Exercises in duality of plane graphs
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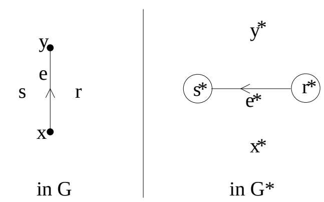
Recall a *plane graph* $G = (V, E)$ is a multigraph (multiple edges, self-loops allowed) with an embedding in the plane $\mathbb{R}^2$. The complement $\mathbb{R}^2 \setminus G$ has connected components called *faces* or *regions*. Denote its

$$\text{vertices } V = \{x_1, \dots, x_n\},$$
$$\text{edges } E = \{e_1, \dots, e_m\},$$
$$\text{faces } F = \{r_1, \dots, r_f\}.$$

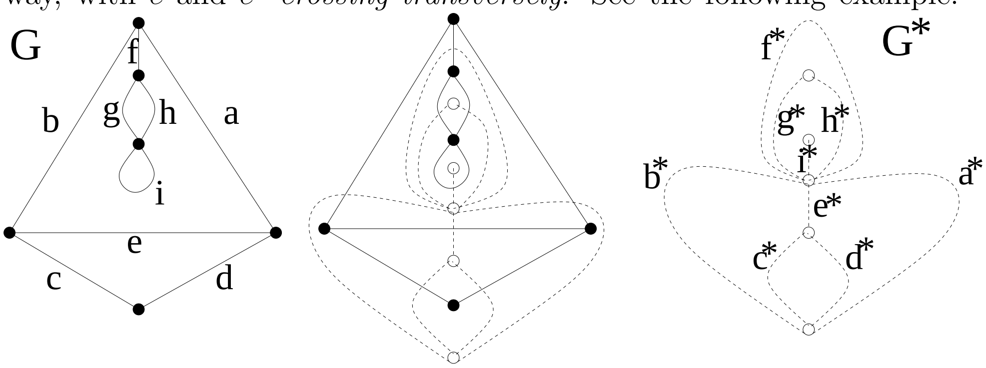
This leads to its *plane dual graph* $G^* = (V^*, E^*)$ with bijections

$$\begin{array}{rll} \text{dual vertices} & V^* = \{r_1^*, \dots, r_f^*\} & \leftrightarrow F, \\ \text{dual edges} & E^* = \{e_1^*, \dots, e_m^*\} & \leftrightarrow E, \\ \text{dual faces} & F^* = \{x_1^*, \dots, x_n^*\} & \leftrightarrow V. \end{array}$$

The bijection $E \longrightarrow E^*$ sending $e_i \leftrightarrow e_i^*$ is set up as follows: having (arbitrarily) picked an orientation $x \to y$ for an edge $e = \{x, y\}$, as one traverses e from x to y, there are faces, r, s of G lying to the right and to the left of e, allowing for the possbility that $r = s$ if e is a bridge/isthmus of G. Then one creates in G^* an edge $e^* = \{r, s\}$, and when one is keeping track of orientations, one orients e^* as $r \to s$.

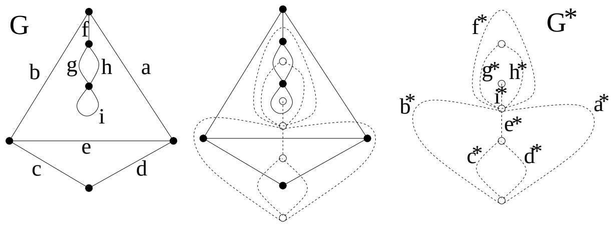


One can show that if G has been embedded in $\mathbb{R}^2$ so that each edge is embedded smoothly, then G^* can also be embeddeded in this way, with e and e^* *crossing transversely*. See the following example:

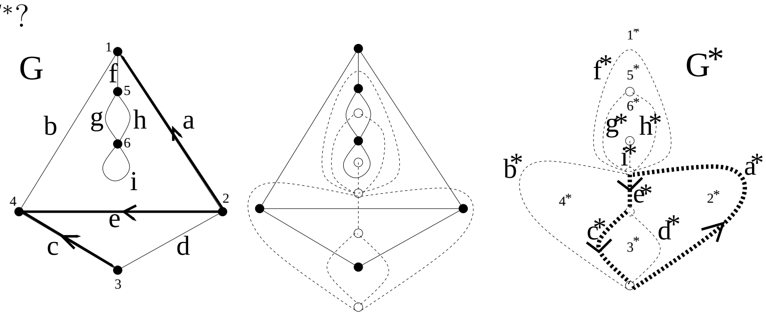


We wish to explore concepts for G corresponding to concepts for G^*.

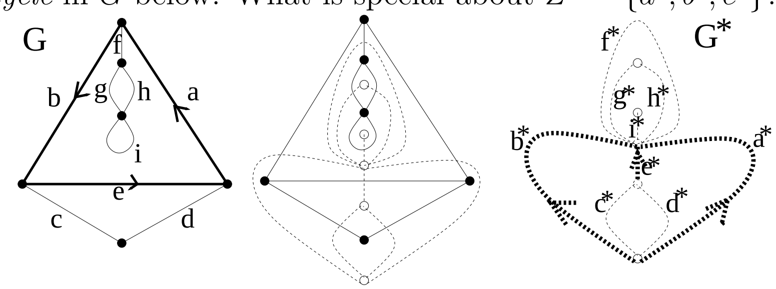
1. (a) What is special about the edge i. And its dual edge i^*?
(b) What about edge f and its dual edge f^*?
(c) What general statement do (a),(b) suggest, and can you prove it?



2. Number the vertices $V = \{1, 2, 3, 4, 5, 6\}$ in G as shown, with corresponding faces $F^* = \{1^*, 2^*, 3^*, 4^*, 5^*, 6^*\}$ in G^*.
(a) Consider edges $C = \{a, c, e\}$ shown darkened below, and oriented as a *minimal directed cut* $C = A(S, T)$ from[1] $S = \{2, 3\}$ to $T = \{1, 4, 5, 6\}$.
What is special about the corresponding edges $C^* = \{a^*, c^*, e^*\}$ of G^*?



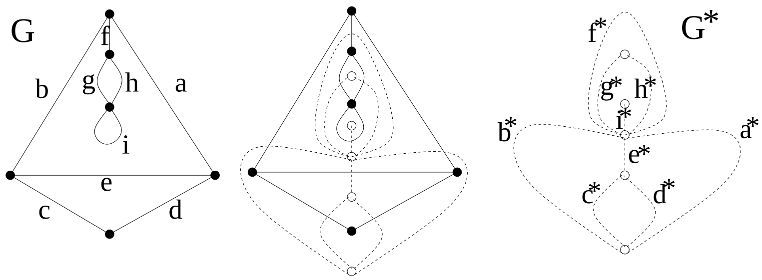
(b) Consider the edges $Z = \{a, b, e\}$ darkened and oriented as a *directed cycle* in G below. What is special about $Z^* = \{a^*, b^*, e^*\}$?

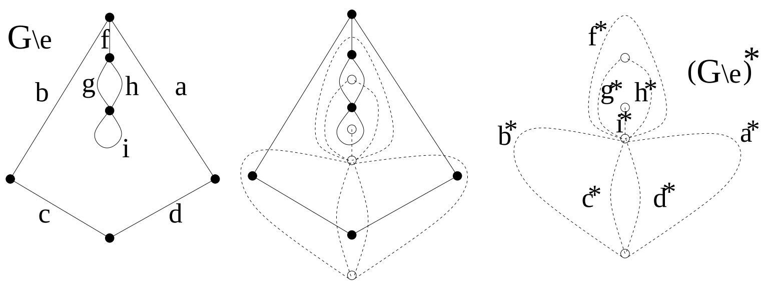


(c) What general statement do (a),(b) suggest, and can you prove it? Does your statement take into account orientations?

[1]Here *minimality* of the cut C means that removing it increases the number of connected components in the graph, but this fails for any proper subset of C. Equivalently, both S, T induce connected subgraphs of G.

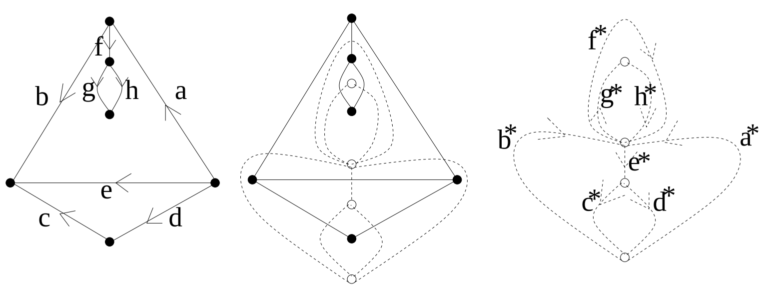
3. The figure below shows G and G^* followed by the *deletion* $G\backslash e$ and its dual $(G\backslash e)^*$.





How does $(G\backslash e)^*$ relate to G^*?
What general statement does this suggest, and can you prove it?

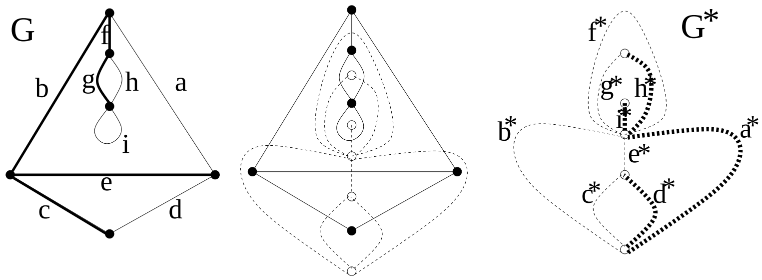
4. Delete the loop i from G, so that one can *acyclically orient* it as shown below, and consider the corresponding orientation of the dual, also shown below.



What is special about this dual orientation?
What general statement does this suggest, and can you prove it?

5. Can you explain why the plane dual G^* of G, as we have defined it, will always have G^* *connected*? Can you use this to construct examples where the *double dual* $(G^*)^*$ is *not* isomorphic to G?

6. Consider the *spanning tree* $T = \{b, c, e, f, g\}$ in G shown darkened below, and the *complementary* set of dual edges $E^* \backslash T^* = \{a^*, d^*, h^*, i^*\}$ shown darkened in G^*.



(a) What is special about the set $E^* \setminus T^*$?
What general statement does this suggest, and can you prove it?

The statement from (a) leads to a pleasing proof of Euler's formula $n - m + f = 2$ for any connected planar graph G that has n vertices, m edges, and f faces. Here is how it works ...

(b) How many edges should a spanning tree T in G contain, in terms of the parameter n?

(c) Based on the special property of $E^* \setminus T^*$ from part (a), how many edges should $E^* \setminus T^*$ contain, in terms of the parameter f.

(d) Explain why the numbers of edges in parts (a) and part (b) should sum up to m. Write down the resulting equation that asserts this. How does it lead to Euler's formula?