Name: ______________________________

Signature: ______________________________

Section and TA: ______________________________

Math 1272. Lecture 010 (V. Reiner) Midterm Exam III
Thursday, April 22, 2010

This is a 50 minute exam. No books, notes, calculators, cell phones or other electronic devices are allowed. There are a total of 100 points. To get full credit for a problem you must show the details of your work. Answers unsupported by an argument will get little credit. Do all of your calculations on this test paper.

Problem	Score
1.	
2.	
3.	
4.	
Total:	

Problem 1. *(28 points total; 7 points each) For each of the following series, indicate whether they converge or diverge, with explanation.*

a. $\sum_{n=0}^{\infty} e^{-5n+3}$

b. $\sum_{n=0}^{\infty} \frac{1}{n^3+n^2+100}$

c. $\sum_{n=0}^{\infty} \frac{n^5-7}{2n^5+5n^4-6}$

d. $\sum_{n=0}^{\infty} \frac{4^n}{9^n+8}$

Problem 2. *(27 points total) Compute the following exactly.*

a. (6 points) $\lim_{n\to\infty} 4^{\frac{n^5-7}{2n^5+5n^4-6}}$

b. (6 points) $\sum_{n=0}^{\infty} 7 \cdot (-1)^n \cdot \frac{2^n}{3^n}$

c. (8 points) *The* **radius** *of convergence for the power series* $\sum_{n=0}^{\infty} \frac{(-6x)^n}{n+10}$.

d. (7 points) *The* **interval** *of convergence for the power series* $\sum_{n=0}^{\infty} \frac{(-6x)^n}{n+10}$.

Problem 3. *(20 points)*
Find the quadratic Taylor polynomial $T_2(x)$ *approximating the function* $f(x) = \sec(x)$ *about* $x = 0$.

Problem 4. *(25 points total; 5 points each) For the two vectors in* $\mathbf{R}^3$
$A = \langle 1, 0, 1\rangle$
$B = \langle 1, 1, 0\rangle$
compute the following.

a. *The magnitude* $|A|$.

b. *The unit vector pointing in the direction of* B.

c. *The dot product* $A \cdot B$.

d. *The angle between A and B,* **exactly**, *in radians.*

e. *The magnitude of A's projection onto the line in the direction of B.*

Brief solutions

1. (a.) $\sum_{n=0}^{\infty} e^{-5n+3}$ converges by integral test since the improper integral

$$\int_0^\infty e^{-5x+3}dx = \left[-\frac{1}{5}e^{-5x+3}\right]_0^\infty$$
$$= \lim_{b\to\infty} -\frac{1}{5}e^{-5b+3} - \left(-\frac{1}{5}e^{-5\cdot 0+3}\right)$$
$$= \frac{1}{5}e^3$$

converges.
(b.) $\sum_{n=0}^{\infty} \frac{1}{n^3+n^2+100}$ converges by comparison to $\sum_{n=0}^{\infty} \frac{1}{n^3}$ since $n^3+n^2+100 \geq n^3$ for $n \geq 0$ implies $\frac{1}{n^3+n^2+100} \leq \frac{1}{n^3}$, and since $\sum_{n=0}^{\infty} \frac{1}{n^3}$ converges by integral test (or the special case sometimes called "p-test").
(c.) $\sum_{n=0}^{\infty} \frac{n^5-7}{2n^5+5n^4-6}$ diverges since

$$\lim_{n\to\infty} \frac{n^5-7}{2n^5+5n^4-6} = \lim_{n\to\infty} \frac{1-\frac{7}{n^5}}{2+\frac{5}{n}-\frac{6}{n^5}} = \frac{1}{2} \neq 0.$$

(d). $\sum_{n=0}^{\infty} \frac{4^n}{9^n+8}$ converges by comparison to the convergent geometric series $\sum_{n=0}^{\infty} \frac{4^n}{9^n} = \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^n$ which has ratio $\frac{4}{9} < 1$
2.(a.)

$$\lim_{n\to\infty} 4^{\frac{n^5-7}{2n^5+5n^4-6}} = 4^{\lim_{n\to\infty} \frac{n^5-7}{2n^5+5n^4-6}} = 4^{\lim_{n\to\infty} \frac{1-\frac{7}{n^5}}{2+\frac{5}{n}-\frac{6}{n^5}}} = 4^{\frac{1}{2}} = 2$$

(b.)

$$\sum_{n=0}^{\infty} 7\cdot(-1)^n \cdot \frac{2^n}{3^n} = 7\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n = 7\left(\frac{1}{1-(-\frac{2}{3})}\right) = \frac{21}{5}$$

(c.) The **radius** of convergence for the power series $\sum_{n=0}^{\infty} \frac{(-6x)^n}{n+10}$ is computed by finding which values of x make the sum absolutely convergent via ratio test:

$$\lim_{n\to\infty} \left|\frac{(-6x)^{n+1}}{(n+1)+10} / \frac{(-6x)^n}{n+10}\right| = \lim_{n\to\infty} \left|\frac{(-6x)^{n+1}(n+11)}{(-6x)^n(n+10)}\right|$$
$$= |-6x| \lim_{n\to\infty} \frac{n+11}{n+10}$$
$$= |6x|$$

Thus ratio test shows absolute convergence for $|6x| < 1$ or $|x| < \frac{1}{6}$, and divergence for $|x| > \frac{1}{6}$. This means the radius of convergence is $\frac{1}{6}$.
(d.) The **interval** of convergence for the power series $\sum_{n=0}^{\infty} \frac{(-6x)^n}{n+10}$ is obtained by checking the endpoints $x = \pm\frac{1}{6}$.

For $x = \frac{1}{6}$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+10}$ converges (conditionally) by the alternating series test, as $\frac{1}{n+10}$ is a decreasing function of n and it approaches 0 as $n \to \infty$.

For $x = \frac{-1}{6}$, the series $\sum_{n=0}^{\infty} \frac{1}{n+10}$ diverges, either by integral test ($\int_0^{\infty} \frac{1}{x+10} dx = [\ln(x+10)]_0^{\infty}$ diverges) or by comparison test: $\frac{1}{n+10} \geq \frac{1}{2n}$ for $n \geq 10$ since $n + 10 \leq 2n$ in that case, and $\sum_{n=0}^{\infty} \frac{1}{2n}$ diverges by integral test (or by the special case called p-test).

3. The quadratic Taylor polynomial $T_2(x)$ approximating the function $f(x) = \sec(x)$ about $x = 0$ is

$$T_2(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f'(0)}{2!}x^2.$$

Since

$$\begin{aligned} f(0) &= \sec(0) = 1 \\ f'(x) &= \sec(x)\tan(x), && \text{so } f'(0) = 0 \\ f''(x) &= \sec(x)\tan(x)\tan(x) + \sec(x)\sec^2(x), && \text{so } f'(0) = 1 \end{aligned}$$

one has

$$T_2(x) = 1 + \frac{0}{1!}x^1 + \frac{1}{2!}x^2 = 1 + \frac{x^2}{2}.$$

4. For $A = \langle 1, 0, 1\rangle, B = \langle 1, 1, 0\rangle$ one has ... (a.) The magnitude $|A| = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{2}$.
(b.) The unit vector pointing in the direction of B is

$$\frac{B}{|B|} = \frac{1}{\sqrt{1^2 + 1^2 + 0^2}}B = \frac{1}{\sqrt{2}}\langle 1, 1, 0\rangle = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle.$$

(c.) The dot product $A \cdot B = 1 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 1$.
(d.) The angle between A and B is

$$\arccos\left(\frac{A \cdot B}{|A||B|}\right) = \arccos\left(\frac{1}{\sqrt{2}\sqrt{2}}\right) = \arccos\frac{1}{2} = \frac{\pi}{3}.$$

(e.) The magnitude of A's projection onto the line in the direction of B is

$$|A|\cos(\theta) = \frac{A \cdot B}{|B|} = \frac{1}{\sqrt{2}}.$$