MATH 23b, SPRING 2005
THEORETICAL LINEAR ALGEBRA
AND MULTIVARIABLE CALCULUS

The Inverse Function Theorem

The Inverse Function Theorem. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be continuously differentiable on some open set containing $\mathbf{a}$, and suppose $\det Jf(\mathbf{a}) \neq 0$. Then there is some open set V containing $\mathbf{a}$ and an open W containing $f(\mathbf{a})$ such that $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ which is differentiable for all $\mathbf{y} \in W$.

Note: As matrices, $J(f^{-1})(\mathbf{y}) = [(Jf)(f^{-1}(\mathbf{y}))]^{-1}$.

Lemma: Let $A \subset \mathbb{R}^n$ be an open rectangle, and suppose $f : A \longrightarrow \mathbb{R}^n$ is continuously differentiable. If there is some $M > 0$ such that
$\left|\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right| \leq M, \ \forall \mathbf{x} \in A$, then $||f(\mathbf{y}) - f(\mathbf{z})|| \leq n^2 \cdot M \cdot ||\mathbf{y} - \mathbf{z}||, \ \forall \mathbf{y}, \mathbf{z} \in A$.

Proof: We write

$$\begin{aligned} f_i(\mathbf{y}) - f_i(\mathbf{z}) &= f_i(y_1, \ldots, y_n) - f_i(z_1, \ldots, z_n) \\ &= \sum_{j=1}^{n} [f(y_1, \ldots, y_j, z_{j+1}, \ldots, z_n) - f(y_1, \ldots, y_{j-1}, z_j, z_{j+1}, \ldots, z_n)] \\ &= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\mathbf{x}_{ij})(y_j - z_j) \end{aligned}$$

for some $\mathbf{x}_{ij} = (y_1, \ldots, y_{j-1}, c_j, z_{j+1}, \ldots, z_n)$ where, for each $j = 1, \ldots, n$, we have c_j is in the interval (y_j, z_j), by the single-variable Mean Value Theorem.

Then

$$\begin{aligned} ||f(\mathbf{y}) - f(\mathbf{z})|| &\leq \sum_{i=1}^{n} ||f_i(\mathbf{y}) - f_i(\mathbf{z})|| \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} \left|\frac{\partial f_i}{\partial x_j}(\mathbf{x}_{ij})\right| \cdot |y_j - z_j| \\ &\leq \sum_{i=1}^{n} \sum_{j=1}^{n} M \cdot ||\mathbf{y} - \mathbf{z}|| \\ &= n^2 \cdot M \cdot ||\mathbf{y} - \mathbf{z}|| \end{aligned}$$

$\square$

Proof of the Inverse Function Theorem:

(borrowed principally from Spivak's *Calculus on Manifolds*)

Let $L = Jf(\mathbf{a})$. Then $\det(L) \neq 0$, and so L^{-1} exists. Consider the composite function $L^{-1} \circ f : \mathbb{R}^n \to \mathbb{R}^n$. Then:

$$\begin{array}{rcl} J(L^{-1} \circ f)(\mathbf{a}) & = & J(L^{-1})(f(\mathbf{a})) \circ Jf(\mathbf{a}) \\ & = & L^{-1} \circ Jf(\mathbf{a}) \\ & = & L^{-1} \circ L \end{array}$$

which is the identity. Since L is invertible, the theorem is equally true or false for both $L^{-1} \circ f$ and f simultaneously, and hence we prove it in the case when $L = I$.

Suppose $f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a})$. Then $\dfrac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})|}{|\mathbf{h}|} = \dfrac{|\mathbf{h}|}{|\mathbf{h}|} = 1.$

On the other hand, we have have $\displaystyle\lim_{||\mathbf{h}|| \to 0} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - L(\mathbf{h})}{||\mathbf{h}||} = \mathbf{0},$

which is a contradiction, and hence there must be some open neighborhood/rectangle U around $\mathbf{a}$ in which $f(\mathbf{a}+\mathbf{h}) \neq f(\mathbf{a}), \ \ \forall \mathbf{a}+\mathbf{h} \in U, \ \ \mathbf{h} \neq \mathbf{0}$.

Furthermore, we may choose this neighborhood U small enough so that:

- $\det(Jf(\mathbf{x})) \neq 0, \ \ \forall \mathbf{x} \in U$
- $\left| \dfrac{\partial f_i}{\partial x_j}(\mathbf{x}) - \dfrac{\partial f_i}{\partial x_j}(\mathbf{a}) \right| < \dfrac{1}{2n^2}, \ \ \forall i, j, \ \forall \mathbf{x} \in U$

since these are conditions on $n^2 + 1$ continuous functions!

Claim 1: $||\mathbf{x}_1 - \mathbf{x}_2|| \leq 2 \cdot ||f(\mathbf{x}_1) - f(\mathbf{x}_2)||, \ \ \forall \mathbf{x}_1, \mathbf{x}_2 \in U$

Proof of Claim 1: First, we let $g(\mathbf{x}) = f(\mathbf{x}) - \mathbf{x}$. By construction and the second fact above, we have $\left| \dfrac{\partial g_i}{\partial x_j}(\mathbf{x}) \right| = \left| \dfrac{\partial f_i}{\partial x_j}(\mathbf{x}) - \dfrac{\partial f_i}{\partial x_j}(\mathbf{a}) \right| \leq \dfrac{1}{2n^2},$

and so we apply the Lemma with $M = \dfrac{1}{2n^2}$:

$$\begin{array}{rcl} ||\mathbf{x}_1 - \mathbf{x}_2|| - ||f(\mathbf{x}_1) - f(\mathbf{x}_2)|| & \leq & ||(f(\mathbf{x}_1) - \mathbf{x}_1) - (f(\mathbf{x}_2) - \mathbf{x}_2)|| \\ & = & ||g(\mathbf{x}_1) - g(\mathbf{x}_2)|| \\ & \leq & \frac{1}{2} \cdot ||\mathbf{x}_1 - \mathbf{x}_2|| \end{array}$$

and so, combining these inequalities, we have

$$\tfrac{1}{2} \cdot ||\mathbf{x}_1 - \mathbf{x}_2|| \ \ \leq \ \ ||f(\mathbf{x}_1) - f(\mathbf{x}_2)||$$

$\square$

Now consider the set ∂U, which is compact since U is bounded. We know by the reasoning in the second paragraph of the proof that if $\mathbf{x} \in \partial U$, then $f(\mathbf{x}) \neq f(\mathbf{a})$. Hence $\exists d > 0$ such that $||f(\mathbf{x}) - f(\mathbf{a})|| \geq d, \quad \forall \mathbf{x} \in \partial U$. (Since both f and the taking of norms are continuous functions, the expression $||f(\mathbf{x}) - f(\mathbf{a})||$ attains its non-zero minimum on the compact set ∂U.)

We construct the set $W \subset \mathbb{R}^n$, thinking of it as a subset of the range of f, as follows:

$$W = \left\{ \mathbf{y} \in \mathbb{R}^n \;\middle|\; ||\mathbf{y} - f(\mathbf{a})|| < \frac{d}{2} \right\} = B_{d/2}(f(\mathbf{a}))$$

By its construction and the use of the positive real number d, we see that if $\mathbf{y} \in W$ and $\mathbf{x} \in \partial U$, then

$$||\mathbf{y} - f(\mathbf{a})|| < ||\mathbf{y} - f(\mathbf{x})||. \tag{1}$$

Claim 2: Given $\mathbf{y} \in W$, there is a unique $\mathbf{x} \in U$ such that $f(\mathbf{x}) = \mathbf{y}$.

Proof of Claim 2:

Existence:
Consider $h : U \to \mathbb{R}$ defined by $h(\mathbf{x}) = ||\mathbf{y} - f(\mathbf{x})||^2$. A straightforward simplification of this expression gives $h(\mathbf{x}) = \sum_{i=1}^{n} (y_i - f_i(\mathbf{x}))^2$. Note that h is continuous and hence attains its minimum on the compact set $\overline{U}$. This minimum does *not* occur on the boundary, ∂U, by the inequality (1), and hence it must occur on the interior. Since h is also differentiable, we must have $\nabla h(\mathbf{x}) = 0$ at the minimum, and hence:

$$0 = \frac{\partial h}{\partial x_j}(\mathbf{x}) = \sum_{i=1}^{n} 2 \cdot (y_i - f_i(\mathbf{x})) \cdot \frac{\partial f_i}{\partial x_j}(\mathbf{x}), \quad \forall j$$

In other words, collecting this information over the various i and j, we have

$$\mathbf{0} = Jf(\mathbf{x}) \cdot (\mathbf{y} - f(\mathbf{x})),$$

but since we have assumed that $\det Jf(\mathbf{x}) \neq 0$ for any $\mathbf{x} \in U$, it follows that $Jf(\mathbf{x})$ is invertible, and hence $\mathbf{y} - f(\mathbf{x}) = \mathbf{0}$.

Uniqueness:
We use Claim 1. Suppose $\mathbf{y} = f(\mathbf{x}_1) = f(\mathbf{x}_2)$.
Then $||\mathbf{x}_1 - \mathbf{x}_2|| \leq 2 \cdot ||f(\mathbf{x}_1) - f(\mathbf{x}_2)|| = 0$, and hence $\mathbf{x}_1 = \mathbf{x}_2$.

$\square$

By Claim 2, if we define $V = U \cap f^{-1}(W)$, then $f : V \to U$ has an inverse! It remains to show that f^{-1} is continuous and differentiable. Even though continuity would follow from differentiability, we do this in two steps because we will use the continuity to help prove the differentiability.

Claim 3: f^{-1} is continuous.

Proof of Claim 3:

For $\mathbf{y}_1, \mathbf{y}_2 \in W$, find $\mathbf{x}_1, \mathbf{x}_2 \in U$ such that $f(\mathbf{x}_1) = \mathbf{y}_1$ and $f(\mathbf{x}_2) = \mathbf{y}_2$. Claim 1 implies that $||\mathbf{x}_1 - \mathbf{x}_2|| \leq 2 \cdot ||f(\mathbf{x}_1) - f(\mathbf{x}_2)||$, or in other words, that $||f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}_2)|| \leq 2 \cdot ||\mathbf{y}_1 - \mathbf{y}_2||$.

It is now easy to see that given $\varepsilon > 0$, we need only choose $\delta = \varepsilon/2$ to guarantee that if $||\mathbf{y}_1 - \mathbf{y}_2|| < \delta$, then $||f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}_2)|| < \varepsilon$.

$\square$

Claim 4: f^{-1} is differentiable.

Proof of Claim 4:

Let $\mathbf{x} \in V$, let $A = Jf(\mathbf{x})$, and let $\mathbf{y} = f(\mathbf{x}) \in W$.
We claim that $Jf^{-1}(\mathbf{y}) = A^{-1}$.
Define $\varphi(\mathbf{x}) = f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - A(\mathbf{h})$.
Then $\displaystyle\lim_{||\mathbf{h}|| \to 0} \frac{||\varphi(\mathbf{h})||}{||\mathbf{h}||} = 0$, by the differentiability of f.
Since $\det(A) = \det Jf(\mathbf{x}) \neq 0$ by hypothesis, we know that A^{-1} exists, and it is linear since A is. Then:

$$\begin{aligned} A^{-1}(f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x})) &= \mathbf{h} + A^{-1}(\varphi(\mathbf{h})) \\ &= [(\mathbf{x} + \mathbf{h}) - \mathbf{x}] + A^{-1}(\varphi(\mathbf{h})) \end{aligned}$$

Letting $\mathbf{y} = f(\mathbf{x})$ and $\mathbf{y}_1 = f(\mathbf{x} + \mathbf{h})$ on both sides yields:

$$A^{-1}(\mathbf{y}_1 - \mathbf{y}) = [f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})] + A^{-1}(\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})))$$

Re-arranging sides:

$$A^{-1}(\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}))) = [f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})] - A^{-1}(\mathbf{y}_1 - \mathbf{y}) \quad (2)$$

To show differentiability, we need:

$$\lim_{||\mathbf{y}_1-\mathbf{y}||\to 0} \frac{||f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}) - A^{-1}(\mathbf{y}_1 - \mathbf{y})||}{||\mathbf{y}_1 - \mathbf{y}||} = 0$$

but by equation (2) above, this is the same as showing:

$$\lim_{||\mathbf{y}_1-\mathbf{y}||\to 0} \frac{||A^{-1}(\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})))||}{||\mathbf{y}_1 - \mathbf{y}||} = 0.$$

Since A^{-1} is linear, it suffices to use the Chain Rule and show that:

$$\lim_{||\mathbf{y}_1-\mathbf{y}||\to 0} \frac{||\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}))||}{||\mathbf{y}_1 - \mathbf{y}||} = 0, \tag{3}$$

so we factor the expression inside the limit as follows:

$$\frac{||\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}))||}{||\mathbf{y}_1 - \mathbf{y}||} = \frac{||\varphi(f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y}))||}{||f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})||} \cdot \frac{||f^{-1}(\mathbf{y}_1) - f^{-1}(\mathbf{y})||}{||\mathbf{y}_1 - \mathbf{y}||}.$$

The first term on the right tends to 0 because of how we defined φ and the fact that the continuity of f^{-1} means that $f^{-1}(\mathbf{y}_1) \to f^{-1}(\mathbf{y})$.

Observing that the second term on the right is less than or equal to 2 (by Claim 1) enables us to use the Squeeze Theorem and conclude that the product on the right tends to 0, which establishes equation (3).

$\square$

End Proof of Inverse Function Theorem.

(borrowed principally from
Spivak's *Calculus on Manifolds*)