Name: ______________________________

Signature: ______________________________

Section and TA: ______________________________

Math 1272. Lecture 010 (V. Reiner) Midterm Exam I
Thursday, February 18, 2010

This is a 50 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
Total:	

Problem 1. *(20 points total; 10 points each) A rod is lying along the x-axis in the interval from $x = 1$ cm to $x = 2$ cm, and at a point x cm from the origin has linear density $\rho(x) = 4 + x^3$ grams/cm.*

a. *What is its total mass?*

b. *Where is its center of mass?*

Problem 2. *(30 points total) Consider the following differential equation:* $\frac{dy}{dx} = -e^x(y-7)^2$.

a. (10 points) *Write down all nonconstant solutions to this differential equation. Make sure that your answer expresses y as a function of x explicitly in the form $y = f(x)$.*

b. (5 points) *Are there any constant solutions to this differential equation? This means solutions of the form $y = y_0$ for some constant y_0. Either write them all down, or explain why none exist.*

c. (5 points) *Write down the unique solution to the initial value problem* $\frac{dy}{dx} = -e^x(y-7)^2$ *with* $y(0) = 1$.

d. (5 points) *For your exact solution to the initial value problem in part (c), what is* $y(1)$*?*
(Do **not** *evalute this numerically in decimals.)*

e. (5 points) *For this same initial value problem* $\frac{dy}{dx} = -e^x(y-7)^2$ *with* $y(0) = 1$ *as in part (c), wite down the* **approximation** *to* $y(1)$ *given by one step of Euler's method using step size* $\Delta x (= dx) = 1$,

Problem 3. *(20 points total; 5 points each)*

a. *Write down the equation of the hyperbola in standard form that intersects the* x*-axis at* $(\pm 3, 0)$ *and has asymptotes* $y = \pm\frac{4}{3}x$.

b. *Find the coordinates for the two foci of this same hyperbola.*

c. *For the ellipse with equation* $\frac{x^2}{9} + \frac{y^2}{16} = 1$*, find the coordinates of the points where it intersects the* x*-axis, and of the points where it intersects the* y*-axis.*

d. *Find the coordinates of the two foci for this same ellipse* $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

Problem 4. *(30 points total; 15 points each) Consider the closed curve given in polar coordinates by $r = \cos(\theta)$ where θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$.* **Without** *converting it to a rectangular coordinate equation, find the following.*

a. *The arc length of the curve.*

b. *The area enclosed by the curve.*

Brief solutions

1. A rod is lying along the x-axis in the interval from $x = 1$ cm to $x = 2$ cm, and at a point x cm from the origin has linear density $\rho(x) = 4 + x^3$ grams/cm.

(a) Its total mass is

$$\begin{aligned} M &= \int_1^2 (4 + x^3)dx \\ &= \left[4x + \frac{x^4}{4}\right]_1^2 \\ &= \left(4 \cdot 2 + \frac{2^4}{4}\right) - \left(4 \cdot 1 + \frac{1^4}{4}\right) \\ &= \frac{31}{4} \end{aligned}$$

(b) Its center of mass is $\bar{x} = \frac{M_0}{M}$ where ...

$$\begin{aligned} M_0 &= \int_1^2 x(4 + x^3)dx \\ &= \int_1^2 (4x + x^4)dx \\ &= \left[2x^2 + \frac{x^5}{5}\right]_1^2 \\ &= \left(2 \cdot 2^2 + \frac{2^5}{5}\right) - \left(2 \cdot 1^2 + \frac{1^5}{5}\right) \\ &= \frac{61}{5} \end{aligned}$$

Hence the center of mass is at

$$\bar{x} = \frac{61}{5} / \frac{31}{4} = \frac{244}{155}$$

along the x-axis.

2. Consider the following differential equation: $\frac{dy}{dx} = -e^x(y - 7)^2$.

(a) To get the nonconstant solutions, separate variables and integrate:

$$\begin{aligned}
\frac{dy}{dx} &= -e^x(y-7)^2 \\
\frac{dy}{(y-7)^2} &= -e^x dx \\
\int \frac{dy}{(y-7)^2} &= \int (-e^x dx) \\
\frac{-1}{y-7} &= -e^x + C \\
\frac{1}{e^x - C} &= y - 7 \\
y &= \frac{1}{e^x - C} + 7
\end{aligned}$$

(b) Constant solutions $y = y_0$ come from roots y_0 of $(y-7)^2 = 0$, so $y = y_0 = 7$ is the only constant solution.

(c) The initival value problem for the same differential equation with $y(0) = 1$ pins down the constant:

$$\begin{aligned}
1 = y(0) &= \frac{1}{e^0 - C} + 7 \\
1 &= \frac{1}{1-C} + 7 \\
-6 &= \frac{1}{1-C} \\
\frac{-1}{6} &= 1 - C \\
C &= \frac{7}{6}
\end{aligned}$$

and hence the unique solution to this IVP is

$$y = \frac{1}{e^x - \frac{7}{6}} + 7$$

(d) Plugging in $x = 1$ in the answer to part (c) gives

$$y(1) = \frac{1}{e^1 - \frac{7}{6}} + 7 = \frac{1}{e - \frac{7}{6}} + 7$$

(e) One step of Euler's method with $dx = 1$ approximates this same $y(1)$ as the y_1 in the pair (x_1, y_1) obtained from from $(x_0, y_0) = (0, 1)$ via

$$x_1 = x_0 + dx = 0 + 1 = 1$$

and

$$\begin{aligned} y_1 &= y_0 + -e^{x_0}(y_0 - 7)^2 dx \\ &= 1 + (-e^0(1-7)^2 \cdot 1 \\ &= 1 + (-36) \\ &= -35. \end{aligned}$$

3.(a) The hyperbola in standard form that intersects the x-axis at $(\pm 3, 0)$ and has asymptotes $y = \pm\frac{4}{3}x$ should have equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a = 3$ and where $b = 4$ since the asymptotes should be $y = \pm\frac{b}{a}x$. Thus it is $\frac{x^2}{9} - \frac{y^2}{16} = 1$

(b) The coordinates for the two foci of this same hyperbola should be $(\pm c, 0)$ where $c^2 = a^2 + b^2$, co $c = \sqrt{9 + 16} = 5$. Thus the foci are at $(\pm 5, 0)$

(c) The ellipse with equation $\frac{x^2}{9} + \frac{y^2}{16} = 1$, intersects the x-axis where $y = 0$, so $\frac{x^2}{9} = 1$, i.e. $x = \pm 3$ or the points $(\pm 3, 0)$. It intersects the y-axis where $x = 0$, so $\frac{y^2}{16} = 1$, i.e. $y = \pm 4$ or the points $(0, \pm 4)$.

(d) The coordinates of the two foci for this same ellipse are $(\pm c, 0)$ where $c^2 = a^2 - b^2$ so $c = \sqrt{16 - 9} = \sqrt{7}$. Thus the foci are at $(\pm\sqrt{7}, 0)$.

4. For the curve given in polar coordinates by $r = \cos(\theta)$ where θ varies from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, one has that...

(a) the arc length is

$$\begin{aligned} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r(\theta)^2 + r'(\theta)^2} d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos^2(\theta) + (-\sin(\theta))^2} d\theta \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta \\ &= [\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \pi, \end{aligned}$$

(b) the area enclosed by the curve is ...

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2} r(\theta)^2 d\theta &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(\theta) d\theta \\ &= \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(2\theta)}{2} d\theta \\ &= \frac{1}{4}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= \frac{\pi}{4}\end{aligned}$$