarrow{\langle \cdot, \cdot \rangle} K$
$(x, y) \longmapsto ev(xy)$

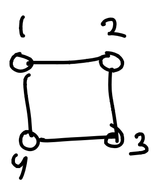
is perfect/nondegenerate

EXAMPLE: P a simplicial polytope
$\Rightarrow \Delta = \partial P$ has $K[\Delta]$ is Gorenstein $\forall K$

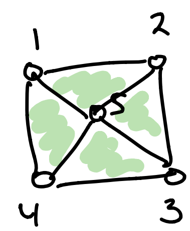
THEOREM: (Stanley 1977) $K[\Delta]$ is Gorenstein $\iff$

Δ is an iterated cone over Δ' a Gorenstein*/K complex

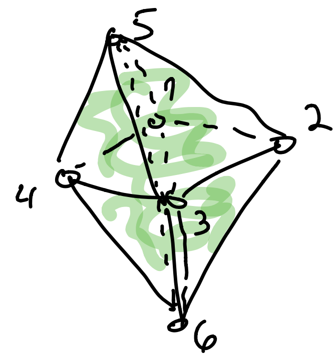
EXAMPLES:

① $\Delta' =$  $\cong \mathbb{S}^1$ is a sphere,
so Gorenstein*/K

so $K[\Delta']$ is a Gorenstein ring $\forall K$

and hence also
$\Delta'' = \text{Cone}(\Delta') = \{5\} * \Delta' =$  $\cong \mathbb{B}^2$

$$\Delta''' = \mathrm{Cone}^2(\Delta') = \mathrm{Cone}(\mathrm{Cone}(\Delta')) =$$



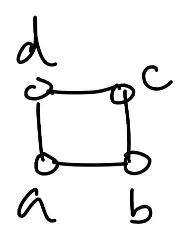
$$\cong \mathbb{B}^3$$

$$= \overline{1256} \cup \overline{2356} \cup \overline{3456} \cup \overline{1456}$$

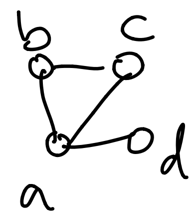
have $K[\Delta'']$, $K[\Delta''']$ Gorenstein rings $\forall K$

$$K[\Delta''] = K[x_5] \otimes K[\Delta']$$

② $\Delta_1 =$



vs. $\Delta_2 =$



have same $\underline{f}$-vector $\underline{f} = (1,4,4)$

$\underline{h}$-vector $\underline{h} = (1,2,1)$

with $K[\Delta_1]$ Gorenstein

but $K[\Delta_2]$ is not, since e.g.

$$\tilde{\chi}(\mathrm{link}_{\Delta_2}\{d\}) = 0 \neq \pm 1$$

$$\tilde{\chi}(\mathrm{link}_{\Delta_2}\{a\}) = \pm 2 \neq \pm 1$$

Check $K[\Delta_1]/(\theta_1, \theta_2)$

$$\begin{tikzcd} d \arrow[r, dash] \arrow[d, dash] & c \arrow[d, dash] \\ a \arrow[r, dash] & b \end{tikzcd}$$

$$\theta_1 = a+c \qquad \theta_2 = b+d \qquad \begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$K[a,b,c,d]/(bd, ac, a+c, b+d)$$

$$\cong K[a,b]/(b^2, a^2)$$

$$= K\text{-span of } \{1, \; a, b, \; ab\} \quad ab \to 1$$

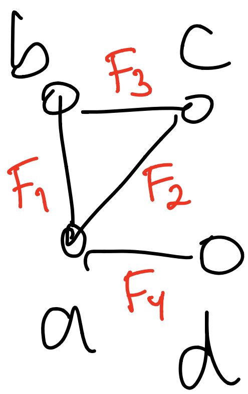
$$A_0 \quad A_1 \quad A_2 \to K$$

Gram matrix for

$$A_1 \times A_1 \to K$$

$$\begin{array}{c|cc} & a & b \\ \hline a & \mathrm{ev}(a^2) & \mathrm{ev}(ab) \\ b & \mathrm{ev}(ab) & \mathrm{ev}(b^2) \end{array} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ non-singular}$$

Check $K[\Delta_2]/(\Theta_1, \Theta_2)$



$\Theta_1 = a+b$

$\Theta_2 = b+c+d$

$$\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$A := K[a,b,c,d]/(abc, bd, cd, a+b, b+c+d)$

has K-basis $\left\{ \overset{x^{G_1}}{1},\ \overset{x^{G_2}}{c},\ \overset{x^{G_3}}{\underset{\substack{\downarrow ev \\ 1}}{bc}},\ \overset{x^{G_4}}{d} \right\}$

Gram matrix for $A_1 \times A_1 \longrightarrow K$ is

$$\begin{array}{c|cc} & c & d \\ \hline c & & \\ d & & \end{array} \begin{bmatrix} ev(c^2) & ev(cd) \\ ev(cd) & ev(d^2) \end{bmatrix} = \begin{bmatrix} ev(c^2) & 0 \\ 0 & 0 \end{bmatrix}$$

singular, not perfect.

$d^2 = d \cdot d = -d(b+c)$
$= -bd - bc = 0$

Piecewise polynomials on fans

(Refs: Two papers by Brion on syllabus, Fleming-Karu)

For $\Delta = \partial P$, P a simplicial polytope, we've proven Poincaré duality.

How to get the other parts of the Kähler package:

Hard Lefschetz (HL)

Hodge-Riemann-Minkowski (HRM) ?

We use a disguised version of $\mathbb{R}[\Delta]$!

Comes from think of Δ as arising from the fan $\underset{\text{face fan}}{\Sigma = \mathcal{F}(P)} = \underset{\text{normal fan}}{\mathcal{N}(P^\Delta)}$

The geometric $(\underline{\Theta}) = (\underline{\Theta}_\Sigma)$ becomes much more natural.

DEFIN: Given a fan $\Sigma \subset \mathbb{R}^d$ with support $|\Sigma| = \bigcup_{\text{cones } \sigma \in \Sigma} \sigma$

say that $f : |\Sigma| \longrightarrow \mathbb{R}$ is piecewise polynomial on Σ if

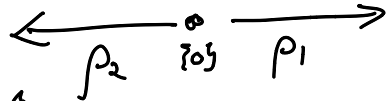
$\forall\, \sigma \in \Sigma \quad \exists$ some polynomial function

$$f_\sigma : \underbrace{\mathrm{Lin}(\sigma)} \longrightarrow \mathbb{R}$$

$= \mathbb{R}\sigma$

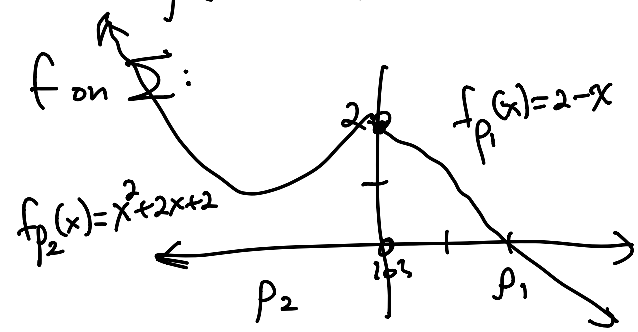
$=$ smallest $\mathbb{R}$-linear subspace containing σ

EXAMPLES:

① $\Sigma =$ 

$\subset \mathbb{R}^1$

has this p.p. f on Σ:



Let $R_\Sigma := \{$ all piecewise polynomial functions $f : |\Sigma| \to \mathbb{R} \}$

as a ring (and an $\mathbb{R}$-algebra) with pointwise $+$ and $\times$

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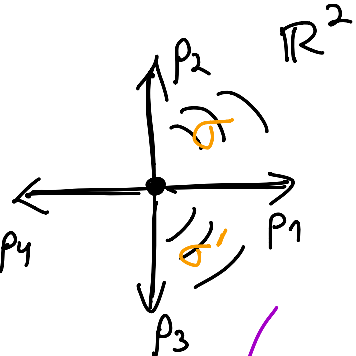
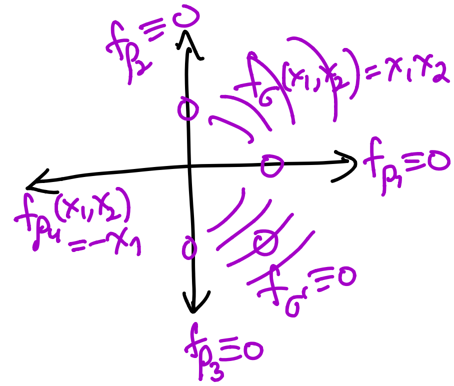
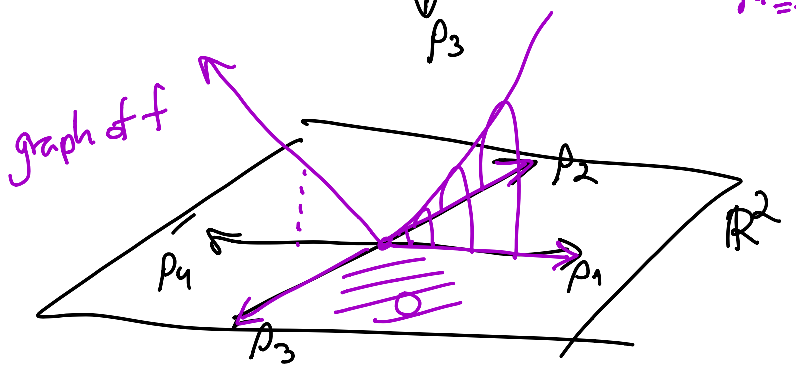
EXAMPLES:

② $\Sigma =$



has this $f \in R_\Sigma$:





Some preliminary observations:

(1) To be well-defined as $f : |\Sigma| \to \mathbb{R}$,

if $\sigma' \subset \sigma$ then $f_\sigma\big|_{\sigma'} = f_{\sigma'}$

$\Rightarrow$ f is continuous

$\Rightarrow$ f is completely determined by

$(f_\sigma)_{\sigma \text{ a maximal cone of } \Sigma}$

and the compatibility required is

$$f_\sigma\big|_{\sigma\cap\sigma'} \equiv f_{\sigma'}\big|_{\sigma\cap\sigma'} \quad \forall \text{ max cones } \sigma, \sigma'$$

e.g. in EXAMPLE 2 above,

needed $f_\sigma - f_{\sigma'} = x_1 x_2 - 0$ to vanish

on $\rho_1 = \sigma \cap \sigma' = x_1$-axis

so needed it to be divisible by x_2

(2) If we define $f \in R_\Sigma$ to be homogeneous of degree d if each f_σ is a homog. polynomial function of deg d on $\mathrm{Lin}(\sigma)$ $\forall$ cones $\sigma \in \Sigma$.

Then we claim that

$$R_\Sigma = \bigoplus_{d \geq 0} (R_\Sigma)_d$$

and this makes R_Σ a graded $\mathbb{R}$-algebra.

In other words,

PROP: $f = (f_\sigma)_{\text{max cones } \sigma} \in R_\Sigma$

then $\left((f_\sigma)_d \right)_{\text{max cones } \sigma} \in (R_\Sigma)_d$ $\forall d$.

EXAMPLE: ① $\Sigma \quad \rho_2 \longleftarrow \circ \longrightarrow \rho_1$

$$f = \quad \xleftarrow{f_{\rho_2}(x) = x^2+2x+2} \circ \xrightarrow{f_{\rho_1}(x) = 2-x} \qquad f_{\{0\}} = 2$$

$$= \begin{cases} f_0 & \xleftarrow{\;2\;} \circ \xrightarrow{\;2\;} \in (R_\Sigma)_0 \\ + & \\ f_1 & \xleftarrow{\;2x\;} \circ \xrightarrow{\;-x\;} \in (R_\Sigma)_1 \\ + & \\ f_2 & \xleftarrow{\;x^2\;} \circ \xrightarrow{\;0 \cdot x^2 = 0\;} \in (R_\Sigma)_2 \end{cases}$$

proof: Use the fact that cones σ are closed under $\underline{\mathbb{R}_{\geq 0}\text{-scaling}}$.

So $\quad (f_\sigma)_{\substack{\text{max cones} \\ \sigma}} \in R_\Sigma$

$$\iff (f_\sigma - f_{\sigma'})\Big|_{\sigma \cap \sigma'} \equiv 0 \quad \forall \text{ max cones } \sigma, \sigma'$$

$$\iff f_\sigma(v) - f_{\sigma'}(v) = 0 \quad \forall v \in \sigma \cap \sigma'$$

$$\iff f_\sigma(tv) - f_{\sigma'}(tv) = 0 \quad \forall t \in \mathbb{R}_{\geq 0} \quad \forall v \in \sigma \cap \sigma'$$

$$\| $$

$$\sum_{d \geq 0} \left[(f_\sigma)_d(tv) - (f_{\sigma'})_d(tv) \right]$$

$$= \sum_{d \geq 0} t^d \left[(f_\sigma)_d(v) - (f_{\sigma'})(v) \right]$$

$$\iff (f_\sigma)_d(v) - (f_{\sigma'})(v) \quad \forall v \in \sigma \cap \sigma'$$

$$\iff ((f_\sigma)_d) \in R_\Sigma \quad \forall d$$

$\blacksquare$

EXERCISE:
$g(t) \in \mathbb{R}[t]$
$g(t) = 0 \ \forall t \in \mathbb{R}_{\geq 0}$
$\iff g \equiv 0$

To make the identification of $R_\Sigma = (R_\Sigma)_0 \oplus (R_\Sigma)_1 \oplus \cdots$ with $R[\Delta_\Sigma]$ when Σ is simplicial

$$R[\Delta_\Sigma] = \mathbb{R}[x_1, \ldots, x_n] / I_{\Delta_\Sigma}$$

let's identify where the x_i map to in R_Σ.
(in $R[\Delta_\Sigma]$)

DEF'N: Given Σ a simplicial fan in $\mathbb{R}^d$
with rays $\rho_1, \ldots, \rho_n$, define
the Courant function / tent $g_{\rho_i} \in (R_\Sigma)_1$

by 1st choosing vectors $v_1, \ldots, v_n$ spanning $\rho_1, \ldots, \rho_n$

so $\rho_i = \mathbb{R}_{\geq 0} \cdot v_i$

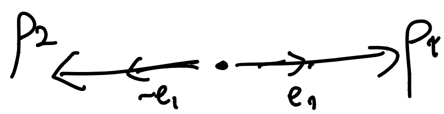
and then defining $g_{\rho_i}(v_i) = 1$

$g_{\rho_i}(v_j) = 0$ if $j \neq i$

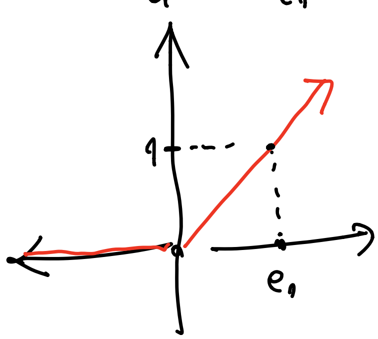
or equivalently $g_{\rho_i} \equiv 0$ on cones $\sigma \not\ni \rho_i$

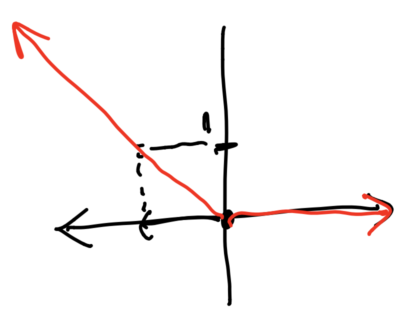
and imposing piecewise-linearity (PL-ness).

EXAMPLES:

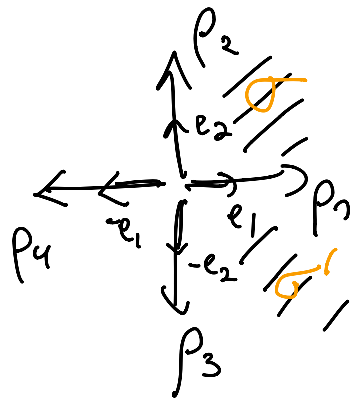
① Σ 

$\subset \mathbb{R}^1$

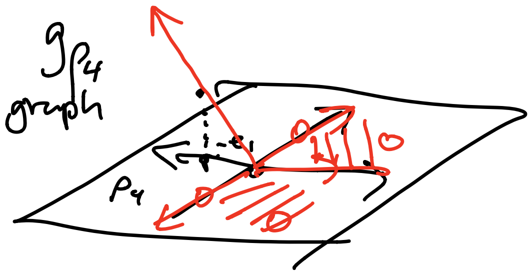
g_{ρ_1} 

g_{ρ_2} 

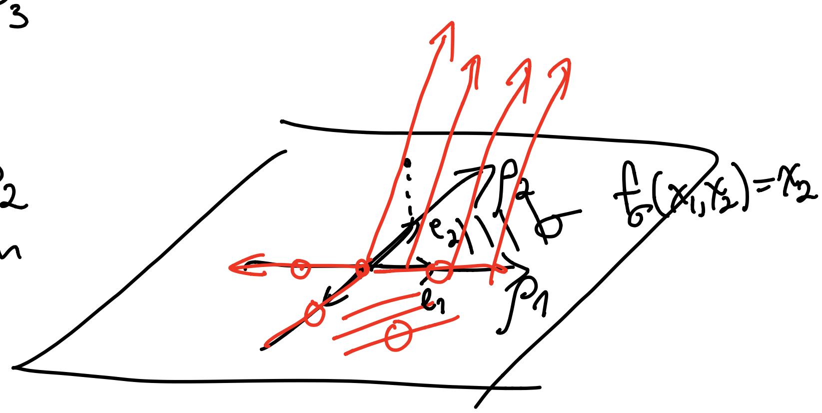
② $\Sigma \cap \mathbb{R}^2$



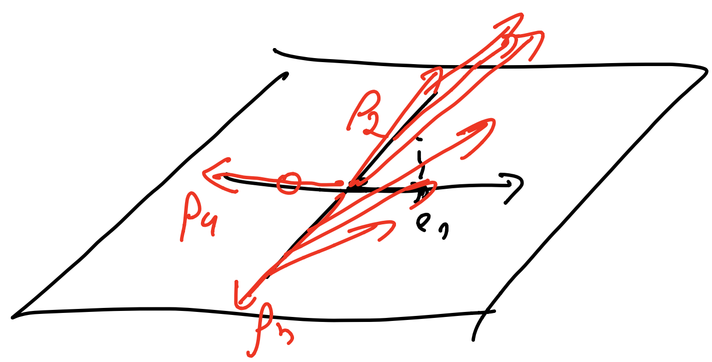
has g_{ρ_4} graph



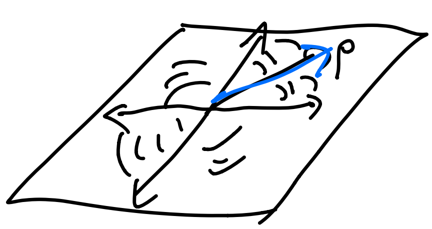
but g_{ρ_2} has graph

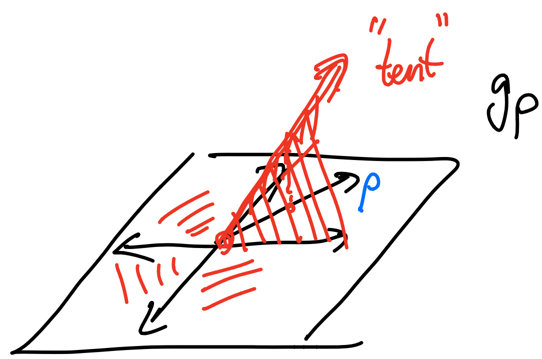


g_{ρ_1} has graph



Why "tent" function?





Where will other squarefree monomials go?

<u>DEF'N</u>: For any cone $\sigma \in \Sigma$ a simplicial fan,

$$g_\sigma := g_{\rho_{i_1}} \cdot g_{\rho_{i_2}} \cdots g_{\rho_{i_k}}$$ if σ has rays $\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}$

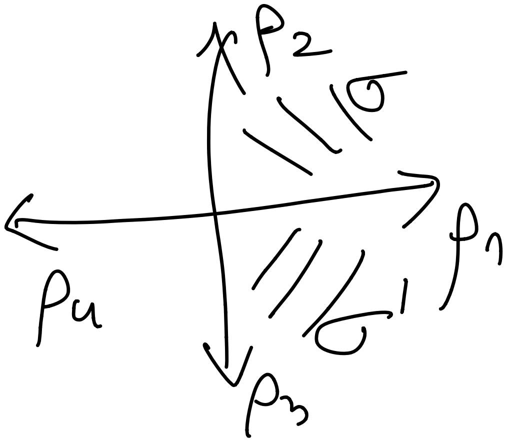
so $\sigma = \mathbb{R}_{\geq 0} v_{i_1} + \cdots + \mathbb{R}_{\geq 0} v_{i_k}$

Note that g_σ is supported on the <u>star</u> of σ:

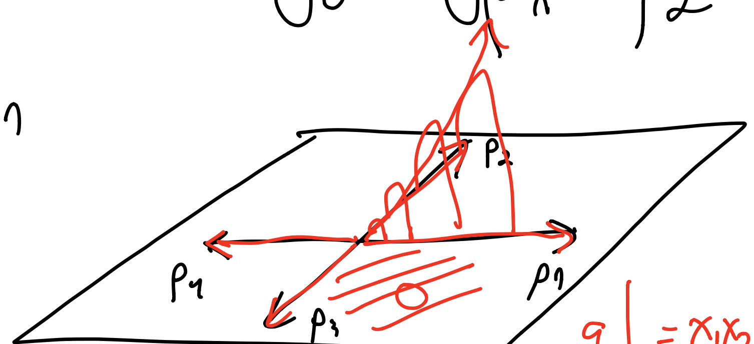
$$\operatorname{star}_\Sigma(\sigma) = \bigcup_{\substack{\text{cones} \\ \sigma' \supseteq \sigma}} \sigma'$$

means $g_\sigma \big|_\tau \equiv 0$ if $\tau \not\supseteq \sigma$

<u>EXAMPLE</u>:



has $g_\sigma = g_{\rho_1} \cdot g_{\rho_2}$



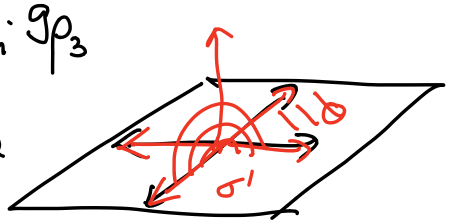
$g_\sigma \big|_\sigma = x_1 x_2$

because

$g_{\rho_1}\big|_\sigma = x_1$, $g_{\rho_2}\big|_\sigma = x_2$

$g_{\sigma'} = g_{\rho_1} \cdot g_{\rho_3}$

$g_{\sigma'}\big|_{\sigma'} = -x_1 x_2$



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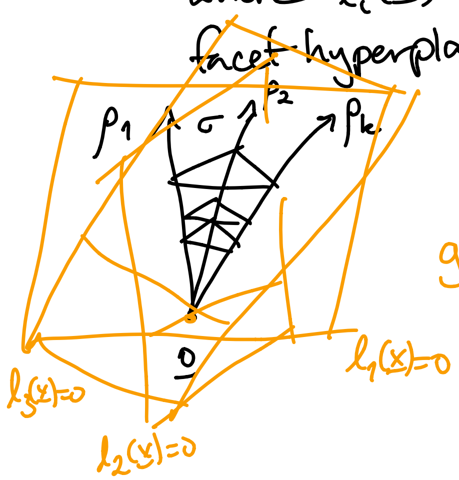
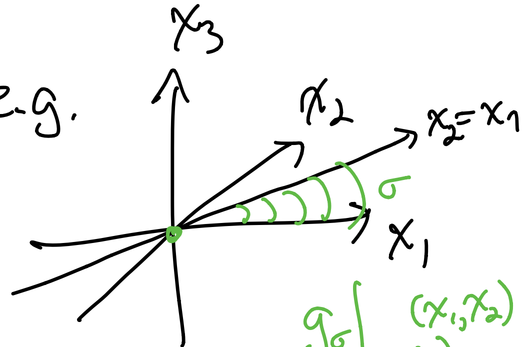
PROP: When σ is a ~~maximal~~ any! one in Σ a simplicial fan, with $\dim(\sigma)=k$, then $g_\sigma\big|_\sigma$ is the unique polynomial function on $\mathrm{Lin}(\sigma)$ of degree k, up to a nonzero scalar, that vanishes on $\partial\sigma$, and divides any other such polynomial function.

Specifically $g_\sigma\big|_\sigma = c \cdot \ell_1 \ell_2 \cdots \ell_k$ for $c \in \mathbb{R}-\{0\}$ where $\ell_i(\underline{x})$ are linear functions defining the facet hyperplanes of σ.



$g_\sigma\big|_\sigma = c \cdot \ell_1(\underline{x})\,\ell_2(\underline{x})\,\ell_3(\underline{x})$

e.g.

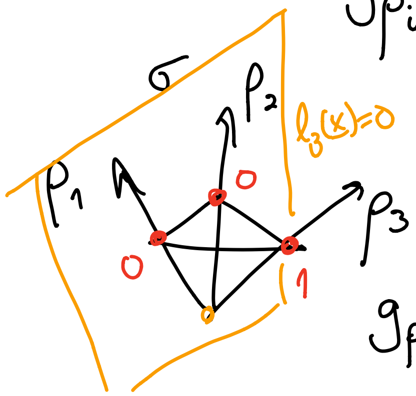


$$g_\sigma\Big|_{\underbrace{\mathrm{Lin}(\sigma)}_{x_1x_2\text{-plane}}}(x_1,x_2) = c\cdot \underbrace{(x_2-x_1)}_{\ell_1(\underline{x})}\ \underbrace{x_2}_{\ell_2(\underline{x})}$$

proof: In fact, if σ has $\rho_{i_1}, \ldots, \rho_{i_k}$ as rays

then $g_{\rho_{i_j}}\Big|_\sigma = c_j \cdot l_j(\underline{x})$ for some $c_i \in \mathbb{R} - \{0\}$

where $l_i(\underline{x})$



$g_{\rho_3}\Big|_\sigma = c_3 \cdot l_3(\underline{x})$

And then

$g_\sigma\Big|_\sigma = c_1 \cdots c_k \; l_1(\underline{x}) \cdots l_k(\underline{x})$

And any polynomial $f(x_1, \ldots, x_k)$ on $\mathrm{Lin}(\sigma)$ that vanishes on $\partial\sigma$ must vanish each of the hyperplanes defined by $l_j(\underline{x}) = 0$

Hence f is divisible by $l_j(\underline{x})$ for $j = 1, 2, \ldots, k$

EXERCISE!

so f is divisible by $\underbrace{l_1(\underline{x}) \cdots l_k(\underline{x})}_{= \vec{c} \cdot g_\sigma|_\sigma}$

$\blacksquare$

Now we'll use this to set up a short exact seq. parallel to one we saw for $\mathbb{R}[\Delta]$ or $K[\Delta]$....

Recall for any subcomplex $\Delta' \subset \Delta$, we had a surjection

$$\begin{array}{ccc} K[\Delta] & \xrightarrow{\pi} & K[\Delta'] \\ \| & & \| \\ K[\underline{x}]/I_\Delta & & K[\underline{x}]/I_{\Delta'} \end{array} \quad \text{since } I_{\Delta'} \supseteq I_\Delta$$

and when we specialized to $\Delta' = \Delta - \{F\}$ for F a maximal face of Δ

we had identified the kernel of π

principal ideal

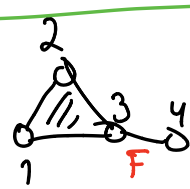
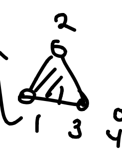
$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\underline{x}^F) & \hookrightarrow & K[\Delta] & \xrightarrow{\pi} & K[\Delta - \{F\}] & \rightarrow & 0 \\ & & \uparrow \wr \ \text{K-vector space iso.} & & & & & & \\ & & K[x_j]_{j \in F} & & & & & & \end{array}$$

$f(\underline{x}_j) \cdot \underline{x}^F$

$\uparrow$

$f(x_j)$

is short exact as K-vector spaces

e.g. $0 \rightarrow (x_3 x_4) \hookrightarrow K\Big[$  $\Big] \xrightarrow{\pi} K\Big[$  $\Big] \rightarrow 0$

$x_3 x_4 f(x_3, x_4)$ $\quad \uparrow \wr$

$\uparrow$

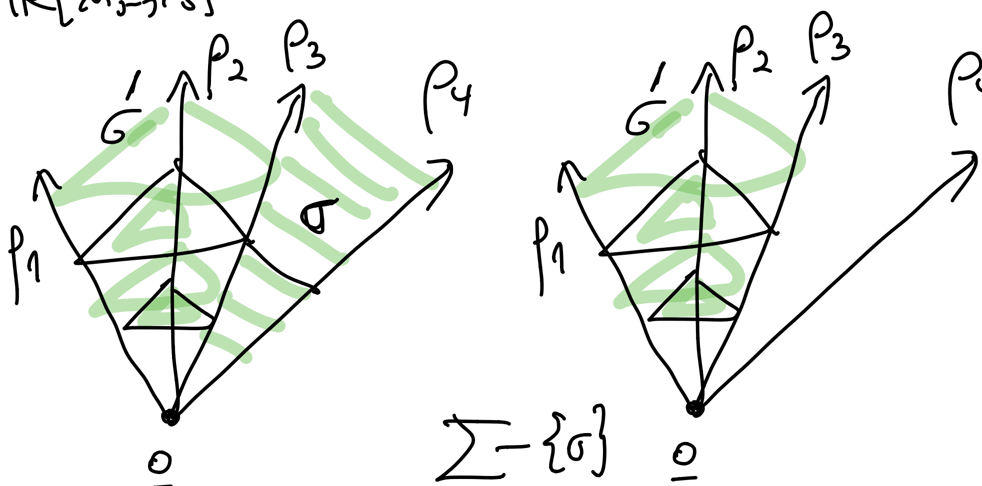
$f(x_3, x_4) \quad K[x_3, x_4]$

PROP: (Brion '96 §1.2) For σ a maximal cone in Σ a simplicial fan, say with rays $\rho_1, \dots, \rho_s$ for σ, one has a short exact sequence of $\mathbb{R}$-vector spaces

$$\begin{array}{ccccccc}
 & \text{principal ideal} & & & & & \\
0 \longrightarrow & (g_\sigma) & \hookrightarrow & R_\Sigma & \xrightarrow{\text{res}} & R_{\Sigma - \{\sigma\}} & \longrightarrow 0 \\
 & \uparrow\wr & f(g_{\rho_1}, \dots, g_{\rho_s}) \cdot g_\sigma & f & \longmapsto & f & \\
 & & \updownarrow & \shortparallel & & \shortparallel & \\
 & & f(x_1, \dots, x_s) & (f_\tau)_{\tau \in \Sigma} & & (f_\tau)_{\tau \in \Sigma - \{\sigma\}} & \\
 & \mathbb{R}[x_1, \dots, x_s] & & & & &
\end{array}$$

and a $\mathbb{R}$-vector space iso.

e.g. $\Sigma \subset \mathbb{R}^3$



<u>proof</u>: The previous PROP actually told us that $\ker\left(R_\Sigma \xrightarrow{\text{res}} R_{\Sigma - \{\sigma\}}\right) = (g_\sigma)$

and the other part of the PROP gave us the iso.

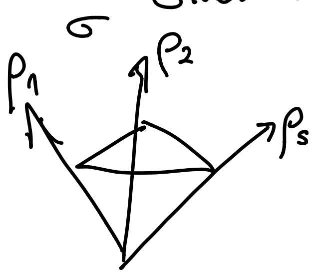
$$\begin{aligned} \mathbb{R}[x_1, \dots, x_s] &\longrightarrow (g_\sigma) \\ f(x_1, \dots, x_s) &\longmapsto f(g_{\rho_1}, \dots, g_{\rho_s}) \cdot g_\sigma \end{aligned}$$

We're still missing surjectivity of $R_\Sigma \xrightarrow{res} R_{\Sigma - \{\sigma\}}$.

Given some $f = (f_I)$ well-defined on every subcone $\sigma_I := \mathbb{R}_{\geq 0}\rho_{i_1} + \dots + \mathbb{R}_{\geq 0}\rho_{i_k}$

$$f_I = f_I(x_I)$$

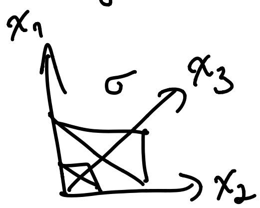
for $I = \{i_1, \dots, i_k\} \subsetneq \{1, 2, \dots, s\}$



then we can define f as a polynomial on $\mathrm{Lin}(\sigma)$:

$$f(x_1, \dots, x_s) = \sum_{I \subsetneq \{1,2,\dots,s\}} (-1)^{s-1-\#I} f_I(x_I)$$

e.g. $s=3$



$$f(x_1,x_2,x_3) := f_{12}(x_1,x_2) + f_{13}(x_1,x_3) + f_{23}(x_2,x_3) - \big(f_1(x_1) + f_2(x_2) + f_3(x_3)\big) + f_\emptyset(\{0\})$$

e.g. on x_1-x_2 plane we have

$$f(x_1,x_2,0) = f_{12}(x_1,x_2) + \cancelto{0}{f_{13}(x_1,0) - f_1(x_1)} + \cancelto{0}{f_{23}(x_2,0) - f_2(x_2)} \; \cancelto{0}{- f_3(0) + f_\emptyset(\{0\})}$$

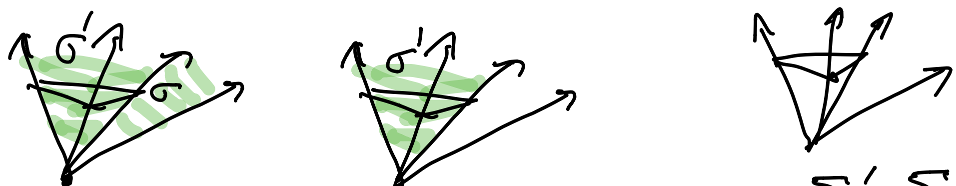
∎

Math 8680 March 26, 2021

Some COROLLARIES:

COR 1: For any simplicial fan $\Sigma \subset \mathbb{R}^d$ and any subfan $\Sigma' \subset \Sigma$, the restriction map $R_\Sigma \xrightarrow{res} R_{\Sigma'}$ surjects.

proof: Repeatedly remove maximal cones σ in $\Sigma - \Sigma'$ with surjectivity at each step from the previous result $\blacksquare$



$\Sigma \quad \supset \quad \Sigma - \{\sigma\} \quad \supset \quad \Sigma' = \Sigma - \{\sigma, \sigma'\}$

COR 2: (Brion '96 §1.3, Billera '89) Given Σ a simplicial fan in $\mathbb{R}^d$ with rays $\rho_1, \rho_2, \ldots, \rho_n$

consider the ring homomorphism

$$\mathbb{R}[x_1, \ldots, x_n] \xrightarrow{\varphi} R_\Sigma$$
$$x_i \longmapsto g_{\rho_i} = \text{tent/courant function}$$

Then it will induce a ring isomorphism

$$\mathbb{R}[\Delta_\Sigma] = K[X]/I_\Delta \xrightarrow[\sim]{\varphi} R_\Sigma$$

Stanley-Reisner ring

proof: 1st note that if G is a non-face of Δ_Σ
$G \subset \{1,2,\dots,n\}$

then $\underline{x}^G = \prod_{j\in G} x_j \overset{\varphi}{\longmapsto} \prod_{j\in G} g_{\rho_j} \equiv 0$

$\uparrow$ by defn of Δ_Σ and g_{ρ_j} is supported on $\mathrm{star}_\Sigma(\rho_i)$.

so $I_\Delta = (\underline{x}^G)_{G \text{ non-face}} \subset \ker\varphi$

and φ descends to a ring homom.

$$\mathbb{R}[\Delta_\Sigma] \xrightarrow{\varphi} R_\Sigma$$

Now compare the two short exact sequences:
for a choice of max cone $\sigma \in \Sigma$
with corr. facet $F \in \Delta_\Sigma$

$$\begin{tikzcd}
0 \arrow[r] & (\underline{x}^F) \arrow[r] \arrow[d, phantom, "\cong"] \arrow[dd, bend left, "\text{iso?}"] & \mathbb{R}[\Delta_\Sigma] \arrow[r] \arrow[dd, "\varphi"] & \mathbb{R}[\Delta_\Sigma - \{F\}] \arrow[r] \arrow[dd, "\varphi", "\cong"'] & 0 \\
& \mathbb{R}[x_j]_{j\in F} \arrow[d, phantom, "\cong"] & & & \\
0 \arrow[r] & (g_\sigma) \arrow[r] & R_\Sigma \arrow[r, "\mathrm{res}"] & R_{\Sigma - \{\sigma\}} \arrow[r] & 0
\end{tikzcd}$$

$f(\underline{x})\cdot \underline{x}^F \mapsfrom f(\underline{x}) \mapsto f(g_{\rho_1},\dots,g_{\rho_s})\cdot g_\sigma$

$x_i \mapsto g_{\rho_i}$

This is an isomorphism by induction on $\#\Sigma$

Hence by the 5-lemma, the middle vertical map is an $\mathbb{R}$-vector space isomorphism, and a ring isomorphism. $\blacksquare$

This isomorphism interacts well with the ~~l.s.o.p.~~ linear elements (see correction below)

$$(\underline{\theta}_\Sigma) = (\theta_{f_1}, \dots, \theta_{f_d})$$

where $f_1, \dots, f_d$ were an $\mathbb{R}$-basis for $(\mathbb{R}^d)^*$

and $$\theta_f := \sum_{j=1}^{n} f(v_j) \cdot x_j$$

for $f \in (\mathbb{R}^d)^*$

where $v_1, v_2, \dots, v_n$ were chosen to span rays $\rho_1, \rho_2, \dots, \rho_n$ of Σ

i.e. $\rho_j = \mathbb{R}_{\geq 0} \cdot v_j$

This raises the point that, inside R_Σ, we have the $\left\{ \begin{array}{c} \text{global polynomial functions} \\ f: \mathbb{R}^d \to \mathbb{R} \end{array} \right\}$ ~~as an~~ mapping to an $\mathbb{R}$-subalgebra of R_Σ

$$A := \cong \mathbb{R}[f_1, f_2, \dots, f_d]$$

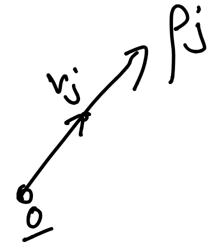
We'll regard R_Σ as a ($\mathbb{N}$-graded) module over this ~~sub~~($\mathbb{N}$-graded) algebra A

PROP: Given any $f \in (\mathbb{R}^d)^*$ the ring isomorphism

$$\mathbb{R}[\Delta_\Sigma] \xrightarrow{\varphi} R_\Sigma$$

sends

$$\begin{array}{ccc} \Theta_f = \sum_{j=1}^{n} f(v_j)\cdot x_j & \xmapsto{\varphi} & f \\ \cap & & \cap \\ \mathbb{R}[\Delta_\Sigma]_1 & & A_1 \\ & & \cap \\ & & A \end{array}$$



and hence sends $\mathbb{R}[\underline{\Theta}] = \mathbb{R}[\Theta_{f_1}, \ldots, \Theta_{f_d}] \longrightarrow A$ = global polynomial functions on Σ

proof:

$$\Theta_f = \sum_{j=1}^{n} f(v_j)\cdot x_j \xmapsto{\varphi} \varphi(\Theta_f) = \sum_{j=1}^{n} f(v_j)\, g_{\rho_j}$$

has $\varphi(\Theta_f)(v_i) = \sum_{j=1}^{n} f(v_j) \underbrace{g_{\rho_j}(v_i)}_{\delta_{ij}}$ for $i = 1, \ldots, n$

$$= f(v_i)$$

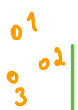
Since $\varphi(\Theta_f)$ is PL on Σ

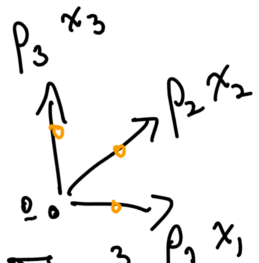
and f is linear globally,

and they agree on all $v_1, \dots, v_n$,

they <u>coincide</u>. $\blacksquare$

REMARK: Sam Hopkins asked a good question, prompting this correction

$\Delta_\Sigma =$ 



$\Sigma \subset \mathbb{R}^3$

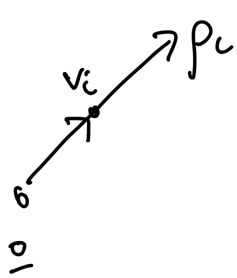
a good example to ponder

① $\Theta_{f_1}, \dots, \Theta_{f_d}$ are <u>not</u> an l.s.o.p., whenever $d > \dim(\Delta_\Sigma)+1$

AND ...

②
$$\begin{tikzcd}
R_\Sigma & \left\{\substack{\text{functions } f: |\Sigma| \to \mathbb{R} \\ \text{that are restrictions of} \\ \text{global polynomials} \\ \mathbb{R}^d \to \mathbb{R}}\right\} \arrow[l, "\text{subalgebra}"'] \\
A = \mathbb{R}[f_1, f_2, \dots, f_d] \arrow[u, "\times", "\text{not a subalgebra!}"'] \arrow[r, phantom, "\cong"] & \left\{\substack{\text{global polynomial} \\ f: \mathbb{R}^d \to \mathbb{R}}\right\} \arrow[u, "\text{res}"']
\end{tikzcd}$$

<u>REMARK:</u> Θ_{f_i}'s looked very <u>noncanonical</u>, like they depend on the choice of v_i spanning rays ρ_i



But replacing v_i by $c_i v_i$ with $c_i \in \mathbb{R} - \{0\}$ has the effect in $\mathbb{R}[\Delta_\Sigma]$ of scaling variable x_i by $\frac{x_i}{c_i}$