Root and weight semigroup rings for signed posets

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Dedication

For my grandad.

Abstract

We consider a pair of semigroups associated to a signed poset, called the root semigroup and the weight semigroup, and their semigroup rings, R_P^{rt} and R_P^{wt}, respectively.

Theorem 4.1.5 gives generators for the toric ideal of affine semigroup rings associated to signed posets and, more generally, oriented signed graphs. These are the subrings of Laurent polynomials generated by monomials of the form $t_i^{\pm 1}, t_i^{\pm 2}, t_i^{\pm 1}t_j^{\pm 1}$. This result appears to be new and generalizes work of Boussicault, Féray, Lascoux, and Reiner [9], Gitler, Reyes, and Villarreal [26] and Villarreal [59]. Theorem 4.2.12 shows that strongly planar signed posets P have rings R_P^{rt}, $R_{P^\vee}^{\mathrm{rt}}$ which are complete intersections, with Corollary 4.2.20 showing how to compute Ψ_P in this case. Theorem 5.2.3 gives a Gröbner basis for the toric ideal of R_P^{wt} in type B, generalizing Féray and Reiner [20, Proposition 6.4]. Theorems 5.3.10 and 5.3.21 giving two characterizations (via forbidden subposets versus via inductive constructions) of the situation where this Gröbner basis gives a complete intersection presentation for its initial ideal, generalizing Féray and Reiner [20, Theorems 10.5, 10.6].

Contents

Acknowledgements **i**

Dedication **ii**

Abstract **iii**

List of Tables **vii**

List of Figures **viii**

1 Introduction **1**

1.1 A root system perspective on posets 2

1.2 The signed poset story 7

1.3 Summary of the Main Results 12

1.4 A remark on notation 13

2 Some Background **14**

2.1 Polyhedral Cones 14

2.1.1 Rational Functions and Cones 16

2.2 Semigroups 18

2.2.1 Semigroup Rings and Toric Ideals 19

2.2.2 A Rational Function 22

2.2.3 Complete Intersections 26
2.3 Root Systems 28
2.3.1 The Weyl Group and Signed Permutations 30

3 Signed Posets **33**
3.1 Representing Signed Posets 34
3.2 Ideals and P-partitions in Signed Posets 38
3.3 Linear Extensions 43
3.4 Embedding Posets as Signed Posets 43
3.5 The Root and Weight Cones and Their Extreme Rays 44
3.5.1 Dual Characterizations 49
3.5.2 Matroids 50
3.5.3 When the root cone is full-dimensional and simplicial 51
3.6 Two Rational Functions 57

4 The Root Cone Semigroup **64**
4.1 The Toric Ideal of an Oriented Signed Graph 65
4.1.1 The toric ideal of the root cone semigroup 73
4.1.2 Toric ideals of posets and directed graphs 75
4.1.3 Toric ideals of graphs 77
4.2 Some Complete Intersection Rings R_P^{rt} 80
4.2.1 A few reductions 82
4.2.2 Signed Notches 85
4.2.3 Strongly Planar Signed Posets 96
4.2.4 Computing Ψ_P 117

5 The Weight Cone Semigroup **121**
5.1 Generators of the Semigroup 122

5.2 The Toric Ideal . . . 123

5.2.1 Term Orders and Gröbner Bases . . . 125

5.2.2 The Proof of Theorem 5.2.3 . . . 126

5.3 Complete intersections and the sum over linear extensions . . . 130

5.3.1 Reducing to connected $\widehat{G}(P)$ . . . 132

5.3.2 Characterizing the initial complete intersections . . . 134

5.3.3 Constructing the complete intersection posets . . . 142

5.3.4 Computing Rational Functions . . . 150

6 Unfinished Business **154**

6.1 Two triangulations of the weight cone . . . 154

6.2 The type C weight cone . . . 161

6.3 On characterizing the R_P^{rt} complete intersections . . . 163

List of Tables

3.1 Association between bidirected edges and elements of a signed poset 35

3.2 Relations defining $\widehat{G}_B(P)$ 36

3.3 Relations defining $\widehat{G}_C(P^\vee)$ 37

4.1 Correspondence between edges in Σ and $\widetilde{\Sigma}$ 68

5.1 Linear extensions for the signed poset of Figure 5.10, their descents and major index. 152

6.1 Type B and C ideals for $\widehat{G}(P)$ from Figure 6.5 162

List of Figures

1.1 A poset . . . 1

1.2 A signed poset . . . 8

1.3 A signed poset . . . 10

2.1 Cone spanned by $(1, 1, 1)$, $(-1, 1, 1)$, $(-1, -1, 1)$ and $(1, -1, 1)$ . . . 15

2.2 The cone spanned by $(1, 0)$ and $(0, 1)$ intersected with $\mathbb{Z}^2$ . . . 19

2.3 Cone spanned by $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(1, 1, 1)$ . . . 25

2.4 Root and Weight Lattices . . . 31

3.1 A signed graph and a bidirection . . . 35

3.2 The Hasse diagram of $\{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$ . . . 36

3.3 $\widehat{G}_B(P)$ and $\widehat{G}_C(P)$ for $P^\star = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$. 37

3.4 $J(P^\star)$ for $P^\star = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$ . . . 42

3.5 The poset $P = \{e_1 - e_2, e_1 - e_3\}$ and its embedding as a signed poset . . . 44

3.6 $\widehat{G}(P)$ for $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$ . . . 46

3.7 The root cone and weight cone of $\{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$ 47

3.8 Hasse diagram of $P = \{+e_1 - e_2, +e_3 - e_2, +e_1 + e_3, e_3 - e_5, e_3 - e_4, -e_4 - e_5, +e_6 - e_4, +e_6 - e_5\}$ and associated signed graph . . . 52

3.9 Σ_P and $\widehat{G}(P)$ for $P = \{+e_1 + e_2, +e_2 - e_3, -e_3 - e_4, +e_1 - e_4\}$ . . . 56

3.10 Hasse diagram and $\widehat{G}(P)$ for $P = \{e_1 + e_2, e_1 - e_2, e_1 + e_3, e_1, e_3)\}$ . . . 62

3.11 A forest and its embedding as a signed poset . . . 63

4.1 An oriented signed graph . . . 66

4.2 Computing a toric ideal with Macaulay2 . . . 67

4.3 Signed covering of the oriented signed graph in Figure 4.1 . . . 69

4.4 Another signed covering . . . 69

4.5 $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ for $P = \{+e_1 - e_2, +e_1 + e_3, +e_1 - e_3, +e_1 + e_2, +e_1 - e_4, +e_1 + e_4, +e_1, +e_4\}$ . . . 74

4.6 The directed (multi-)graph from the proof of Theorem 4.1.5 . . . 74

4.7 A digraph, its associated oriented signed graph and signed covering . . . 76

4.8 A graph and its signed covering . . . 78

4.9 A graph for which the generating sets from Theorem 4.1.5 and Theorem 4.1.11 differ . . . 80

4.10 $\widehat{G}(P)$ for $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_1 + e_3, +e_1 - e_4, +e_1 - e_5, +e_1 - e_6, +e_2 - e_3, +e_2 - e_4, +e_2 - e_5, +e_2 - e_6, +e_4 - e_6, +e_5 - e_6, +e_1\}$ with biconnected components indicated by dashed and solid lines. . . . 83

4.11 $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ from Figure 4.10 after closing the signed notch $(2, 3, 5)$ and $(-2, -3, -5)$. . . . 87

4.12 After closing two notches in the proof of Proposition 4.2.7 . . . 88

4.13 $\widehat{G}_C(P^\vee)$ in case 1 of the proof of Proposition 4.2.8 . . . 90

4.14 $\widehat{G}_C(P^\vee)$ in case 1(d) in the proof of Proposition 4.2.8 . . . 91

4.15 P_B^+ and P_C^+ for $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ from Figure 4.11 . . . 94

4.16 $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ for $P = \{+e_1 + e_2, +e_1, +e_2\}$ . . . 97

4.17 A signed poset with $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ both strongly planar . . . 98

4.18 Having opened one notch to obtain P' . . . 99

4.19 P' broken into biconnected components . . . 100

4.20 $\widehat{G}_B(P_3)$ . . . 100

4.21 The situation in the last claim of the proof of Proposition 4.2.14 . . . 104

4.22 Two regions exchanged by the involution separated by a chain, c, fixed by the involution . . . 107
4.23 The first two cases of the proof of Theorem 4.2.12 . . . 112
4.24 The chain $c_{j_1} \lessdot \cdots \lessdot c_{j_\ell}$ in the proof of Theorem 4.2.12. . . . 113
4.25 $J(P)$ for P in Figure 4.17 . . . 119
5.1 $P = \{+e_1, +e_1 - e_2, +e_1 - e_3, +e_1 - e_4, +e_2 - e_3\}$ . . . 125
5.2 The excluded posets, part one . . . 137
5.2 The excluded posets, part two . . . 138
5.3 A signed poset P where R_P^{wt} is a complete intersection, but P is not an initial complete intersection. . . . 140
5.4 The path described in the second half of case (b)(i) in the proof of Theorem 5.3.10 141
5.5 Obtaining a new poset by hanging a single element 4 below 2 . . . 143
5.6 Obtaining a new poset by duplicating the hanger 2 . . . 144
5.7 Obtaining a signed poset by the type B hanging of 3 above 1 . . . 145
5.8 The posets from which the initial complete intersections are built . . . 146
5.9 An induced subposet when $c > a, c \not> a', c < -c$ in case (b.ii) of the proof of Theorem 5.3.21 . . . 149
5.10 A signed poset . . . 151
6.1 The two different triangulations of K_P^{wt} . . . 156
6.2 $P = \{+e_1 - e_2, +e_1 + e_2, +e_1\}$ . . . 159
6.3 The zonotop $\mathcal{Z}[\Sigma]$ and its Newton polytope for Σ from Figure 6.2 . . . 159
6.4 The complete signed graph $\pm K_2^B$, its acyclotope and the normal fan of the acyclotope . . . 160
6.5 $P = \{e_1 - e_2, e_1 + e_2, e_1\}$ . . . 162
6.6 $\widehat{G}(P^\vee)$ for $P^\vee = \{e_1 - e_2, e_1 + e_2, e_1 + e_3, e_3 - e_2, 2e_1\}$ . . . 163
6.7 A poset which has R_P^{rt} a complete intersection but is not gluing of diamonds along chains . . . 164

6.8 The poset of Figure 6.7 broken into unicycle components 165
6.9 A signed poset which is not strongly planar, but has R_P^{rt} a complete intersection 166

Chapter 1

Introduction

A partially ordered set, or *poset*, is a set P together with a binary relation $<$ such that

- $<$ is antisymmetric: if $x < y$ then $y \not< x$ for all $x, y \in P$, and

- $<$ is transitive: if $x < y$ and $y < z$ then $x < z$ for all $x, y, z \in P$.

Consider the poset in Figure 1.1. The underlying set is $P = \{1, 2, 3, 4, 5\}$ and $x < y$ if there is a path from x to y travelling upwards along each edge in the path. For example, $1 < 3$,

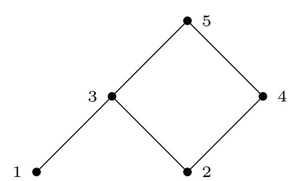


Figure 1.1: A poset

but $1 \not< 4$.

An element of a poset $y \in P$ is said to *cover* $x \in P$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. One writes $x \lessdot y$ to emphasize that the relation between x and y is a *covering relation*.

A *linear extension* of a poset is an extension $\prec$ of $<$ to a total order, i.e. a linear order of the elements of P by $\prec$ so that if $x < y$ then $x \prec y$. The linear extensions of our example are

$$1 \prec 2 \prec 3 \prec 4 \prec 5$$
$$1 \prec 2 \prec 4 \prec 3 \prec 5$$
$$2 \prec 1 \prec 3 \prec 4 \prec 5$$
$$2 \prec 1 \prec 4 \prec 3 \prec 5$$
$$2 \prec 4 \prec 1 \prec 3 \prec 5$$

The set of linear extensions of P is denoted $\mathcal{L}(P)$. When P is a poset on $[n] := \{1, 2, \ldots, n\}$, it is natural to regard the linear extensions as permutations of $[n]$. When P is the poset in Figure 1.1 one then has $\mathcal{L}(P) = \{12345, 12435, 21345, 21435, 24135\}$.

While finding a linear extension of a finite poset is straightforward—it is what is known as *topological sorting* in computer science and can be done in linear time (see, for instance [15, §22.4])—counting the number of linear extensions proves rather more difficult. Brightwell and Winkler showed in [10] that the problem is $\#P$-complete. As a consequence, computing rational functions which are sums over linear extensions proves difficult. This leads to two basic questions:

- When and how can the linear extensions of a poset be counted without listing them?
- When and how can such a rational function be computed without listing all the linear extensions?

1.1 A root system perspective on posets

In [9], Boussicault, Féray, Lascoux and Reiner used a view of posets as sets of type A roots to explain how a pair of rational functions which are sums over linear extensions can be evaluated, at least in certain cases. They transform a poset P into the collection of type A roots $\{e_i - e_j \colon i <_P j\}$. Under this scheme, P from Figure 1.1 corresponds to $\{e_1 - e_3, e_1 - e_5, e_2 - e_3, e_2 - e_4, e_2 - e_5, e_3 - e_5, e_4 - e_5\}$. The two rational functions they considered are, for a poset P on $[n] \coloneqq \{1, 2, \ldots, n\}$,

$$\Psi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)}\right)$$

and

$$\Phi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1 + x_2)\cdots(x_1 + \cdots + x_n)}\right),$$

where the linear extensions are viewed as permutations acting on the indices of x_i.

The function Ψ_P had previously been considered by Greene [30], where he gave an evaluation for *strongly planar* posets, i.e. those posets whose Hasse diagrams remain planar after the addition of minimum and maximum elements $\hat{0}$ and $\hat{1}$.

Theorem 1.1.1 (Greene, [30])**.** *Suppose P is a strongly planar poset. Then*

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i \lessdot j}(x_i - x_j)},$$

where ρ runs over all the bounded regions enclosed by the Hasse diagram of P and $i \lessdot j$ runs over all covering relations of the poset.

In our example, computing Ψ_P as a sum over the linear extensions gives

$$\begin{aligned}
\Psi_P(\boldsymbol{x}) = &\sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)}\right) \\
=&\frac{1}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)(x_4 - x_5)} + \frac{1}{(x_1 - x_2)(x_2 - x_4)(x_4 - x_3)(x_3 - x_5)} \\
&+ \frac{1}{(x_2 - x_1)(x_1 - x_3)(x_3 - x_4)(x_4 - x_5)} + \frac{1}{(x_2 - x_1)(x_1 - x_4)(x_4 - x_3)(x_3 - x_5)} \\
&+ \frac{1}{(x_2 - x_4)(x_4 - x_1)(x_1 - x_3)(x_3 - x_5)} \\
=&\frac{x_2 - x_5}{(x_1 - x_3)(x_2 - x_3)(x_2 - x_4)(x_3 - x_5)(x_4 - x_5)}.
\end{aligned}$$

On the other hand,

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i<j}(x_i - x_j)} = \frac{x_2 - x_5}{(x_1 - x_3)(x_2 - x_3)(x_2 - x - 4)(x_3 - x_5)(x_4 - x_5)}.$$

The function Φ_P was considered in the case of forests by Chapoton, Hivert, Novelli, and Thibon [14], who proved the following.

Theorem 1.1.2 (Chapoton, Hivert, Novelli, Thibon, [14])**.** *Suppose P is a forest (i.e. every element is covered by at most one other element). Then*

$$\Phi_P(\boldsymbol{x}) = \prod_{i=1}^{n} \frac{1}{\sum_{j \leq_P i} x_j}.$$

Boussicault, Féray, Lascoux and Reiner then defined a pair of dual cones, the root cone $K_P^{\mathrm{rt}} = \mathbb{R}_+ P$ and the weight cone $K_P^{\mathrm{wt}} = \mathbb{R}_+\{\chi_J \colon J \in J(P)\}$, where $J(P)$ is the set of *order ideals* J, subsets $J \subset P$ with the condition that if $y \in J$ and $x <_P y$, then $x \in J$, and χ_J is the characteristic vector of J. They made two important realizations:

- Ψ_P and Φ_P are the Laplace transform valuations of K_P^{rt} and K_P^{wt}, and
- Ψ_P and Φ_P can be recovered from the Hilbert series of the semigroup rings

$$R_P^{\mathrm{rt}} = k[K_P^{\mathrm{rt}} \cap \mathbb{Z}^n] \quad \text{and} \quad R_P^{\mathrm{wt}} = k[K_P^{\mathrm{wt}} \cap \mathbb{Z}^n],$$

respectively.

These two observations enabled Boussicault, Féray, Lascoux and Reiner in [9] and Féray and Reiner in [20] to use Proposition 2.2.14 to obtain the following two results.

Theorem 1.1.3 (Boussicault, Féray, Lascoux, Reiner)**.** *Suppose P is a strongly planar* *poset. Then R_P^{rt} is a complete intersection,*

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - x_{\min(\rho)} x_{\max(\rho)}^{-1})}{\prod_{i \lessdot_P j} (1 - x_i x_j^{-1})}$$

and

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i \lessdot_P j} (x_i - x_j)}.$$

In [20], Féray and Reiner described a class of posets generalizing forests called *forests with duplication*, which are precisely those posets such that R_P^{wt} is a complete intersection, and compute Φ_P in this case.

Theorem 1.1.4 (Féray and Reiner)**.** *A poset P is a forest with duplication if and only if R_P^{wt} is a complete intersection, in which case*

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, \boldsymbol{x}) = \frac{\prod_{\{J,K\} \in \Pi(P)} (1 - \boldsymbol{x}^J \boldsymbol{x}^k)}{\prod_{J \in J_{\mathrm{conn}}(P)} (1 - \boldsymbol{x}^J)}$$

and

$$\Phi_P(\boldsymbol{x}) = \frac{\prod_{\{J,K\} \in \Pi(P)} \langle \boldsymbol{x}, \chi_{J+K} \rangle}{\prod_{J \in J_{\mathrm{conn}}(P)} \langle \boldsymbol{x}, \chi_J \rangle},$$

where $J_{\mathrm{conn}}(P)$ is the set of connected order ideals of P, $\Pi(P)$ is the set of pairs of connected order ideals which intersect nontrivially ($J \cap K \neq \emptyset$ and neither $J \subset K$ nor $K \subset J$) and $\langle \boldsymbol{x}, \chi_J \rangle = \sum_{i \in J} x_i$.

Return again to the poset in Figure 1.1. One has that the root cone semigroup ring is presented as

$$k[U_{13}, U_{23}, U_{24}, U_{35}, U_{45}]/(U_{23}U_{35} - U_{24}U_{45})$$

with the map $U_{ij} \mapsto x_i x_j^{-1} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, and the weight cone semigroup ring is presented as

$$k[U_1, U_2, U_{24}, U_{123}, U_{1234}, U_{12345}]/(U_{123}U_{24} - U_2 U_{1234})$$

with the map $U_J \mapsto \prod_{i \in J} x_i \in k[x_1, \ldots, x_n]$. Since both of the presentation ideals, $(U_{23}U_{35} - U_{24}U_{45})$ and $(U_{123}U_{24} - U_2U_{1234})$, are principal, both R_P^{rt} and R_P^{wt} are complete intersections. Both rings are naturally $\mathbb{Z}^5$-graded, and this $\mathbb{Z}^5$-grading coincides with an $\mathbb{N}^5$-grading of R_P^{wt}. One then has

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \frac{(1 - x_2x_5^{-1})}{(1 - x_1x_3^{-1})(1 - x_2x_3^{-1})(1 - x_2x_4^{-1})(1 - x_3x_5^{-1})(1 - x_4x_5^{-1})},$$

so

$$\Psi_P(\boldsymbol{x}) = \frac{(x_2 - x_5)}{(x_1 - x_3)(x_2 - x_3)(x_2 - x_4)(x_3 - x_5)(x_4 - x_5)},.$$

as seen above. On the other hand,

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, \boldsymbol{x}) = \frac{(1 - x_1x_2^2x_3x_4)}{(1 - x_1)(1 - x_2)(1 - x_2x_4)(1 - x_1x_2x_3)(1 - x_1x_2x_3x_4)(1 - x_1x_2x_3x_4x_5)},$$

so

$$\Phi_P(\boldsymbol{x}) = \frac{x_1 + 2x_2 + x_3 + x_4}{x_1x_2(x_2 + x_4)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + x_4 + x_5)}.$$

Computing $\Phi_P(\boldsymbol{x})$ as a sum over linear extensions, one has

$$\begin{aligned}
&\Phi_P(\boldsymbol{x}) = \\
&\quad \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1 + x_2)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + x_4 + x_5)}\right) \\
&\qquad = \frac{1}{x_1(x_1 + x_2)(x_1 + x_2 + x_3)(x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + x_4 + x_5)} \\
&\qquad + \frac{1}{x_1(x_1 + x_2)(x_1 + x_2 + x_4)(x_1 + x_2 + x_4 + x_3)(x_1 + x_2 + x_4 + x_3 + x_5)} \\
&\qquad + \frac{1}{x_2(x_2 + x_1)(x_2 + x_1 + x_3)(x_2 + x_1 + x_3 + x_4)(x_2 + x_1 + x_4 + x_3 + x_5)} \\
&\qquad + \frac{1}{x_2(x_2 + x_1)(x_2 + x_4 + x_1)(x_2 + x_1 + x_4 + x_3)(x_2 + x_1 + x_4 + x_3 + x_5)}
\end{aligned}$$

$$+\frac{1}{x_2(x_2+x_4)(x_2+x_4+x_1)(x_2+x_1+x_4+x_3)(x_2+x_4+x_1+x_3+x_5)}$$
$$=\frac{x_1+2x_2+x_3+x_4}{x_1x_2(x_1+x_2+x_3)(x_2+x_4)(x_1+x_2+x_3+x_4)(x_1+x_2+x_3+x_4+x_5)},$$

matching the above computation.

Féray and Reiner also observed that when the $\mathbb{N}^n$-grading on R_P^{wt} is collapsed to an $\mathbb{N}$-grading, taking $\deg x_i = 1$ for all i, one has

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, q) = \frac{\sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

In particular, when P is a forest, this recovers the q-hook formula of Björner and Wachs [5]. For the poset in Figure 1.1, one has

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, q) = \frac{1-q^5}{(1-q)^2(1-q^2)(1-q^3)(1-q^4)(1-q^5)}$$
$$= \frac{1+q^3+q+q^4+q^2}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)} = \frac{\sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}.$$

1.2 The signed poset story

The story for signed posets is really quite similar to that for posets and we will generalize all of the above results to this context. Signed posets will be defined formally in Definition 3.0.2. They come in pairs, $P \subset \Phi_{B_n}$ and $P^\vee \subset \Phi_{C_n}$, since Φ_{B_n} and Φ_{C_n} are dual root systems. Inspecting the definitions of Ψ_P and Φ_P in type A, one sees that the denominators of the fraction on which w acts correspond to a choice of simple roots and the corresponding fundamental dominant coweights, respectively. This observation leads one to define

$$\Psi_P(\boldsymbol{x}) = \sum_{w\in\mathcal{L}(P)} w\left(\frac{1}{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)x_n}\right) \text{ and}$$
$$\Psi^\vee_{P^\vee}(\boldsymbol{x}) = \sum_{w\in\mathcal{L}(P^\vee)} w\left(\frac{1}{(x_1-x_2)(x_2-x_3)\cdots(x_{n-1}-x_n)2x_n}\right)$$

to parallel Ψ in type A and

$$\Phi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1+x_2)\cdots(x_1+\cdots+x_n)}\right) \text{ and}$$
$$\Phi^{\vee}_{P^{\vee}}(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P^{\vee})} w\left(\frac{1}{x_1(x_1+x_2)+\cdots+(x_1+\cdots+x_{n-1})(\frac{1}{2}x_1+\cdots+\frac{1}{2}x_n)}\right)$$

to parallel Φ in type A.

One can likewise define root and weight cones for signed posets (see Section 3.5) and use the corresponding semigroup rings to evaluate Ψ and Φ in some cases where the semigroup ring is a complete intersection.

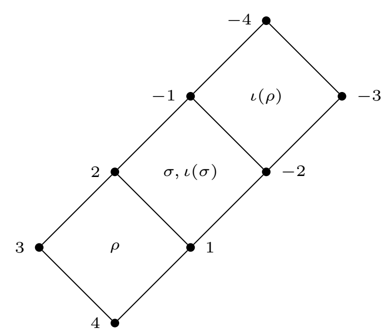


Figure 1.2: A signed poset

By way of example, consider Figure 1.2. As will be explained in Section 3.1, it is a representation of a signed poset $P^{\vee} \subset \Phi_{C_n}$, with $P^{\vee} = \{+e_1 - e_2, +e_1 + e_2, +e_3 - e_2, +e_4 - e_3, +e_4 - e_1, +e_4 - e_2, +e_2 + e_4, +e_3 + e_4, +e_1 + e_3, +e_1 + e_4, +2e_1, +2e_4\}$. Note that there is an involutive poset anti-automorphism, ι, exchanging i and $-i$. The poset is strongly planar (this will mean $P^{\vee}$ is strongly planar) and the three regions it encloses fall into two orbits under the involution: $\{\rho, \iota(\rho)\}$ and $\{\sigma = \iota(\sigma)\}$. Theorem 4.2.12 will show that this implies the following complete intersection presentation for $R^{\mathrm{rt}}_{P^{\vee}}$:

$$R^{\mathrm{rt}}_{P^{\vee}} \cong k[U_{12}, U_{1\bar{2}}, U_{\bar{1}4}, U_{\bar{2}3}, U_{\bar{3}4}]/(U_{\bar{1}4}U_{1\bar{2}} - U_{\bar{3}4}U_{\bar{2}3}),$$

with the map $U_{ij} \mapsto x_{|i|}^{\mathrm{sgn}(i)} x_{|j|}^{\mathrm{sgn}(j)} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where $\mathrm{sgn}(i) = 1$ and $\mathrm{sgn}(\bar{i}) = -1$, i.e.

$$\begin{aligned} U_{12} &\mapsto x_1 x_2 \\ U_{1\bar{2}} &\mapsto x_1 x_2^{-1} \\ U_{\bar{1}4} &\mapsto x_1^{-1} x_4 \\ U_{\bar{2}3} &\mapsto x_2^{-1} x_3 \\ U_{\bar{3}4} &\mapsto x_3^{-1} x_4. \end{aligned}$$

Suppose the polynomial ring is $\mathbb{Z}^4$-graded with

$$\begin{aligned} \deg U_{12} &= (1, 1, 0, 0) \\ \deg U_{1\bar{2}} &= (1, -1, 0, 0) \\ \deg U_{\bar{1}4} &= (-1, 0, 0, 1) \\ \deg U_{\bar{2}3} &= (0, -1, 1, 0) \\ \deg U_{\bar{3}4} &= (0, 0, -1, 1). \end{aligned}$$

Then

$$\mathrm{Hilb}(R_{P^\vee}^{\mathrm{rt}}, \boldsymbol{x}) = \frac{1 - x_2^{-1} x_4}{(1 - x_1 x_2)(1 - x_1 x_2^{-1})(1 - x_1^{-1} x_4)(1 - x_2^{-1} x_3)(1 - x_3^{-1} x_4)}.$$

Looking at the numerator, one sees that it corresponds to $1 - x_{|\max(\rho)|}^{-\mathrm{sgn}(\max(\rho))} x_{|\min(\rho)|}^{\mathrm{sgn}(\min(\rho))}$, and the terms of the denominator correspond to (pairs of) covering relations in the poset of Figure 1.2, paralleling Theorem 1.1.3. This is Theorem 4.2.12, and Section 4.2.4 will explain the computation of Ψ for certain signed posets in types B and C.

As in type A, the weight cone semigroup will allow one to calculate Φ, but there is a wrinkle. Consider the poset shown in Figure 1.3. The weight cone semigroup ring R_P^{wt} is

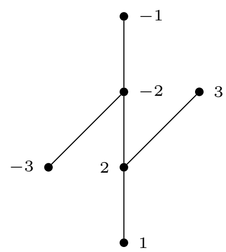


Figure 1.3: A signed poset

presented as

$$R_P^{\mathrm{wt}} \cong k[U_1, U_{12}U_{123}, U_{\bar{3}}]/(U_{123}U_{\bar{3}} - U_{12}),$$

via the map $U_J \mapsto \boldsymbol{x}^J = \prod_{j \in J} x_{|j|}^{\mathrm{sgn}(j)} \in k[x_1^{\pm 1}, \ldots, x_n^{\pm n}]$, i.e.

$$\begin{aligned} U_1 &\mapsto x_1 \\ U_{12} &\mapsto x_1 x_2 \\ U_{123} &\mapsto x_1 x_2 x_3 \\ U_{\bar{3}} &\mapsto x_3^{-1}. \end{aligned}$$

When the polynomial ring is graded by $\mathbb{Z}^3$ with

$$\begin{aligned} \deg U_1 &= (1, 0, 0) \\ \deg U_{12} &= (1, 1, 0) \\ \deg U_{123} &= (1, 1, 1) \\ \deg U_{\bar{3}} &= (0, 0, -1), \end{aligned}$$

the Hilbert series is

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, \boldsymbol{x}) = \frac{1 - x_1 x_2}{(1 - x_1)(1 - x_1 x_2)(1 - x_1 x_2 x_3)(1 - x_3^{-1})}.$$

The ideals of the signed poset are $\{1\}, \{1,2\}, \{1,2,3\}, \{-3\}$ (see Section 3.2), corresponding to each term of the denominator of the Hilbert series. The numerator corresponds to the one pair of ideals which intersect nontrivially (to be defined in Section 5.2), $\{1,2,3\}$ and $\{-3\}$, which combine to form $\{1,2\}$. While this parallels Theorem 1.1.4, it turns out that the "correct" thing to do is to consider an initial ideal, as doing so allows one to migrate through various specializations of the grading—the toric ideal from Theorem 5.2.3 is not necessarily homogeneous in gradings other than the one by $\mathbb{Z}^n$.

Chapter 2 summarizes necessary background on cones, semigroups and root systems. Further background information will be introduced as needed. Chapter 3 reviews the definition of signed posets, as well as ideals, P-partitions and linear extensions of signed posets. Section 3.1 explains how signed posets can be represented as posets and oriented signed graphs. Fischer's representation of a signed poset $P \subset \Phi_{B_n}$ from [23] is modified for purposes of Chapter 4. Section 3.5 defines the root and weight cones of a signed poset and gives dual characterizations for when they are each pointed, full-dimensional and simplicial.

Chapters 4 and 5 proceed independently of one another. Chapter 4 discusses the root cone semigroup. Section 4.1 discusses the toric ideal of the semigroup associated to a signed graph (Theorem 4.1.5) and uses that result to describe generating sets for the toric ideals of the root cone semigroup, posets/digraphs and graphs. Section 4.2 describes a situation when R_P^{rt} is a complete intersection and Section 4.2.4 computes Ψ_P.

Chapter 5 discusses the weight cone semigroup in type B. Section 5.2.3 gives a presentation for the weight cone semigroup ring. Section 5.3 turns its attention to computing Φ_P and

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}.$$

This is again a question of complete intersections, but not of the weight cone semigroup ring. Instead, modding out by an initial ideal of the toric ideal preserves the Hilbert series, but allows a presentation involving an $\mathbb{N}$-graded homogeneous ideal. Signed posets with the property that modding out by the initial ideal of the toric ideal gives a complete intersection will be called *initial complete intersections.* Section 5.3.2 characterizes these signed posets as those avoiding certain induced subposets, while Section 5.3.3 explains how these posets can be constructed via any sequence of three moves.

Chapter 6 discusses a few loose ends: understanding two triangulations of the weight cone, the type C weight cone and characterizing when R_P^{rt} is a complete intersection.

1.3 Summary of the Main Results

- Theorem 4.1.5 giving generators for the toric ideal of affine semigroup rings associated to signed posets and, more generally, oriented signed graphs. These are the subrings of Laurent polynomials generated by monomials of the form $t_i^{\pm 1}, t_i^{\pm 2}, t_i^{\pm 1} t_j^{\pm 1}$. This result appears to be new and generalizes work of Boussicault, Féray, Lascoux, and Reiner [9], Gitler, Reyes, and Villarreal [26] and Villarreal [59].

- Theorem 4.2.12 showing that strongly planar signed posets P have rings R_P^{rt}, $R_{P^\vee}^{\mathrm{rt}}$ which are complete intersections, with Corollary 4.2.20 showing how to compute Ψ_P in this case.

- Theorem 5.2.3 giving a Gröbner basis for the toric ideal of R_P^{wt} in type B, generalizing Féray and Reiner [20, Proposition 6.4].

- Theorems 5.3.10 and 5.3.21 giving two characterizations (via forbidden subposets versus via inductive constructions) of the situation where this Gröbner basis gives a complete intersection presentation for its initial ideal, generalizing Féray and Reiner [20, Theorems 10.5, 10.6].

1.4 A remark on notation

- To save space and improve readability, negative numbers will sometimes be written as $\bar{1}$ rather than -1.
- The parentheses and brackets will sometimes be dropped from vectors and sets in figures, e.g. 100 instead of $(1, 0, 0)$.
- There will be many vertices of graphs that come in pairs $i, -i$ for $i \in [n]$ and polynomial rings where the variables are indexed by $[n]$. If v is a variable, understand x_v to mean the variable $x_{|v|}$.
- The characteristic vector of a set $J \subset \{\pm 1, \ldots, \pm n\}$ such that J does not contain both i and $-i$ for any i is the vector χ_J whose coordinates are defined by

$$(\chi_J)_i = \begin{cases} 1 & \text{if } i \in J \\ -1 & \text{if } -i \in J \\ 0 & \text{else} \end{cases}$$

- If $J \subset \{\pm 1, \ldots, \pm n\}$ is such that there is no i such that both $i, -i \in J$, then

$$\langle \boldsymbol{x}, J \rangle = \langle \boldsymbol{x}, \chi_J \rangle = \sum_i (\chi_J)_i x_i.$$

Chapter 2

Some Background

This chapter reviews requisite material on polyhedral cones, semigroups and root systems. Chapter 3 will fit these ideas together in the discussion of signed posets. Further background material will be introduced as needed.

2.1 Polyhedral Cones

Associated to a signed poset will be two polyhedral cones, the root cone and the weight cone. Consequently, this section reviews some basic facts about polyhedral cones. One can also refer to Fulton [25] for further information on polyhedral cones.

Definition 2.1.1. A *polyhedral cone* $K \subset \mathbb{R}^n$ is the intersection of finitely many half-spaces determined by hyperplanes $H_\alpha = \{x \colon \langle x, \alpha \rangle = 0\}$. The H_α are the *supporting hyperplanes* of K. Alternatively, a cone may be characterized as the positive span of some finite collection of vectors, W, with the positive span being denoted $\mathbb{R}_+W$. A set of *extreme rays* of a cone K are vectors comprising a set W, minimal with respect to inclusion, such that $K = \mathbb{R}_+W$.

There are a number of properties that can be used to describe a cone.

Definition 2.1.2. The *dimension* of a cone K, denoted $\dim K$ is the dimension of the vector space spanned by its extreme rays, and K is said to be *full-dimensional* if $K \subset \mathbb{R}^n$ and $\dim K = n$. A cone is said to be *pointed* when it does not contain a line. It is said to be *rational* with respect to a full-rank lattice $L \subset \mathbb{R}^n$ when the α determining the supporting hyperplanes lie in L. It is *simplicial* if the extreme rays are linearly independent. A (simplicial) cone is *unimodular* with respect to a lattice if the primitive vectors along the extreme rays form a basis of the lattice.

For example, consider the cone in Figure 2.1. The extreme rays are $(1,1,1)$, $(-1,1,1)$,

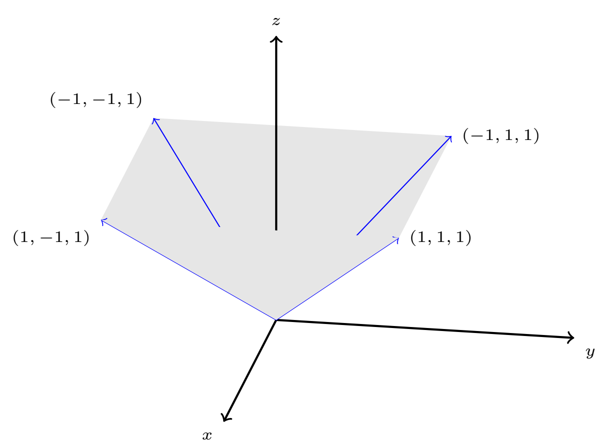


Figure 2.1: Cone spanned by $(1,1,1)$, $(-1,1,1)$, $(-1,-1,1)$ and $(1,-1,1)$

$(-1,-1,1)$ and $(1,-1,1)$ and the cone is pointed and rational. Since it is a cone in $\mathbb{R}^3$ with four extreme rays, it cannot be simplicial. Likewise, it cannot be unimodular with respect to the lattice $\mathbb{Z}^3$.

Definition 2.1.3. If K is a cone, its *dual* or *polar cone* is

$$K^* = \{x \in \mathbb{R}^n \colon \langle x, a \rangle \geq 0 \ \forall a \in K\}.$$

The dual is sometimes defined with a $\leq$ rather than a $\geq$, but the appeal of this choice of

$\geq$ will become clear in Section 3.5 and one can insert a minus sign where appropriate when the $\leq$ definition is more apt. The following facts about cones and their duals are well known.

- A cone is the dual of its dual, i.e. $K^{**} = K$.
- A cone K is full-dimensional if and only if its dual K^* is pointed.
- A cone K is simplicial if and only if its dual K^* is also simplicial.

2.1.1 Rational Functions and Cones

Recall that the goal is to prove some rational function identities for signed posets. A key step will be to understand the functions Ψ and Φ as valuations of cones.

In [3], Barvinok considers an exponential integral and exponential sum over a polyhedral cone, K:

$$\int_K e^{-\langle x,u\rangle}\, du \quad \text{and} \quad \sum_{K\cap\mathbb{Z}^n} e^{-\langle x,u\rangle},$$

where $x \in \mathbb{R}^n$ and du is Lebesgue measure on $\mathbb{R}^n$. Each gives a rational function, in x_i and $X_i = e^{x_i}$, respectively (see Propositions 2.1.4 and 2.2.8).

Proposition 2.1.4 ([3, Proposition 2.4])**.** *Let K be a pointed, full-dimensional polyhedral cone in $\mathbb{R}^n$. Then, for all $x \in \operatorname{Int} K^*$, the integral*

$$\int_K e^{-\langle x,u\rangle}\, du$$

exists and determines a function $s(K;x)$, which is rational in $x \in \mathbb{C}^n$.

Furthermore, if K is not pointed or if K is not full-dimensional, $s(K;x) = 0$.

For example, consider the cone in Figure 2.1 once again. To compute the integral, it is easiest to split K into two simplicial cones K_1 and K_2:

$$K_1 = \operatorname{span}_{\mathbb{R}_+}\{(1,-1,1),(-1,-1,1),(-1,1,1)\}$$

$$K_2 = \operatorname{span}_{\mathbb{R}_+}\{(-1,1,1),(1,-1,1),(1,1,1)\}.$$

Since K_1 and K_2 are both simplicial, every element of each cone can be written uniquely as a linear combination of the extreme rays. Then, if one denotes the extreme rays of K_1 by u_1, u_2, u_3 and the extreme rays of K_2 by v_1, v_2, v_3, for a fixed $x \in \operatorname{Int} K^*$, one can compute the integral as follows.

$$\int_K e^{-\langle x,u\rangle}\, du = \int_{K_1} e^{-\langle x,u\rangle}\, du + \int_{K_2} e^{-\langle x,u\rangle}\, du - \int_{K_1\cap K_2} e^{-\langle x,u\rangle}\, du.$$

The last integral is 0 since $K_1 \cap K_2$ is not full-dimensional. Then

$$\begin{aligned}
\int_{K_1} e^{-\langle x,u\rangle}\, du &= \int_0^\infty \int_0^\infty \int_0^\infty e^{-\langle x, a_1u_1+a_2u_2+a_3u_3\rangle}\, da_3 da_2 da_1 \\
&= \lim_{b_1,b_2,b_3\to\infty} \int_0^{b_1}\int_0^{b_2}\int_0^{b_3} e^{-\langle x,a_1u_1\rangle} e^{-\langle x,a_2u_2\rangle} e^{-\langle x,a_3u_3\rangle}\, da_3 da_2 da_1 \\
&= \lim_{b_1,b_2,b_3\to\infty} \int_0^{b_1} e^{-\langle x,a_1u_1\rangle} \int_0^{b_2} e^{-\langle x,a_2u_2\rangle} \int_0^{b_3} e^{-a_3\langle x,u_3\rangle}\, da_3 da_2 da_1 \\
&= \lim_{b_1,b_2,b_3\to\infty} \int_0^{b_1} e^{-\langle x,a_1u_1\rangle} \int_0^{b_2} e^{-\langle x,a_2u_2\rangle} \left[\frac{-1}{\langle x,u_3\rangle} e^{-a_3\langle x,u_3\rangle}\right]_0^{b_3} da_2 da_1 \\
&= \lim_{b_1,b_2,b_3\to\infty} \left[\frac{-1}{\langle x,u_1\rangle} e^{-a_1\langle x,u_1\rangle}\right]_0^{b_1} \left[\frac{-1}{\langle x,u_2\rangle} e^{-a_2\langle x,u_2\rangle}\right]_0^{b_2} \left[\frac{-1}{\langle x,u_3\rangle} e^{-a_3\langle x,u_3\rangle}\right]_0^{b_3}
\end{aligned}$$

Since $x \in \operatorname{Int} K^*$, one knows that $\langle x,u_1\rangle, \langle x,u_2\rangle, \langle x,u_3\rangle \geq 0$. Therefore,

$$\lim_{b_i\to\infty} e^{-b_i\langle x,u_i\rangle} = 0$$

for $i = 1,2,3$. Thus, one has

$$\int_{K_1} e^{-\langle x,u\rangle}\, du = \frac{1}{\langle x,u_1\rangle}\frac{1}{\langle x,u_2\rangle}\frac{1}{\langle x,u_3\rangle} = \frac{1}{(-x_1+x_2-x_3)}\frac{1}{(x_1+x_2-x_3)}\frac{1}{(x_1-x_2-x_3)}.$$

A similar calculation with K_2 gives

$$\int_{K_2} e^{-\langle x,u\rangle}\,du = \frac{1}{(x_1-x_2-x_3)}\frac{1}{(-x_1+x_2-x_3)}\frac{1}{(-x_1-x_2-x_3)}.$$

One then has

$$\begin{aligned}\int_{K} e^{-\langle x,u\rangle}\,du &= \frac{1}{(-x_1+x_2-x_3)}\frac{1}{(x_1+x_2-x_3)}\frac{1}{(x_1-x_2-x_3)}\\ &\quad+\frac{1}{(x_1-x_2-x_3)}\frac{1}{(-x_1+x_2-x_3)}\frac{1}{(-x_1-x_2-x_3)}\\ &= \frac{1}{(x_1-x_2-x_3)(-x_1+x_2-x_3)}\left(\frac{1}{-x_1-x_2-x_3}+\frac{1}{x_1+x_2-x_3}\right)\end{aligned}$$

Proposition 2.1.5 (Barvinok [3, (2.1)]). *Suppose K is a simplicial cone whose extreme rays are $\{u_1,\ldots,u_n\}$. Then*

$$s(K;\boldsymbol{x}) = |u_1\wedge\cdots\wedge u_n|\prod_{i=1}^{n}\langle \boldsymbol{x},u_i\rangle^{-1},$$

where $|u_1\wedge\cdots\wedge u_n|$ is the volume of the parallelopiped formed by the u_i.

Discussion of the exponential sum will be postponed until the Section 2.2.

2.2 Semigroups

The next important concept is that of the semigroup.

Definition 2.2.1. A *semigroup* is a set together with an associative binary operation. An *affine semigroup* is a semigroup which is isomorphic to a finitely-generated subsemigroup of $\mathbb{Z}^n$ under addition.

In general, unlike monoids, semigroups need not have an identity element. However, the semigroups of interest here will have an identity and the binary operation, $+$, will be commutative. The semigroups considered in Chapters 4 and 5 are affine semigroups as a consequence of Gordan's Lemma (see [19, Proposition 5.14]).

Theorem 2.2.2 (Gordan's Lemma)**.** *Suppose K is a rational polyhedral cone in $\mathbb{R}^n$ and A is a subgroup of $\mathbb{Z}^n$. Then $K \cap A$ is an affine semigroup.*

As an example, suppose K is the cone in Figure 2.2 whose extreme rays are $(1,0)$ and $(0,1)$ and suppose A is $\mathbb{Z}(1,1) \subset \mathbb{Z}^2$. Then $K \cap A = \{(i,i) \colon i \in \mathbb{Z}_{\geq 0}\}$.

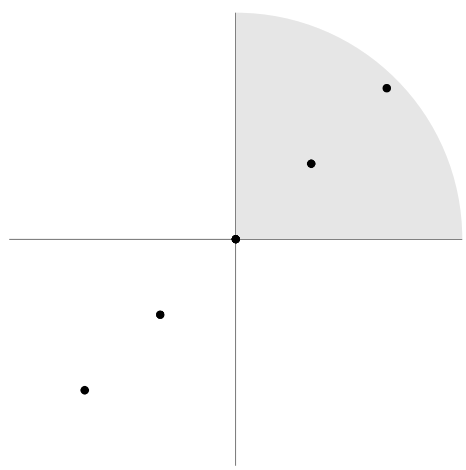


Figure 2.2: The cone spanned by $(1,0)$ and $(0,1)$ intersected with $\mathbb{Z}^2$

2.2.1 Semigroup Rings and Toric Ideals

One can move from the semigroup world to the somewhat more familiar world of rings by considering semigroup rings.

Definition 2.2.3. Suppose $A \subset \mathbb{Z}^n$ is an affine semigroup generated by $\{a_1, \ldots, a_m\}$. Let $L = k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ be the Laurent polynomial ring. The *semigroup ring* of A is the subring of L spanned by $\boldsymbol{t}^{a_i}$, $i = 1, \ldots, m$, where $\boldsymbol{t}^{a_i} = t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_n^{a_{in}}$ when $a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$. Denote the semigroup ring of A by $k[A]$.

It is often more convenient to work with a presentation of the semigroup ring as a quotient by the toric ideal than to view the semigroup ring as a subring of the Laurent polynomial ring.

Definition 2.2.4. Let $S = k[x_1, \ldots, x_m]$ be a polynomial ring in m variables and A a semigroup generated by $\{a_1, \ldots, a_m\}$. Define a map $\varphi \colon S \to L$ by

$$\varphi(x_i) = \boldsymbol{t}^{a_i}.$$

One then has that $k[A] \cong S/\ker\varphi$. The kernel of φ is the ideal known as the *toric ideal* of A, denoted I_A.

Definition 2.2.5. A *binomial* is a polynomial which is a difference of two monomials and a *binomial ideal* is an ideal that is generated by binomials.

Comprehensive discussion of binomial ideals can be found in Eisenbud and Sturmfels [18].

The following definition and notation is useful for understanding the generators of toric ideals.

Definition 2.2.6. If $u \in \mathbb{Z}^n$, define two vectors, the positive and negative supports of u as follows. The *positive support* u^+ is given by

$$u_i^+ = \begin{cases} u_i & \text{if } u_i > 0 \\ 0 & \text{else} \end{cases}$$

Likewise, the *negative support* u^- is

$$u_i^- = \begin{cases} u_i & \text{if } u_i < 0 \\ 0 & \text{else} \end{cases}$$

Then to each $u \in \mathbb{Z}^n$, one can associate a binomial $\boldsymbol{x}^{u^+} - \boldsymbol{x}^{u^-}$.

The following proposition is well known and the proof may be found in Sturmfels [57, Lemma 4.1] or Ene and Herzog [19, Lemma 5.2].

Proposition 2.2.7. *Suppose A is an affine semigroup and I_A its toric ideal. Then I_A is a binomial ideal generated by $\boldsymbol{x}^{u^+} - \boldsymbol{x}^{u^-}$ for $u \in \ker M$ where M is the matrix whose columns are the generators of A.*

As an example, consider the semigroup whose generators are $(1, -1, 1)$, $(-1, -1, 1)$, $(-1, 1, 1)$, $(1, 1, 1)$. (This is the semigroup obtained by intersecting the cone of Figure 2.1 with $\mathbb{Z}^3$.) The matrix M is then

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The kernel of M is one-dimensional and spanned by $(1, -1, 1, -1)^\top$. Let $S = k[x_1, x_2, x_3, x_4]$ and $\varphi\colon S \to k[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}]$ be defined by

$$\begin{aligned} \varphi(x_1) &= t_1 t_2^{-1} t_3 \\ \varphi(x_2) &= t_1^{-1} t_2^{-1} t_3 \\ \varphi(x_3) &= t_1^{-1} t_2 t_3 \\ \varphi(x_4) &= t_1 t_2 t_3 \end{aligned}$$

The kernel of φ is then the principal ideal $(x_1x_3 - x_2x_4)$. Certainly this ideal is contained in the kernel. One knows from Proposition 2.2.7 that $\ker\varphi = (\boldsymbol{x}^{u^+} - \boldsymbol{x}^{u^-})$ for $u = m(1, -1, 1, -1)^\top \in \ker M \cap \mathbb{N}^n$, with $m \in \mathbb{Z}_{>0}$, so one shows that $\boldsymbol{x}^{u^+} - \boldsymbol{x}^{u^-} \in (x_1x_3 - x_2x_4)$. This is straightforward:

$$x_1^m x_3^m - x_2^m x_4^m =$$

$$(x_1x_3 - x_2x_4)((x_1x_3)^{m-1} + (x_1x_3)^{m-2}x_2x_4 + \cdots + x_1x_3(x_2x_4)^{m-2} + (x_2x_4)^{m-1})$$

$$\in (x_1x_3 - x_2x_4),$$

so $(x_1x_3 - x_2x_4) = \ker\varphi$.

2.2.2 A Rational Function

The exponential sum

$$\sum_{K\cap\mathbb{Z}^n} e^{-\langle x,u\rangle}$$

discussed in [3] and mentioned in Section 2.1.1 is then a sum over the elements of an affine semigroup. The following proposition is the analogue of Proposition 2.1.4.

Proposition 2.2.8 (Barvinok [3, Proposition 4.4]). *Suppose K is a pointed rational polyhedral cone in $\mathbb{R}^n$. Then for $x \in \operatorname{Int} K^*$, the series*

$$\sum_{K\cap\mathbb{Z}^n} e^{-\langle x,u\rangle}$$

converges and determines a function $\sigma(K;x)$ which is rational in $X_i = e^{x_i}$, $i = 1,\dots,n$. Furthermore, there exists a representation

$$\sigma(K;x) = \frac{P(x)}{\prod_{i=1}^m (1 - e^{-\langle x,u_i\rangle})},$$

where $P(x)$ is a Laurent polynomial in X_i and the u_i are the extreme rays of K. If K is not pointed, $\sigma(K;x) = 0$.

Moreover, we can understand this sum as the Hilbert series of the semigroup ring.

Definition 2.2.9. Suppose k is a field and R is a finitely-generated k-algebra. Suppose further that R is graded by some index set I equipped with an addition, i.e. $R = \bigoplus_{\alpha\in I} R_\alpha$

with $R_\alpha R_\beta \subset R_{\alpha+\beta}$. Its *Hilbert series* is

$$\mathrm{Hilb}(R,x) = \sum_{\alpha\in I} \dim(R_\alpha) x^\alpha = \sum_{r\in R} x^{\deg r}.$$

A natural grading of the semigroup ring of an affine semigroup A is the $\mathbb{Z}^n$-grading where $\deg(\boldsymbol{x}^a) = (a_1,\ldots,a_n)$ for $a \in A$. Then one has that

$$\mathrm{Hilb}(k[A],\boldsymbol{x}) = \sum_{a\in A} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}.$$

Note that we are abusing notation here and using x_i in both the presentation of the semigroup ring and the Hilbert series, though these are actually different x_i.

Moreover, if one supposes that A arose from Gordan's Lemma as the intersection of a pointed rational cone K with $\mathbb{Z}^n$, one has that

$$\mathrm{Hilb}(k[A],\boldsymbol{x}) = \sum_{a\in A} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = \sum_{a\in K\cap\mathbb{Z}^n} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n},$$

and this last sum is precisely the sum $\sum_{K\cap\mathbb{Z}^n} e^{-\langle X,u\rangle}$ after the change of coordinates $x_i = e^{-X_i}$.

One can compute the Hilbert series of a graded ring from a minimal finite free resolution courtesy of the following well-known fact (see, for example, Stanley [50, Theorem I.11.3]).

Proposition 2.2.10. *Suppose M is a graded A-module and*

$$0 \to F_t \to F_{t-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

is a finite free resolution of M by graded A-modules. Then

$$\mathrm{Hilb}(M,x) = \sum_{i=0}^{t} (-1)^i \mathrm{Hilb}(F_i,x).$$

Consider the cone K having extreme rays $(1,0,0),(1,1,0),(1,0,1),(1,1,1)$. In fact, K is the weight cone of a poset on three elements, so [9, Proposition 7.1] gives that these vectors also (minimally) generate the semigroup $K \cap \mathbb{Z}^n$, which will be denoted by A. Let $S = k[a,b,c,d]$ with $\deg(a) = (1,0,0)$, $\deg(b) = (1,1,0)$, $\deg(c) = (1,0,1)$, $\deg(d) = (1,1,1)$. The semigroup ring $k[A]$ is isomorphic to $S/(bc - ad)$ (see [20, Proposition 6.4]). One then has the following (minimal) finite free resolution for S/I:

$$0 \to S(-(2,1,1)) \xrightarrow{\varphi} S \to S/I \to 0$$

where φ is the map sending 1 to $bc - ad$, and $S(-(2,1,1))$ is S with the grading shifted so that $1 \in S$ has degree $(2,1,1)$. One can then compute the Hilbert series of S/I (i.e. of $k[A]$) as

$$\begin{aligned}
\operatorname{Hilb}(S/I, \boldsymbol{x}) &= \operatorname{Hilb}(S, \boldsymbol{x}) - \operatorname{Hilb}(S(-(2,1,1)), \boldsymbol{x}) \\
&= \frac{1}{(1-x_1)(1-x_1x_3)(1-x_1x_2)(1-x_1x_2x_3)} \\
&\qquad - \frac{x_1^2x_2x_3}{(1-x_1)(1-x_1x_3)(1-x_1x_2)(1-x_1x_2x_3)} \\
&\qquad = \frac{1-x_1^2x_2x_3}{(1-x_1)(1-x_1x_3)(1-x_1x_2)(1-x_1x_2x_3)}
\end{aligned}$$

On the other hand, one can compute the exponential sum directly. Let K_1 be the cone whose extreme rays are $(1,0,0),(1,1,0),(1,0,1)$, K_2 the cone whose extreme rays are $(1,1,0),(1,0,1),(1,1,1)$. Then $K_1 \cap K_2$ is the cone whose extreme rays are $(1,1,0)$ and $(1,0,1)$. (See Figure 2.3.)

One then has

$$\sum_{K\cap\mathbb{Z}^n} e^{-\langle x,u\rangle} = \sum_{K_1\cap\mathbb{Z}^n} e^{-\langle x,u\rangle} + \sum_{K_2\cap\mathbb{Z}^n} e^{-\langle x,u\rangle} - \sum_{K_1\cap K_2\cap\mathbb{Z}^n} e^{-\langle x,u\rangle}.$$

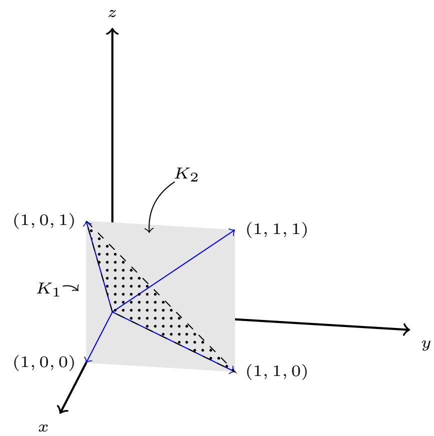


Figure 2.3: Cone spanned by $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(1, 1, 1)$

Making the change of variables so that $x_i = e^{-x_i}$, one has that

$$\begin{aligned}\sum_{K_1 \cap \mathbb{Z}^3} e^{-\langle x,u\rangle} &= \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty} e^{-\langle x, a_1(1,0,0)+a_2(1,1,0)+a_3(1,0,1)\rangle} \\ &= \sum_{a_1=0}^{\infty}\sum_{a_2=0}^{\infty}\sum_{a_3=0}^{\infty} x_1^{a_1}(x_1x_2)^{a_2}(x_1x_3)^{a_3} \\ &= \frac{1}{(1-x_1x_3)(1-x_1x_2)(1-x_1)}.\end{aligned}$$

Similarly,

$$\sum_{K_2 \cap \mathbb{Z}^3} e^{-\langle x,u\rangle} = \frac{1}{(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)}$$

and

$$\sum_{K_1 \cap K_2 \cap \mathbb{Z}^3} e^{-\langle c,x\rangle} = \frac{1}{(1-x_1x_2)(1-x_1x_3)}.$$

One then has that

$$\begin{aligned}\sum_{K \cap \mathbb{Z}^n} e^{-\langle x,u\rangle} = \frac{1}{(1-x_1x_3)(1-x_1x_2)(1-x_1)} &+ \frac{1}{(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)} \\ &- \frac{1}{(1-x_1x_2)(1-x_1x_3)}\end{aligned}$$

$$= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)},$$

matching the Hilbert series computation.

2.2.3 Complete Intersections

Part of the goal of Chapters 4 and 5 will be to identify signed posets for which the computation of the Hilbert series of (one or the other of) the relevant rings is particularly straightforward, namely when the rings are complete intersections.

Definition 2.2.11. Suppose R is a ring. A sequence of elements $\theta_1, \ldots, \theta_k \in R$ is said to be a *regular sequence* if θ_{i+1} is a non-zero divisor in $R/(\theta_1, \ldots, \theta_i)$ for $i = 0, \ldots, k-1$. A quotient ring R/I is said to be a *complete intersection* if I is generated by a regular sequence.

The following proposition simplifies the computation of the Hilbert series of a complete intersection.

Proposition 2.2.12. *Suppose R is a graded ring and θ is a non-zero divisor in R and* $\deg\theta \neq 0$. *Then*

$$\mathrm{Hilb}(R/(\theta), \boldsymbol{x}) = (1 - \boldsymbol{x}^{\deg\theta})\mathrm{Hilb}(R, \boldsymbol{x}).$$

Iterating this relation gives the following.

Corollary 2.2.13. Suppose R is a ring and $(\theta_1, \ldots, \theta_k)$ is a regular sequence. Then

$$\mathrm{Hilb}(R/(\theta_1, \ldots, \theta_k), \boldsymbol{x}) = \mathrm{Hilb}(R, \boldsymbol{x}) \prod_{i=1}^{k} (1 - \boldsymbol{x}^{\deg\theta_i}).$$

Boussicault, Féray, Lascoux and Reiner explain in Section 2.4 of [9] how the valuations $s(K;c)$ from Section 2.1.1 and $\sigma(K;c)$ from Section 2.2.2 are connected via a residue operation. This will be particularly relevant in the case where the semigroup ring is a complete intersection as a result of the following proposition.

Proposition 2.2.14. *Suppose L is a lattice and let K be a pointed L-rational cone for which $R = k[K \cap L]$ is a complete intersection with*

$$R \cong S/I = k[U_1, \ldots, U_k]/(\theta_1, \ldots, \theta_d),$$

where $\theta_1, \ldots, \theta_d$ are L-homogeneous elements of degrees $\delta_1, \ldots, \delta_d$ forming a regular sequence. Then

$$\mathrm{Hilb}(R; \boldsymbol{X}) = \sigma(K; \boldsymbol{X}) = \frac{\prod_{i=1}^{d}(1 - \boldsymbol{X}^{\delta_i})}{\prod_{j=1}^{d}(1 - \boldsymbol{X}^{u_j})},$$

and if K is full-dimensional,

$$s(K; x) = \frac{\prod \langle x, \delta_j \rangle}{\prod \langle x, u_j \rangle},$$

where $\deg U_j = u_j$.

Section 3.6 will show that the two rational functions Ψ and Φ on a signed poset can be understood as the valuation $s(-; x)$ on certain cones. Proposition 2.2.14 enables one to compute these rational functions in some cases by establishing that a certain ring is a complete intersection and finding a regular sequence generating the toric ideal.

As an example, consider the semigroup generated by

$$(1, 0, 0), (0, 0, -1), (1, 1, 0), (1, -1, -1), (1, 1, 1).$$

This will be one of the semigroups considered in Chapter 5. Let K be the cone spanned by the semigroup generators and $S = k[x_1, x_2, x_3, x_4, x_5]$ with $\deg x_1 = (1, 0, 0)$, $\deg x_2 = (0, 0, -1)$, $\deg x_3 = (1, 1, 0)$, $\deg x_4 = (1, -1, -1)$ and $\deg x_5 = (1, 1, 1)$. The toric ideal of the semigroup is, courtesy of Macaulay2, $I = (x_2 x_5 - x_3, x_4 x_5 - x_1^2)$, and one can check that the generators of the toric ideal form a regular sequence, so the semigroup ring $R \cong S/I$ is a complete intersection.

Then, from Corollary 2.2.13 (or the first half of Proposition 2.2.14, which is obtained by repeatedly applying the corollary), one has

$$\mathrm{Hilb}(R, \boldsymbol{x}) = \frac{(1 - x_1x_2)(1 - x_1^2)}{(1 - x_1)(1 - x_3^{-1})(1 - x_1x_2)(1 - x_1x_2^{-1}x_3^{-1})(1 - x_1x_2x_3)}$$

and

$$s(K, x) = \frac{(x_1 + x_2)(2x_1)}{x_1(-x_3)(x_1 + x_2)(x_1 - x_2 - x_3)(x_1 + x_2 + x_3)}.$$

2.3 Root Systems

To extend the notion of posets, one must first recall some definitions regarding root systems.

Definition 2.3.1. A *(crystallographic) root system* is a set $\Phi \subset \mathbb{R}^n$ such that

(a) Φ spans $\mathbb{R}^n$;

(b) if $\alpha \in \Phi$, then $-\alpha \in \Phi$ and $\pm\alpha$ are the only multiples of α in Φ;

(c) Φ is closed under the reflection σ_α across the hyperplane perpendicular to α for each $\alpha \in \Phi$;

(d) for all $\alpha, \beta \in \Phi$, one has
$$2\frac{\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z},$$
where $\langle -, - \rangle$ is the standard inner product on $\mathbb{R}^n$.

Condition (d) is known as the *crystallographic condition*. This condition gives the existence of root and weight lattices, which will be of use later.

Definition 2.3.2. A subset Δ of a root system Φ is a choice of *simple roots* if Δ spans $\mathbb{R}^n$ and partitions the root system into those roots lying in their positive integer span—the *positive roots*, denoted Φ^+—and those lying in their negative integer span—the *negative roots*, Φ^-.

The discussion of signed posets will focus on two root systems, Φ_{B_n} and Φ_{C_n}. The roots are

- $\{\pm e_i \pm e_j \colon i, j \in [n]\} \cup \{\pm e_i \colon i \in [n]\}$ for Φ_{B_n}, and
- $\{\pm e_i \pm e_j \colon i, j \in [n]\} \cup \{\pm 2e_i \colon i \in [n]\}$ for Φ_{C_n}.

The simple roots will be taken to be

- $\{+e_i - e_{i+1} \colon i = 1, \ldots, n-1\} \cup \{+e_n\}$ for Φ_{B_n}, and
- $\{+e_i - e_{i+1} \colon i = 1, \ldots, n-1\} \cup \{+2e_n\}$ for Φ_{C_n}.

This choice of simple roots gives the following positive roots.

- $\{+e_i + e_j \colon i, j \in [n]\} \cup \{+e_i - e_j \colon i < j\} \cup \{+e_i \colon i \in [n]\}$ for Φ_{B_n}, and
- $\{+e_i + e_j \colon i, j \in [n]\} \cup \{+e_i - e_j \colon i < j\} \cup \{+2e_i \colon i \in [n]\}$ for Φ_{C_n}.

Subsequent chapters will parallel work of Boussicault, Féray, Lascoux, and Reiner [9] and Féray and Reiner [20] addressing the type A case. The $\Phi_{A_{n-1}}$ roots are

$$\{e_i - e_j \colon i, j \in [n], i \neq j\},$$

and the choice of simple roots used is $\{e_i - e_{i+1} \colon i = 1, \ldots, n-1\}$, giving $\{e_i - e_j \colon i, j \in [n], i < j\}$ as the positive roots.

Definition 2.3.3. Given a root system Φ, the *(integral) weights* are the $\mu \in \mathbb{R}^n$ such that, for each $\alpha \in \Phi$,

$$2\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}.$$

If one has made a choice of simple roots, say $\{\alpha_1, \ldots, \alpha_n\}$, there is then a distinguished set of weights, the *fundamental dominant weights* $\mu_1, \ldots, \mu_n$ uniquely defined by the conditions

$$2\frac{\langle \mu_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

For Φ_{B_n}, the weights are the elements of $\mathbb{Z}^n$ plus $\left(\pm\frac{m}{2}, \pm\frac{m}{2}, \ldots, \pm\frac{m}{2}\right)$ for $m \in \mathbb{Z}$. For the choice of simple roots made above, the fundamental dominant weights are

$$(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1, 0), \left(\frac{1}{2}, \ldots, \frac{1}{2}\right).$$

For Φ_{C_n}, the weights are the elements of $\mathbb{Z}^n$ and, with the choice of simple roots made above, the fundamental dominant weights are $(1, 0, \ldots, 0)$,$(1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1, 0)$,$(1, \ldots, 1)$.

Definition 2.3.4. Suppose Φ is a root system. Its *dual root system*, $\Phi^\vee$, is the root system whose roots are

$$\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$$

for $\alpha \in \Phi$. The roots of $\Phi^\vee$ are the *coroots* of Φ and the weights of $\Phi^\vee$ are the *coweights* of Φ.

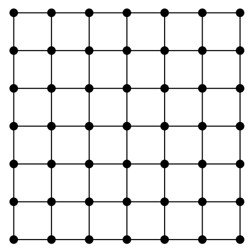
One always has that $\Phi^{\vee\vee} = \Phi$. Note that Φ_{B_n} and Φ_{C_n} are dual root systems. The roots, weights and coweights form lattices, called the *root*, *weight* and *coweight* lattices. As an example, consider the root and weight/coweight lattices for Φ_{B_2} and Φ_{C_2} in Figure 2.4.

2.3.1 The Weyl Group and Signed Permutations

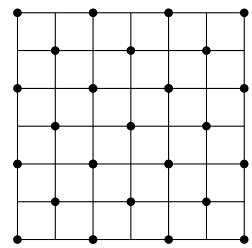
Definition 2.3.5. The reflections σ_α across the hyperplanes H_α perpendicular to $\alpha \in \Phi$ form a group called the *Weyl group* and denoted by W.

If one removes the hyperplanes H_α, one divides $\mathbb{R}^n$ into connected components which are open simplicial cones known as *Weyl chambers*. The Weyl group W acts simply transitively on the Weyl chambers.

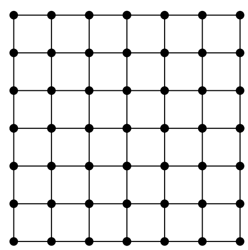
The Weyl group of $\Phi_{A_{n-1}}$ is the familiar symmetric group S_n. Since Φ_{B_n} and Φ_{C_n} are dual root systems they share the same reflections and thus the same Weyl group. This is the group of *signed permutations*, known as the *hyperoctahedral group*.



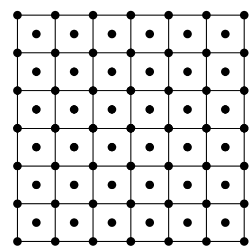
(a) The Φ_{B_2} root lattice



(b) The Φ_{C_2} root lattice



(c) The Φ_{B_2} coweight/Φ_{C_2} weight lattice



(d) The Φ_{B_2} weight/Φ_{C_2} coweight lattice

Figure 2.4: Root and Weight Lattices

Definition 2.3.6. A signed permutation of $[n]$ is a permutation w of $\{\pm 1, \ldots, \pm n\}$ such that $w(-i) = -w(i)$.

Consequently, signed permutations may be written in a two-line notation specifying the images of $\{1, \ldots, n\}$. For example,

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -3 & 4 & -1 \end{pmatrix} \tag{2.1}$$

is the signed permutation where $w(1) = 2$, $w(2) = -3$, $w(3) = 4$, $w(4) = -1$ and $w(-i) = -w(i)$ for $i = 1, \ldots, 4$.

The signed permutation analogues of a well-known permutation statistic will prove useful in Chapter 5. Name our choice of simple roots as $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots, n-1$ and $\alpha_n = e_n$.

Definition 2.3.7. A signed permutation w is said to have a *descent* at i if $w(\alpha_i) \in \Phi^-$. The set of descents of w is denoted $\mathrm{Des}(w)$. The *major index* of w is

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$$

For example, the signed permutation w shown in (2.1) has descents at 2 and 4, so $\mathrm{maj}(w) = 6$.

Chapter 3

Signed Posets

This chapter will discuss background on signed posets and introduce the root cone and weight cone.

Signed posets were introduced by Reiner in [46] and [47]. Before defining signed posets, it is useful to define one piece of notation which will reappear later.

Definition 3.0.1. Suppose Φ is a root system and $S \subset \Phi$. Then the *positive linear closure*, $\overline{S}^{PLC}$, is the set of $\alpha \in \Phi$ which are non-negative linear combinations of elements of S, i.e. $\overline{S}^{PLC} = \mathbb{R}_+ S \cap \Phi$.

Definition 3.0.2. If Φ is a root system, a Φ-*poset* is a subset $P \subset \Phi$ such that

(a) if $\alpha \in P$, then $-\alpha \notin P$

(b) $P = \overline{P}^{PLC}$.

A poset P can be understood as the $\Phi_{A_{n-1}}$-poset $\{e_i - e_j : i <_P j\}$. This view of posets reveals condition (a) in the definition as the analogue of antisymmetry and condition (b) as the analogue of transitivity.

The Φ-posets are the *parsets* of [46]. Since Φ_{B_n} and Φ_{C_n} are dual root systems, $P \subset \Phi_{B_n}$ is a Φ_{B_n}-poset if and only if $P^\vee = \{\alpha^\vee : \alpha \in P\} \subset \Phi_{C_n}$ is a Φ_{C_n}-poset. Signed posets were defined to be the Φ_{B_n}-posets, but in light of this duality, both Φ_{B_n}- and Φ_{C_n}-posets will be called *signed posets*, with Φ_{B_n} or Φ_{C_n} being specified when necessary.

One can check that $P = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\} \subset \Phi_{B_3}$ is a signed poset. In this case, $P^\vee = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +2e_1\} \subset \Phi_{C_3}$. On the other hand, $P \setminus \{+e_1\}$ is *not* a signed poset as it is not closed under positive linear combinations remaining in Φ_{B_3} since $\frac{1}{2}((e_1 - e_3) + (e_1 + e_3)) = e_1$.

Definition 3.0.3. Two signed posets P and P' are *isomorphic* if there is a signed permutation w such that $wP = P'$.

Note that, strictly speaking, the definition of Φ-poset allows a poset $P \subset \Phi \subset \mathbb{R}^m$ where P is supported on $\{e_i : i \in A\}$ for some $A \subsetneq [m]$. In this case, P is isomorphic to some $P' \subset \Phi' \subset \mathbb{R}^n$ for $|A| = n$.

3.1 Representing Signed Posets

There are two representations of signed posets, one as an oriented signed graph and the other as a poset on $\pm[n]$, that prove useful in different contexts. First, though, one needs to define oriented signed graphs.

Definition 3.1.1. A *signed graph* Σ is a pair (Γ, σ) where Γ is a graph with vertex set V and edge set E and σ is a map $\sigma : E \to \{\pm\}$, assigning a sign to each edge.

Definition 3.1.2. An *oriented signed graph* is a signed graph Σ together with a *bidirection*, τ, assigning signs to the incidences of Γ in such a way as to be compatible with σ, i.e. $\tau : I(\Gamma) \to \{\pm\}$ such that

$$\sigma(e) = -\tau(v, e)\tau(w, e)$$

when e is an edge between vertices v and w (v and w need not be distinct).

As an example, consider the signed graph Σ and an orientation of Σ by the bidirection τ in Figure 3.1. Notice that the bidirected edges of Σ correspond to elements of Φ_{C_n}: the edge (u, v) corresponds to $\tau(u)e_u + \tau(v)e_v$.

(a) The signed graph Σ (b) Σ oriented by τ

Figure 3.1: A signed graph and a bidirection

To construct the Hasse diagram of a signed poset, one first constructs an oriented signed graph as follows. If $P \subset \Phi_{B_n}$ (resp. $P \subset \Phi_{C_n}$), the vertices of the graph are $[n]$. The bidirected edges are as in Table 3.1. However, because signed posets are closed under positive

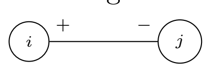
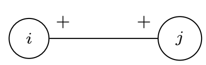
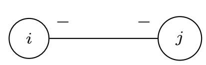
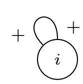
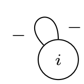
edge	poset element
	$e_i - e_j \in P$
	$e_i + e_j \in P$
	$-e_i - e_j \in P$
	$e_i \in P$, $2e_i \in P^\vee$
	$-e_i \in P$, $-2e_i \in P^\vee$

Table 3.1: Association between bidirected edges and elements of a signed poset

linear combinations, the presence of some elements is implied by others.

Definition 3.1.3. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Let Γ be the oriented signed graph whose edges are obtained from Table 3.1. Let Σ_P be the oriented

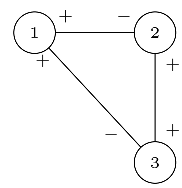


Figure 3.2: The Hasse diagram of $\{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$

signed graph obtained by removing the implied edges from Γ. Then Σ_P is the *Hasse diagram* of P.

It is not immediately obvious that the Hasse diagram should be well-defined. That it is well-defined is explained by Reiner [47, p. 329].

Consider the oriented signed graph in Figure 3.1(b). It corresponds to the signed poset $\{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$. The Hasse diagram is shown in Figure 3.2.

The other representation of a signed poset was introduced by Fischer [23] and then again independently by Ando, Fujishige, and Nemoto [2].

Definition 3.1.4. Suppose $P \subset \Phi_{B_n}$ and $P^\vee \subset \Phi_{C_n}$ are a pair of dual signed posets. Define a poset $\widehat{G}_B(P)$ on $[-n, n] = \{-n, \ldots, 1, 0, 1, \ldots, n\}$ by taking the transitive closure of the relations determined by Table 3.2. Similarly, define a poset $\widehat{G}_C(P^\vee)$ on $\pm[n] = \{\pm 1, \ldots, \pm n\}$ by taking the transitive closure of the relations determined by Table 3.3.

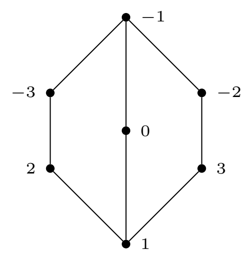
$i < j$	and	$-j < -i$	for	$+e_i - e_j \in P$
$i < -j$	and	$j < -i$	for	$+e_i + e_j \in P$
$-i < j$	and	$-j < i$	for	$-e_i - e_j \in P$
$i < 0$	and	$0 < -i$	for	$+e_i \in P$
$-i < 0$	and	$0 < i$	for	$-e_i \in P$

Table 3.2: Relations defining $\widehat{G}_B(P)$

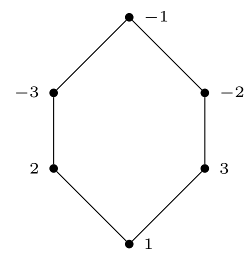
Understand $\widehat{G}(P)$ to mean either $\widehat{G}_B(P)$ or $\widehat{G}_C(P^\vee)$ as appropriate. In Chapter 5 it will be convenient to revert to Fischer's original definition and use $\widehat{G}_C(P^\vee)$ to represent a $P \subset \Phi_{B_n}$.

$i < j$	and	$-j < -i$	for	$+e_i - e_j \in P$
$i < -j$	and	$j < -i$	for	$+e_i + e_j \in P$
$-i < j$	and	$-j < i$	for	$-e_i - e_j \in P$
$\pm i < \mp i$			for	$\pm 2e_i \in P$

Table 3.3: Relations defining $\widehat{G}_C(P^\vee)$



(a) $\widehat{G}_B(P^\star)$



(b) $\widehat{G}_C(P^{\star\vee})$

Figure 3.3: $\widehat{G}_B(P)$ and $\widehat{G}_C(P)$ for $P^\star = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$

Proposition 3.2.3 shows that this abuse is acceptable. We will call $\widehat{G}(P)$ the *Fischer poset* of the signed poset. The Fischer posets for $P^\star = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$ are shown in Figure 3.3. (This poset is being named $P^\star$ since it will feature as an example in this section and the next.) Both Fischer and Ando, Fujishige and Nemoto defined $\widehat{G}_C(P)$ as being associated to $P \subset \Phi_{B_n}$. However, adding the 0 vertex in type B and defining $\widehat{G}_B(P)$ and $\widehat{G}_C(P)$ separately simplifies Chapter 4.

Both $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ are equipped with involutions, denoted ι, sending i to $-i$ and the edge $i \to j$ to $-j \to -i$, making ι a poset anti-automorphism. In $\widehat{G}_B(P)$, this involution fixes 0, and in $\widehat{G}_C(P^\vee)$ it fixes edges of the form $i \to -i$. That $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ are equipped with this involution means that they connect to the theory of coverings of signed graphs. This will be discussed in Section 4.1. Somewhat confusingly, the oriented signed graphs which they cover are *not* necessarily the Hasse diagram just described in Definition 3.1.3. The Hasse diagram will be useful in Section 3.5 and then dispensed with until Chapter 6, as the Fischer poset will prove to be the more convenient notion, lending itself to analogies to the type A case of [9] and [20].

Note that when P does not contain an element of the form $\pm e_i$, the element 0 is not comparable to any other elements of $\widehat{G}(P)$.

The following two propositions characterizing $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ are immediate consequences of the definitions.

Proposition 3.1.5. *A poset on $[-n, n]$ is $\widehat{G}_B(P)$ of a Φ_{B_n}-poset P if and only if the following two conditions hold:*

- *for all $i, j \in \{-n, \ldots, n\}$, if $i < j$ then $-j < -i$.*
- *for all $i \in \{-n, \ldots, n\}$, if $i < -i$ then $i < 0 < -i$.*

Proposition 3.1.6. *A poset on $\pm[n]$ is $\widehat{G}_C(P^\vee)$ for a Φ_{C_n}-poset $P^\vee$ if and only if the following two conditions hold:*

- *if $i < j$ then $-j < -i$ for all $i, j \in \pm[n]$ and*
- *if $i < -i$ and $j < -j$ then $i < -j$ and $j < -i$ for all $i, j \in \pm[n]$.*

Definition 3.1.7. A signed poset P' is an *induced subposet* of another signed poset P if $P' \subset P$ and there is an $A \subsetneq [n]$ such that $P' \subset P \cap \mathrm{span}_{\mathbb{R}}\{e_i \colon i \in A\}$.

Equally, P' is an induced subposet of P if $\widehat{G}(P')$ is an induced subposet of $\widehat{G}(P)$.

3.2 Ideals and P-partitions in Signed Posets

Recall that an order ideal of a poset P is a subset $I \subset P$ such that if $x \in I$ and $y < x$, then $y \in I$. On the other hand, when one views a poset P on $\{1, \ldots, n\}$ as a collection of roots in $\Phi_{A_{n-1}}$, the characteristic vectors of the ideals are precisely those $J \in \{1, 0\}^n$ such that $\langle J, \alpha \rangle \geq 0$ for all $\alpha \in P$. This view motivates Boussicault, Féray, Lascoux and Reiner's definition of the weight cone in [9].

In [47], Reiner defined an *ideal* of a signed poset $P \subset \Phi_{B_n}$ to be a $J \in \{1, -1, 0\}^n$ such that $\langle J, \alpha \rangle \geq 0$ for all $\alpha \in P$. The two key observations to generalize these definitions to Φ-posets are

- the projections of $\{1, 0\}^n$ to the hyperplane $\sum_{i=1}^n x_i = 0$ (i.e. the hyperplane of $\mathbb{R}^n$ in which $\Phi_{A_{n-1}}$ lives) form the orbit of the $\Phi_{A_{n-1}}$-fundamental coweights under the action of the Weyl group (recall that $\Phi_{A_{n-1}}$ is dual to itself, so its weights and coweights are the same), and

- $\{1, -1, 0\}^n$ is the orbit of the fundamental coweights of Φ_{B_n} under the action of the Weyl group.

Definition 3.2.1. Suppose $P \subset \Phi$ is a Φ-poset. An *ideal* of P is an f in the orbit of the fundamental weights under the action of the Weyl group such that $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in P$. The set of ideals of P is denoted $J(P)$, as for posets.

As an example, consider once again the poset $P^\star$ shown in Figure 3.3. The ideals of $P^\star$ are the elements of the poset shown in Figure 3.4. One will note that these are the characteristic vectors of some of the order ideals of $\widehat{G}(P)$.

Definition 3.2.2. Suppose $P \subset \Phi_{B_n}$ (resp. $P \subset \Phi_{C_n}$) is a signed poset. An order ideal, I, of $\widehat{G}(P)$ is said to be *isotropic* when the following two conditions hold:

- $0 \notin I$ (when $P \subset \Phi_{B_n}$)

- if $i \in I$, then $-i \notin I$ for $i \in \pm[n]$

Fischer and later Ando, Fujishige and Nemoto showed that the isotropic order ideals correspond to the ideals of a signed poset. It turns out that it is more convenient to look only at $\widehat{G}_C(P^\vee)$ when talking about the ideals. The correspondence between ideals of a signed poset and isotropic order ideals and the fact that examining $\widehat{G}_C(P^\vee)$ suffices is encapsulated in the following proposition.

Proposition 3.2.3. *Suppose $P \subset \Phi_{B_n}$ is a signed poset and $P^\vee \subset \Phi_{C_n}$ is the poset consisting of the corresponding dual roots. Then*

(a) the ideals of P are the characteristic vectors of the isotropic order ideals of $\widehat{G}_B(P)$,

(b) the ideals of $P^\vee$ are the characteristic vectors of the isotropic order ideals of $\widehat{G}_C(P^\vee)$ with the exception that the order ideals of size n correspond to one half times their characteristic vectors and

(c) the set of isotropic order ideals of $\widehat{G}_B(P)$ is the same as the set of isotropic order ideals of $\widehat{G}_C(P^\vee)$, and the set of connected isotropic order ideals of $\widehat{G}_B(P)$ is the same as the set of connected isotropic order ideals of $\widehat{G}_C(P^\vee)$.

In light of this view of the ideals of a signed poset, the vector of an ideal and the corresponding (isotropic) set of elements of $\pm[n]$ will be used interchangeably.

Proof. (a) Suppose J is an isotropic order ideal of $\widehat{G}_B(P)$. Suppose χ_J is not an ideal of f. Then there is an $\alpha \in P$ such that $\langle \chi_J, \alpha \rangle < 0$. For ease of notation, let $\epsilon = \chi_J$. Note that for all i, $\epsilon_i \in \{0, 1, -1\}$. There are five cases.

- **Suppose $\alpha = e_i - e_j$.** Then $\epsilon_i - \epsilon_j < 0$. If $\epsilon_i = 0$, then $\epsilon_j = 1$ so $j \in J$. However, one knows (from α) that $i < j$ in $\widehat{G}_B(P)$. If $\epsilon_i = -1$, then $\epsilon_j \geq 0$. Therefore, $-i \in J$. However, $-j < -i$ in $\widehat{G}_B(P)$, so $-j \in J$, contradicting that J was isotropic.

- **Suppose $\alpha = e_i + e_j$.** Then $\epsilon_i + \epsilon_j < 0$. Without loss of generality suppose $\epsilon_i = -1$, so $-i \in J$. Therefore, since $j < -i$ in $\widehat{G}_B(P)$, so $j \in J$, meaning $\epsilon_j = 1$, a contradiction since then $\epsilon_i + \epsilon_j = 0$.

- **Suppose $\alpha = +e_i$.** Then $\epsilon_i < 0$, so $-i \in J$. Since $i < -i$ in $\widehat{G}_B(P)$, one has $i \in J$, contradicting that J was isotropic.

- **Suppose** $\alpha = -e_i - e_j$**.** The symmetric argument to the case of $\alpha = e_i + e_j$ shows this is impossible.
- **Suppose** $\alpha = -e_i$**.** The symmetric argument to the case of $\alpha = e_i$ shows this is impossible.

Now suppose $f \in L_B^{\text{cowt}}$ is an ideal. By construction, the set in $\widehat{G}_B(P)$ corresponding to f, call it J, is isotropic. Suppose J is not an order ideal. Then there is $i, j \in [n]$ and $\delta, \epsilon \in \{\pm\}$ such that $\delta i \lessdot \epsilon j$ and $\epsilon j \in J$ and $\delta i \notin J$. In particular, $j \neq 0$. Then $\delta e_i - \epsilon j \in P$ and $\langle \delta e_i - \epsilon e_j, f \rangle < 0$, contradicting that f was an ideal.

(b) The argument for type B also works for type C.

(c) Suppose J is an isotropic order ideal in $\widehat{G}_B(P)$. Since J is isotropic, $0 \notin J$. Consider J as a subset of $\widehat{G}_C(P^\vee)$. Suppose it is not an order ideal. Then there is an $i \in \pm[n]$ and $j \in J$ such that $i < j$ and $i \notin J$. Without loss of generality, one may assume $i, j > 0$. Then, since $i < j$, one has that $e_i - e_j \in P^\vee$, meaning $e_i - e_j \in P$. However, since $i \notin J$, one must have $\langle e_i - e_j, \chi_J \rangle = -1$, contradicting that J is an ideal of P.

The argument when J is an isotropic order ideal in $\widehat{G}_C(P^\vee)$ is exactly the same, except one must also account for the possibility that $i = 0$. However, if $i = 0$, one must have $-j < j$ by Proposition 3.1.5.

□

Definition 3.2.4. An ideal of a signed poset $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) will be called *connected* if it corresponds to a connected isotropic order ideal in $\widehat{G}(P)$. Denote the set of connected order ideals by $J_{\text{conn}}(P)$.

See Figure 3.6 for an example of a signed poset where not every ideal is connected.

In [47], an order was defined on the ideals of a signed poset by extending componentwise the order $0 < 1, -1$. This order on the ideals corresponds to ordering the corresponding

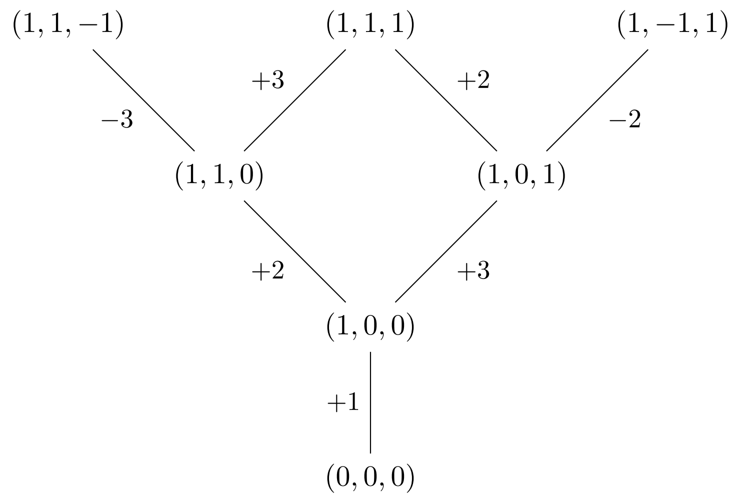


Figure 3.4: $J(P^\star)$ for $P^\star = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$ with the edges annotated with the difference between the ideals

isotropic order ideals by inclusion. As in the case of posets, this order gives a meet-semilattice of ideals, denoted, like the set of ideals, $J(P)$. Figure 3.4 shows $J(P)$ for $P = \{+e_1 - e_2, +e_1 - e_3, +e_2 + e_3, +e_1 + e_3, +e_1\}$.

The definition of P-partition for a signed poset is the analogue of that for a poset when a poset is viewed as an $\Phi_{A_{n-1}}$-poset. Recall that, from this perspective, a P-partition is an $f \in \mathbb{N}^n$ such that $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in P$. In [47], Reiner defined a P-partition for a signed poset $P \subset \Phi_{B_n}$ to be an $f \in \mathbb{Z}^n$ such that $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in P$.

One makes a similar pair of observations regarding the P-partitions as one did regarding ideals:

- the projections of $\mathbb{N}^n$ to the hyperplane $\sum_{i=0}^{n} x_i = 0$ in $\mathbb{R}^n$ are the $\Phi_{A_{n-1}}$ coweights, and
- the $\mathbb{Z}^n$ are the Φ_{B_n} coweights.

Definition 3.2.5. Suppose P is a Φ-poset. A P-partition is an f in the coweight lattice of Φ such that $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in P$. The set of P-partitions is denoted $\mathcal{A}(P)$.

3.3 Linear Extensions

Definition 3.3.1. A *linear extension* of a Φ-poset P is an element w of the Weyl group of Φ such that $P \subset w\Phi^+$. The set of linear extensions is denoted $\mathcal{L}(P)$.

In the case of signed posets, the Weyl group consists of the signed permutations, as explained in Section 2.3.

Definition 3.3.2. A signed poset is said to be *naturally labelled* if the identity signed permutation is a linear extension.

Note that since every signed poset has a linear extension (see the proof of [47, Theorem 3.3]), every signed poset is isomorphic to a naturally labeled signed poset. Thus, it is not deceptive that examples will usually be naturally labelled.

The linear extensions can be read off $J(P)$ by recording the difference between successive ideals in the maximal chains of $J(P)$. In the running example of $P^\star$, these differences were noted in Figure 3.4. One sees that $P^\star$ is naturally labelled and the linear extensions are

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & -3 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \end{pmatrix}$$

In [47], Reiner proved an analogue of Stanley's Fundamental Theorem of P-partitions for Φ-posets.

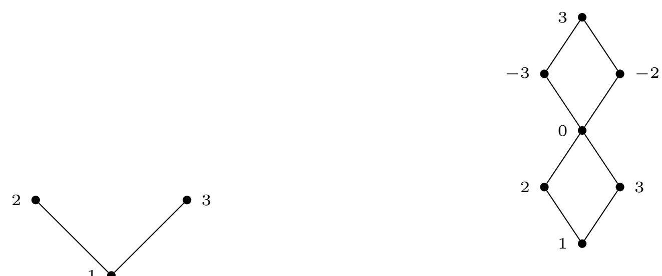
Proposition 3.3.3 ([47, Theorem 3.3])**.** *Suppose $P \subset \Phi_{B_n}$ is a signed poset. Then*

$$\mathcal{A}(P) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{A}(w\Phi^+).$$

3.4 Embedding Posets as Signed Posets

In [47], Reiner defined the embedding of a poset as a signed poset. Recall that a poset P on $[n]$ can be viewed as a $\Phi_{A_{n-1}}$-poset by taking $P = \{e_i - e_j : i <_P j\}$. The poset P is then embedded in Φ_{B_n} as the signed poset $P_B = P \cup \{+e_i : i \in [n]\} \cup \{+e_i + e_j : i, j \in [n]\}$. Then $P_C = (P \cup \{+e_i : i \in [n]\} \cup \{+e_i + e_j : i, j \in [n]\})^\vee = P \cup \{+2e_i : i \in [n]\} \cup \{+e_i + e_j : i, j \in [n]\}$. Call signed posets arising in this manner *type A signed posets*. The ideals of P_B are precisely the ideals of P. Likewise, P_B and P share the same linear extensions (in the sense that any linear extension of P_B and of P have the same action on $[n]$).

Proposition 3.4.1. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is such that $i < -i$ in $\widehat{G}(P)$ for all $i \in [n]$. Then P is a type A signed poset.*



(a) P

(b) $P \cup \{+e_1 + e_2, +e_1 + e_3, +e_2 + e_3, +e1, +e_2, +e_3\}$

Figure 3.5: The poset $P = \{e_1 - e_2, e_1 - e_3\}$ and its embedding as a signed poset

As an example, consider the posets shown in Figure 3.5. P is the intersection of $P \cup \{+e_1 + e_2, +e_1 + e_3, +e_2 + e_3, +e1, +e_2, +e_3\}$ with the hyperplane $\{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$. The ideals of P as a signed poset are precisely the ideals of P as a poset.

3.5 The Root and Weight Cones and Their Extreme Rays

Associated to a signed poset $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) are two polyhedral cones: the *root cone*, denoted K_P^{rt}, and the *weight cone*, denoted K_P^{wt}.

Definition 3.5.1. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Its *root cone* is the positive linear span of its elements: $K_P^{\mathrm{rt}} = \mathbb{R}_+ P$.

Its *weight cone* is the dual to the root cone: $K_P^{\mathrm{wt}} = \{f \in \mathbb{R}^n : \langle f, \alpha \rangle \geq 0 \ \forall \alpha \in P\}$.

Observe that if $P \subset \Phi_{B_n}$ and $P^\vee \subset \Phi_{C_n}$ are dual posets, P and $P^\vee$ share the same root and weight cones. The definition of Hasse diagram gives a characterization of the extreme rays of the root cone.

Proposition 3.5.2. *Suppose P is a signed poset. Its root cone is an affine polyhedral cone whose extreme rays are given by the edges of the Hasse diagram.*

Proposition 3.5.2 generalizes the type A result of Boussicault, Féray, Lascoux, and Reiner [9, Proposition 5.1(i)].

Proposition 3.5.3. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a type A signed poset. Let $P' = P \cap \Phi_{A_{n-1}}$ be the poset on $[n]$ which P embeds. Then*

$$K_{P'}^{\mathrm{rt}} = K_P^{\mathrm{rt}} \cap \{x \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 0\},$$

and furthermore, extreme rays of $K_{P'}^{\mathrm{rt}}$ are extreme rays of K_P^{rt}.

Proof. Inspecting the definitions, one sees immediately that $K_{P'}^{\mathrm{rt}} \subset K_P^{\mathrm{rt}} \cap \{x \in \mathbb{R}^n \colon x_1 + \cdots + x_n = 0\}$.

The extreme rays of $K_{P'}^{\mathrm{rt}}$ correspond to the covering relations of the Hasse diagram of P'. To show that each covering relation $\alpha \in P'$ is an extreme ray of K_P^{rt}, it suffices to check that there cannot exist $\beta, \gamma \in P$ such that $\gamma \in \Phi_{B_n}^+$ (resp. $\gamma \in \Phi_{C_n}^+$) and $\alpha = \beta + \gamma$. However, since $\alpha = e_i - e_j$, this is impossible. □

Characterizing the extreme rays of K_P^{wt} requires an additional definition and a lemma regarding ideals.

Definition 3.5.4. An ideal J of P is said to be *extensible* if there is a set $I \subset [n]$ such that $J \cup I$ and $J \cup -I$ are both ideals of P. An ideal which is not extensible is called *non-extensible.*

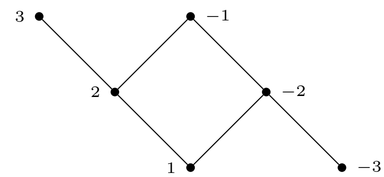


Figure 3.6: $\widehat{G}(P)$ for $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$

For example, in $P^\star$ (Figure 3.3), the ideal $\{1, 2\}$ is extensible because both $\{1, 2, 3\}$ and $\{1, 2, -3\}$ are ideals of P^*. However, it is not obligatory that $J \cup I$ and $J \cup -I$ be connected, nor that J be connected. If one considers instead $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$ (see Figure 3.6 for $\widehat{G}(P)$ and recall that when thinking about ideals, one can always refer to $\widehat{G}_C(P^\vee)$), the ideal $\{1, 2\}$ is extensible, as $\{1, 2, 3\}$ and $\{1, 2, -3\}$ are ideals, though $\{1, 2, -3\}$ is not connected. Similarly, $\{1, 3\}$ is extensible and disconnected.

Proposition 3.5.5. *Suppose P is a signed poset. Its weight cone K_P^{wt} is an affine polyhedral cone whose extreme rays correspond to the connected, non-extensible ideals of P.*

Proposition 3.5.5 generalizes [9, Proposition 5.1(ii)].

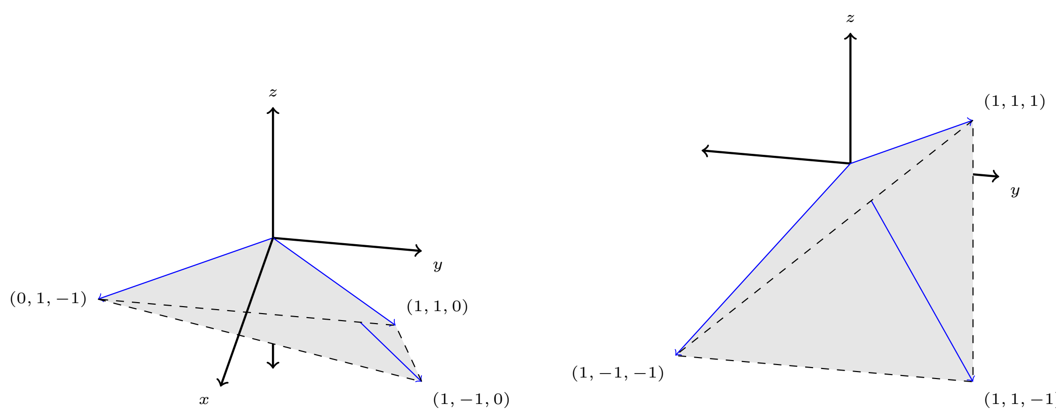
Proposition 3.5.6. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a type A signed poset. Let $P' = P \cap \Phi_{A_{n-1}}$. Then $K_P^{\mathrm{wt}} = K_{P'}^{\mathrm{wt}}$.*

Proof. It suffices to show that the ideals of P' are precisely the ideals of P. (If one is considering $P^\vee \subset \Phi_{C_n}$, let P be the dual poset.) Certainly ideals of P are ideals of P' since $P' \subset P$. Suppose f is an ideal of P'. meaning $\langle f, \alpha \rangle \geq 0$ for all $\alpha \in \Phi_{B_n}^+$. Since $P = P' \cup \Phi_{B_n}^+$, one then has that f is an ideal of P. Thus $K_P^{\mathrm{wt}} = K_{P'}^{\mathrm{wt}}$. □

Figure 3.7 shows the root and weight cones of $\{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$.

The proof of Proposition 3.5.5 requires the following lemma.

Lemma 3.5.7. *Suppose $P \subset \Phi_{B_n}$ is a signed poset and $f = (f_1, f_2, \ldots, f_n) \in K_P^{\mathrm{wt}}$. Let*



(a) K_P^{rt} (b) K_P^{wt}

Figure 3.7: The root cone and weight cone of $\{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_2 - e_3, +e_1\}$

$c_1 \leq c_2 \leq \cdots \leq c_k$ *be the* nonzero $|f_i|$ *arranged in increasing order. Then*

$$J_k = \{\mathrm{sgn}(f_i)i \colon i \in [n], |f_i| \geq c_k\}$$

is an ideal.

As an example, consider the signed poset in Figure 3.6. One has $c_1 = 1$, $c_2 = 2$ and $c_3 = 3$. Writing J_1, J_2, J_3 in a tower as shown in [47], one has

$$\begin{array}{cr} J_3 & (1, 0, \quad 0) \\ J_2 & (1, 1, \quad 0) \\ J_1 & +(1, 1, -1) \\ \hline f & (3, 2, -1) \end{array}$$

Definition 3.5.8. The ideal J_1 from Lemma 3.5.7 is the *signed support* of f, denoted $\mathrm{supp}_{\pm}(f)$.

Proof. Suppose J_k is not an ideal, so it is does not correspond to an order ideal in $\widehat{G}(P)$ (it is isotropic by construction). Then there is a $\pm k$ such that $\pm i \in J_k$ with $\pm k < \pm i$. Since

$\pm k \notin J_k$, $|f_i| \geq c_k > |f_k|$. One must consider several cases, one for each of the possible combinations of signs of $\pm i$ and $\pm k$.

- **Suppose** $+k < +i$**.** Then $e_k - e_i \in P$. Then $\langle f, e_k - e_i \rangle = f_k - f_i \leq |f_k| - f_i < 0$, a contradiction.

- **Suppose** $-k < +i$**.** Then $-e_k - e_i \in P$. Then $\langle f, -e_k - e_i \rangle = -f_k - f_i < 0$, since $0 < |f_k| < f_i$, a contradiction.

- **Suppose** $+k < -i$**.** Then $e_k + e_i \in P$. In this case, $f_i < 0$ and $|f_i| > |f_k|$. Then $\langle f, e_k + e_i \rangle = f_k + f_i = f_k - |f_i| < 0$, a contradiction.

- **Suppose** $-k < -i$**.** Then $e_i - e_k \in P$. Then $\langle f, e_i - e_k \rangle = f_i - f_k \leq -|f_i| + |f_k| < 0$, a contradiction.

Thus, J_k must be closed under going down in $\widehat{G}(P)$. Recall that, it is isotropic by construction, so J_k is an ideal of P. $\square$

Proof of Proposition 3.5.5. First, one shows that K_P^{wt} is spanned by the ideals of P, then that the connected ideals suffice to span K_P^{wt} and, finally, that the connected nonextensible ideals suffice.

Suppose $f = (f_1, \ldots, f_n) \in K_P^{\mathrm{wt}}$ and c_i and J_i are as in Lemma 3.5.7. Then

$$f = \sum_{k=1}^{n} (c_k - c_{k-1}) \chi_{J_k},$$

taking $c_0 = 0$. The J_k can be decomposed into their connected components, showing K_P^{wt} is spanned by the connected ideals of P. Since there are only finitely many ideals, the definition of extensibility means every extensible ideal can be written as a positive linear combination of connected nonextensible ideals. Thus, the nonextensible connected ideals span K_P^{wt}.

To show that the nonextensible connected ideals of P are the extreme rays of K_P^{wt}, one can show that every such ideal lies in the intersection of $n-1$ (linearly independent) hyperplanes, each supporting K_P^{wt}. Let J be a connected nonextensible ideal and consider a spanning tree T of its Hasse diagram (as a subposet of $\widehat{G}(P)$). Then J lies in the intersection of the following $n-1$ hyperplanes:

$$\begin{aligned} x_i &= 0 && \text{for } \pm i \notin J \\ x_i &= x_j && \text{for } +i \lessdot +j \text{ or } -j \lessdot -i \in T \ (+e_i - e_j \in P) \\ x_i &= -x_j && \text{for } +i \lessdot -j \text{ or } -i \lessdot +j \in T \ (\pm(e_i + e_j) \in P) \end{aligned}$$

The above could, *a priori*, fail to specify $n-1$ hyperplanes in one of two ways. First, it may specify both $x_i = x_j$ and $x_i = -x_j$ as hyperplanes, which really would specify only the hyperplane $x_i = 0$. However, this situation cannot arise since J is isotropic. The second possible problem is that a single hyperplane is specified by two different edges in T. Again, this is not possible since J is isotropic, so for each pair (i, j) there can only be one edge in T involving $\pm i$ and $\pm j$. Thus the above truly specifies $n-1$ linearly independent hyperplanes supporting K_P^{wt} such that J lies in their intersection, meaning J is an extreme ray of K_P^{wt}, as claimed. □

3.5.1 Dual Characterizations

Since the root cone and weight cone are dual to one another, one should have that a characterization of when K_P^{rt} is pointed gives a characterization of when K_P^{wt} is full-dimensional and vice versa. Equally, characterizing when K_P^{rt} is simplicial gives a characterization of when K_P^{wt} is simplicial and vice versa.

As a consequence of the antisymmetry condition in the definition of signed poset, K_P^{rt} is always pointed, meaning K_P^{wt} is always full-dimensional. One can also exhibit n linearly independent ideals of a signed poset P.

Lemma 3.5.9. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. For each $i \in [n]$, at least one of $I_{\leq}(i) = \{j \in \widehat{G}(P) \colon j \leq i\}$ and $I_{\leq}(-i) = \{j \in \widehat{G}(P) \colon j \leq -i\}$ is isotropic.*

This lemma introduces some notation that will be used again. In addition to $I_{\leq}(i)$, let $I_{<}(i) = \{i \in \widehat{G}(P)\colon j < i\}$.

Proof of Lemma 3.4.4. Suppose not and neither $I_{\leq}(i)$ nor $I_{\leq}(-i)$ is isotropic. Then there are j and k, not necessarily distinct, such that $j, -j < i$ and $k, -k < -i$. Then, since $j < i$, one has that $-i < -j$ and since $-j < -i$ one has that $-i < j$. Transitivity means that $-i < i$. On the other hand, since $k < -i$, one has that $i < -k$ and since $-k < -i$ one has that $i < k$. Then $i < -i$. It is impossible that both $i < -i$ and $-i < i$, so at least one of $I_{\leq}(i)$ and $I_{\leq}(-i)$ is isotropic. ☐

Proposition 3.5.10. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^{\vee} \subset \Phi_{C_n}$) is a signed poset. Then K_P^{wt} is full-dimensional.*

Proof. Fix a linear extension, $\prec$, of $\widehat{G}(P)$. For each i, pick an ideal J_i as follows:

$$J_i = \begin{cases} I_{\leq}(i) & \text{if } I_{\leq}(-i) \text{ is not isotropic} \\ I_{\leq}(-i) & \text{if } I_{\leq}(i) \text{ is not isotropic} \\ I_{\leq}(i) & \text{if both are isotropic and } i \prec -i \\ I_{\leq}(-i) & \text{if both are isotropic and } -i \prec i \end{cases}$$

The key observation is that J_i contains only elements of $\pm[n]$ that precede i in the $\prec$ order. Writing the J_i as the rows of a matrix (ordered by $\prec$), one has a lower triangular matrix with ± 1 on the diagonal, meaning the J_i are linearly independent. Since K_P^{wt} contains n linearly independent vectors, it must be full-dimensional. ☐

3.5.2 Matroids

In contrast to Proposition 3.5.10, the root cone is not always full-dimensional and the weight cone is not always pointed. Understanding when the root cone is full-dimensional, the weight cone is pointed and when both the root cone and weight cone are simplicial requires some facts from the theory of signed graphic matroids developed by Zaslavsky [62]. This section will review the two equivalent definitions of a matroid that will be useful.

Definition 3.5.11. A *matroid* is a pair $(E, \mathcal{I})$ where E is a finite set called the *ground set* and $\mathcal{I}$ is a collection of subsets called the *independent sets* of E such that

(a) $\mathcal{I}$ is nonempty.

(b) If $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$.

(c) If $I, J \in \mathcal{I}$ and $|I| = |J| + 1$, there is an $x \in I \setminus J$ such that $J \cup \{x\} \in \mathcal{I}$.

Subsets of E which are not independent are said to be *dependent*.

That the elements of $\mathcal{I}$ are called the independent sets is not a coincidence—these are the properties defining the collection of sets of linearly independent columns of a matrix.

Whitney introduced matroids in [60] and gave a number of equivalent definitions. Rather than defining a matroid as a pair $(E, \mathcal{I})$, one can define a matroid as a pair $(E, \mathcal{C})$, where $\mathcal{C}$ is the collection of *circuits*, a collection of subsets of E such that

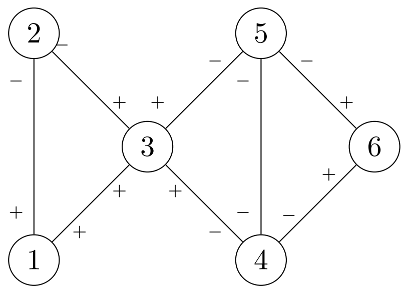
(a) If $I \in \mathcal{C}$ and J is a proper subset of I. Then $J \notin \mathcal{C}$.

(b) If $C_1, C_2 \in \mathcal{C}$, $x \in C_1 \cap C_2$ and $y \in C_1 \setminus C_2$, then there is a $C_3 \in \mathcal{C}$ with $y \in C_3$ and $x \notin C_3$.

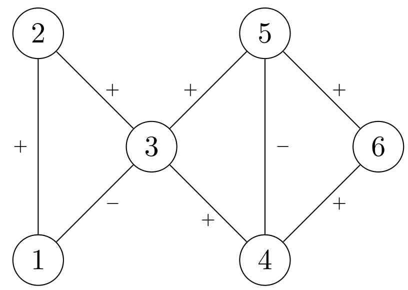
One can show that these two definitions are equivalent by taking the circuits to be the minimal dependent sets. (See Aigner [1, (6.13)].)

3.5.3 When the root cone is full-dimensional and simplicial

Characterization of when the root cone is full-dimensional as well as when it is simplicial relies on the notion of balance in a signed graph. (Signed graphs were discussed in Section 3.1.)



(a) Hasse diagram



(b) Associated signed graph

Figure 3.8: Hasse diagram of $P = \{+e_1 - e_2, +e_3 - e_2, +e_1 + e_3, e_3 - e_5, e_3 - e_4, -e_4 - e_5, +e_6 - e_4, +e_6 - e_5\}$ and associated signed graph

Recall that the Hasse diagram of a signed poset is an oriented signed graph. Denote the underlying signed graph by Σ_P.

Definition 3.5.12. A cycle in a signed graph is said to be *balanced* if it has an even number of edges whose sign is $-$. A cycle containing an odd number of edges with sign $-$ is said to be *unbalanced*. If all cycles in a signed graph are balanced, the graph itself is said to be balanced.

Like unsigned graphs, signed graphs have an associated matroid, introduced by Zaslavsky [62].

Definition 3.5.13. Suppose Σ is a signed graph. The *signed graphic matroid* $\Gamma(\Sigma)$ is the matroid whose circuits are the balanced cycles of Σ and pairs of unbalanced cycles joined by a (possibly empty) path.

Figure 3.8 shows the Hasse diagram of $P = \{+e_1 - e_2, +e_3 - e_2, +e_1 + e_3, e_3 - e_5, e_3 - e_4, -e_4 - e_5, +e_6 - e_4, +e_6 - e_5\}$ and the signed graph it orients, Σ_P. There is one balanced cycle $3-4-6-5$ and two pairs of unbalanced cycles joined by a path: $1-2-3$ and $3-4-5$ are joined by the empty path and $1-2-3$ and $4-5-6$ are joined either by $3-4$ or $3-5$.

On the other hand, any set of vectors can be used to define a matroid by taking the independent sets to be the linearly independent subsets, so the extreme rays of K_P^{rt}, i.e. the edges of the Hasse diagram of P, define a matroid, call it $\mathcal{M}_P$. Thus, one has two matroids arising from the edges of the Hasse diagram of P. However, it is a consequence of a result of Zaslavsky [62] that these two matroids are the same.

Zaslavsky also defines the *incidence matrix* $M(\Sigma)$ of a signed graph.

Definition 3.5.14. Suppose Σ is a signed graph. The *incidence matrix* of Σ is the matrix $M(\Sigma)$ whose columns M_e are indexed by the edges of Σ and

$$\begin{array}{ll} \text{if } e=(v,w) \text{ is an edge,} & m_{ve}=\pm 1 \text{ and } m_{we}=-\sigma(e)m_{ve}, \\ \text{if } e=(v,v) \text{ is a loop,} & m_{ve}=0 \text{ if } \sigma(e)=+ \text{ and } m_{ve}=\pm 2 \text{ if } \sigma(e)=-, \\ \text{if } e \text{ and } v \text{ not incident,} & m_{ve}=0. \end{array}$$

Note that these M_e are the elements of a signed poset $P^\vee \subset \Phi_{C_n}$ corresponding to the edges of the Hasse diagram. (Dividing m_{ve} by two for a loop in type B also does not alter the matroid.) For the poset P in Figure 3.8, one has

$$M(\Sigma_P) = \begin{array}{c} \begin{array}{cccccccc} \rotatebox{90}{$+e_1-e_2$} & \rotatebox{90}{$-e_2+e_3$} & \rotatebox{90}{$+e_1+e_3$} & \rotatebox{90}{$+e_3-e_5$} & \rotatebox{90}{$+e_3-e_4$} & \rotatebox{90}{$-e_4-e_5$} & \rotatebox{90}{$-e_4+e_6$} & \rotatebox{90}{$-e_5+e_6$} \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

That the matroid formed by the columns of $M(\Sigma_P)$ and the signed graphic matroid of Σ_P are the same is a consequence of the observation that the M_e correspond to elements of the signed poset and the following result of Zaslavsky.

Theorem 3.5.15 ([62, Theorem 8B.1])**.** *Let Σ be a signed graph and $f\colon E \to \mathbb{R}^n$ be the mapping $f(e) = M_e$. The matroid structure induced on E by the dependencies among the vectors $f(e)$ is precisely that of the signed graphic matroid $\Gamma(\Sigma)$.*

In other words, the circuits of $\Gamma(\Sigma)$, i.e. balanced cycles and pairs of unbalanced cycles joined by a path in Σ_P, correspond linearly dependent subsets of P which are minimal with respect to inclusion. This allows one to characterize when the root cone is full-dimensional and simplicial.

Proposition 3.5.16. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Its root cone is full-dimensional if and only if each connected component of Σ_P contains an unbalanced cycle.*

The proof requires an alternate characterization of balance due to Harary [32].

Theorem 3.5.17 ([32, Theorem 3])**.** *A signed graph is balanced if and only if its vertices can be partitioned into two sets V_+ and V_- such that every edge of the graph labeled $+$ joins two vertices both in either V_+ or V_- and every edge of the graph labeled $-$ joints a vertex in V_+ to one in V_-.*

Proof of Proposition 3.5.16. Suppose Σ_P contained a component K which is balanced. Then from the theorem of Harary, there is a partition of vertices of K into $V_+(K)$ and $V_-(K)$. Let $f = (f_1, \ldots, f_n) \in \mathbb{R}^n$ be defined by

$$f_i = \begin{cases} 1 & v_i \in V_+(K) \\ -1 & v_i \in V_-(K) \\ 0 & v_i \notin K \end{cases}$$

Recall that an edge of Σ labeled $+$ corresponded to $e_i - e_j$, meaning $f_i = f_j = \pm 1$, so $\langle f, e_i - e_j \rangle = 0$. Similarly, an edge of Σ labeled $-$ corresponds to $\pm(e_i + e_j)$, with $f_i = -f_j$, so $\langle f, \pm(e_i + e_j) \rangle = 0$. Then one has that $\langle f, \alpha \rangle = \langle -f, \alpha \rangle = 0$ for all $\alpha \in P$. In other words, K_P^{rt} lies in $f^\perp$, so cannot be full-dimensional.

In the other direction, suppose every connected component of Σ_P contains an unbalanced cycle. To show that K_P^{rt} is full-dimensional it suffices to show there are k edges corresponding to k linearly independent elements of P in each connected component of Σ_P with k vertices. Suppose K is a connected component of Σ_P with k vertices. A spanning tree of K has $k-1$ edges corresponding to $k-1$ linearly independent elements of P. Adding any other edge of K would create a cycle. Imagine that an edge e is added to K to create a cycle. The only way this edge could fail to correspond to a kth linearly independent element of P is if the edge "completes" both the unbalanced cycle and a balanced cycle. However, this would mean the first $k-1$ edges formed a cycle, which is impossible. Thus, K contains edges corresponding to k linearly independent elements of P, so Σ_P has n such edges, so K_P^{rt} is full-dimensional. □

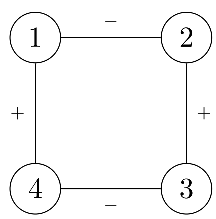
Note that the f found in the proof of Proposition 3.5.16 is an ideal such that $-f$ is also an ideal. In other words, the proof also shows that when K_P^{rt} is not full-dimensional, K_P^{wt} is not pointed.

Proposition 3.5.18. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. K_P^{wt} is pointed if and only if no connected component of $\widehat{G}(P)$ is isotropic.*

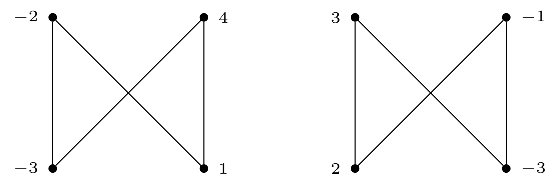
Proof. Suppose $J \subset \widehat{G}(P)$ is a connected component of $\widehat{G}(P)$ which is isotropic. Then $-J$ must also be a connected component of $\widehat{G}(P)$, which is isotropic and thus an ideal. Consequently, K_P^{wt} is not pointed.

In the other direction, suppose J and $-J$ are both isotropic ideals in $\widehat{G}(P)$, i.e. K_P^{wt} is not pointed. Suppose $i \in J$ (so $-i \in -J$) and $i < j$. Then, from the definition of $\widehat{G}(P)$, one must have $-j < -i$, meaning $-j \in -J$, so $j \in J$. Thus J is an order filter. An ideal which is also an order filter must be an entire connected component of the Hasse diagram, meaning $\widehat{G}(P)$ has a connected component which is isotropic, namely J. □

Proposition 3.5.19. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. If $\widehat{G}(P)$ contains an isotropic connected component, then Σ_P contains a balanced connected component.*



(a) Σ_P



(b) $\widehat{G}(P)$

Figure 3.9: Σ_P and $\widehat{G}(P)$ for $P = \{+e_1 + e_2, +e_2 - e_3, -e_3 - e_4, +e_1 - e_4\}$

Proof. Suppose $J \subset \widehat{G}(P)$ is an isotropic connected component. Then Σ_P has a connected component whose vertices are $|i|$ for $i \in J$, call it K. Then the vertices of K can be partitioned into V_+ and V_- by the following rule:

$$V_+ = \{i \colon i \in J\} \quad \text{and} \quad V_- = \{i \colon -i \in J\}.$$

Checking the definition of $\widehat{G}(P)$, one sees that edges labeled $+$ in Σ_P correspond to covering relations between two positive or two negative elements of $\widehat{G}(P)$ and edges labeled $-$ correspond to covering relations between one positive and one negative element in $\widehat{G}(P)$. In other words, the V_+ and V_- coming from J satisfy the requirements of Theorem 3.5.17, so K is a balanced component of Σ_P. $\square$

As an example, consider $P = \{+e_1 + e_2, +e_2 - e_3, -e_3 - e_4, +e_1 - e_4\}$, with Σ_P and $\widehat{G}(P)$ are shown in Figure 3.9. One sees that K_P^{rt} is not full-dimensional as Σ_P contains a balanced cycle. Additionally, one sees that $\widehat{G}(P)$ contains connected components which are isotropic. The corresponding partition of the vertices of Σ_P is $\{1,4\} \sqcup \{2,3\}$ and one sees that all elements of P lie in the hyperplane perpendicular to $(1,-1,-1,1)$, meaning both $(1,-1,-1,1)$ and $(-1,1,1,-1)$ are ideals, so K_P^{wt} is not pointed.

Proposition 3.5.20. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. The cones K_P^{rt} and K_P^{wt} are simplicial if and only if Σ_P does not contain a balanced cycle or two unbalanced cycles joined* *by a path.*

Proof. From Theorem 3.5.15, one has that K_P^{rt} is simplicial if and only if $\Gamma(\Sigma_P)$ contains no dependent sets, as dependent sets in $\Gamma(\Sigma_P)$ correspond precisely to dependent sets of extreme rays of K_P^{rt}. Therefore, K_P^{rt} is simplicial if and only if $\Gamma(\Sigma_P)$ contains no circuits, i.e. no balanced cycles and no pairs of unbalanced cycles joined by a path. One then has that this characterizes when K_P^{wt} is simplicial as well, since K_P^{rt} and K_P^{wt} are either both simplicial or both not simplicial as they are dual cones. $\square$

Corollary 3.5.21. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Its root cone, K_P^{rt}, and weight cone K_P^{wt} are pointed, full-dimensional and simplicial if and only if

- every connected component of Σ_P contains and unbalanced cycle, and
- Σ_P contains a balanced cycle nor two unbalanced cycles joined by a path.

3.6 Two Rational Functions

Recall from the introduction that associated to each signed poset is a pair of rational functions which are sums over the linear extensions. For $P \subset \Phi_{B_n}$, define

$$\Psi_P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)x_n}\right) \quad \text{and}$$
$$\Phi_P(\mathbf{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1 + x_2)\cdots(x_1 + x_2 + \cdots + x_n)}\right),$$

and for $P^\vee \subset \Phi_{C_n}$, define

$$\Psi_{P^\vee}^\vee(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)2x_n}\right) = \frac{1}{2}\Psi_P(\boldsymbol{x}) \quad \text{and} \tag{3.1}$$
$$\Phi_{P^\vee}^\vee(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{x_1(x_1 + x_2)\cdots(x_1 + \cdots + x_{n-1})(\frac{1}{2}x_1 + \frac{1}{2}x_2 + \cdots + \frac{1}{2}x_n)}\right) \tag{3.2}$$

$$= 2\Phi_P(\boldsymbol{x}).$$

These functions are the analogues of the Ψ and Φ considered in [30], [9] and [20] and discussed in Section 1.2 above. Note that when $P \subset \Phi_{B_n}$ is a type A signed poset and $P' = P \cap \Phi_{A_{n-1}}$ the corresponding poset, one has $\Phi_P(\boldsymbol{x}) = \Phi_{P'}(\boldsymbol{x})$, where Φ_P is calculated with the type B definition and $\Phi_{P'}$ is calculated with the type A definition, since P and P' have the same ideals by Proposition 3.5.6 and thus the same linear extensions.

As in [9], one can understand these functions as the valuation $s(-;\boldsymbol{x})$ from Section 2.1.1.

Theorem 3.6.1. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Then*

$$\begin{aligned}
\Psi_P(\boldsymbol{x}) = s(K_P^{\mathrm{rt}};\boldsymbol{x}) &= \int_{K_P^{\mathrm{rt}}} e^{-\langle \boldsymbol{x},u\rangle}\, du\\
\Psi^\vee_{P^\vee}(\boldsymbol{x}) = s^\vee(K_P^{\mathrm{rt}};\boldsymbol{x}) &= \int_{K_P^{\mathrm{rt}}} e^{-\langle \boldsymbol{x},u\rangle}\, \frac{1}{2}du\\
\Phi_P(\boldsymbol{x}) = s(K_P^{\mathrm{wt}};\boldsymbol{x}) &= \int_{K_P^{\mathrm{wt}}} e^{-\langle \boldsymbol{x},u\rangle}\, du\\
\Phi^\vee_{P^\vee}(\boldsymbol{x}) = s^\vee(K_P^{\mathrm{wt}};\boldsymbol{x}) &= \int_{K_P^{\mathrm{wt}}} e^{-\langle \boldsymbol{x},u\rangle}\, 2du
\end{aligned}$$

where du is Lebesgue measure.

The measure used for calculating $\Psi^\vee$ and $\Phi^\vee$ varies to give the parallelopiped spanned by the simple roots and fundamental dominant weights, respectively, volume one.

Proof. In all cases, the proof proceeds by induction on the number of pairs $\{i,j\} \subset \pm[n]$ such that i and j are incomparable in $\widehat{G}(P)$. The proof for Ψ_P and Φ_P is given here; the proofs for $\Psi^\vee_{P^\vee}$ and $\Phi^\vee_P$ follow (3.1) and (3.2). In the base case, suppose all pairs $\{i,j\}$ are comparable, i.e. $\widehat{G}(P)$ is a chain. Then $\mathcal{L}(P)$ must consist of a single signed permutation, call it w.

Applying Proposition 2.1.5 in each case, one has

$$s(K_P^{\mathrm{rt}};\boldsymbol{x}) = |w(\alpha_1)\wedge\cdots\wedge w(\alpha_n)|\prod_{i=1}^{n}\langle \boldsymbol{x}, w(\alpha_i)\rangle^{-1}$$
$$= w\left(\frac{1}{(x_1-x_2)\cdots(x_{n-1}-x_n)x_n}\right) = \Psi_P(\boldsymbol{x})$$
$$s(K_P^{\mathrm{wt}};\boldsymbol{x}) = |w(\mu_1)\wedge\cdots\wedge w(\mu_n)|\prod_{i=1}^{n}\langle \boldsymbol{x}, w(\mu_i)\rangle^{-1}$$
$$= w\left(\frac{1}{x_1(x_1+x_2)\cdots(x_1+\cdots+x_n)}\right) = \Phi_P(\boldsymbol{x})$$

where $|v_1\wedge\cdots\wedge v_n|$ is the volume of the parallelopiped they form.

For the induction step, suppose $\{i,j\}\subset\pm[n]$ are incomparable in $\widehat{G}(P)$. Let

$$P_{i<j} = \overline{P\cup\{\mathrm{sgn}(i)e_i - \mathrm{sgn}(j)e_j\}}^{PLC} \quad\text{and}\quad P_{j<i} = \overline{P\cup\{\mathrm{sgn}(j)e_j - \mathrm{sgn}(i)e_i\}}^{PLC}.$$

(If $i=j$, one needs to divide $\mathrm{sgn}(i)e_i-\mathrm{sgn}(j)e_j$ and $\mathrm{sgn}(j)e_j-\mathrm{sgn}(i)e_i$ by 2 in type B.) The only way $P_{i<j}$ could fail to be a signed poset is if $\mathrm{sgn}(j)e_j-\mathrm{sgn}(i)e_i\in P$, a contradiction since it would mean i and j were comparable in $\widehat{G}(P)$. A symmetric argument means $P_{j<i}$ is also a signed poset. Next, observe that, by construction,

$$\mathcal{L}(P) = \mathcal{L}(P_{i<j})\sqcup\mathcal{L}(P_{j<i}),$$

meaning

$$\Psi_P(\boldsymbol{x}) = \Psi_{P_{i<j}}(\boldsymbol{x}) + \Psi_{P_{j>i}}(\boldsymbol{x})$$
$$\Phi_P(\boldsymbol{x}) = \Phi_{P_{i<j}}(\boldsymbol{x}) + \Phi_{P_{j>i}}(\boldsymbol{x})$$

Recall that $s(-;\boldsymbol{x})$ is a valuation, so one wants to write K_P^{rt} and K_P^{wt} as a sum of $K_{P_{i<j}}^{\mathrm{rt}}$, $K_{P_{j<i}}^{\mathrm{rt}}$, $K_{P_{i<j}}^{\mathrm{wt}}$ and $K_{P_{j<i}}^{\mathrm{wt}}$ and apply the induction assumption to compute $s(-;\boldsymbol{x})$ for each of these cones and use that to compute $s(K_P^{\mathrm{rt}};\boldsymbol{x})$ and $s(K_P^{\mathrm{wt}};\boldsymbol{x})$. To that end, define a set $P_{i=j} = \overline{P\cup\{\mathrm{sgn}(i)e_i-\mathrm{sgn}(j)e_j, \mathrm{sgn}(e_j)e_j-\mathrm{sgn}(i)e_i\}}^{PLC}$. The set $P_{i=j}$ is of course, not a

signed poset, but the definitions of root cone and weight cone still make sense.

Next, observe that

$$\begin{aligned} K_P^{\mathrm{rt}} &= K_{P_{i<j}}^{\mathrm{rt}} \cap K_{P_{j<i}}^{\mathrm{rt}} \\ K_{P_{i=j}}^{\mathrm{rt}} &= K_{P_{i<j}}^{\mathrm{rt}} \cup K_{P_{j<i}}^{\mathrm{rt}} \\ K_{P_{i=j}}^{\mathrm{wt}} &= K_{P_{i<j}}^{\mathrm{wt}} \cap K_{P_{j<i}}^{\mathrm{wt}} \\ K_P^{\mathrm{wt}} &= K_{P_{i<j}}^{\mathrm{wt}} \cup K_{P_{j<i}}^{\mathrm{wt}}. \end{aligned}$$

One then has by the valuative property that

$$\begin{aligned} s(K_{P_{i=j}}^{\mathrm{rt}}; \boldsymbol{x}) &= s(K_{P_{i<j}}^{\mathrm{rt}}; \boldsymbol{x}) + s(K_{P_{j<i}}^{\mathrm{rt}}; \boldsymbol{x}) - s(K_P^{\mathrm{rt}}; \boldsymbol{x}) \\ s(K_P^{\mathrm{wt}}; \boldsymbol{x}) &= s(K_{P_{i<j}}^{\mathrm{wt}}; \boldsymbol{x}) + s(K_{P_{j<i}}^{\mathrm{wt}}; \boldsymbol{x}) - s(K_{P_{i=j}}^{\mathrm{wt}}; \boldsymbol{x}) \end{aligned}$$

Since $K_{P_{i=j}}^{\mathrm{rt}}$ is not pointed and $K_{P_{i=j}}^{\mathrm{wt}}$ is not full-dimensional,

$$s(K_{P_{i=j}}^{\mathrm{rt}}; \boldsymbol{x}) = s(K_{P_{i=j}}^{\mathrm{wt}}; \boldsymbol{x}) = 0.$$

Then, applying the induction assumption one has

$$\begin{aligned} s(K_P^{\mathrm{rt}}; \boldsymbol{x}) &= \Psi_{P_{i<j}}(\boldsymbol{x}) + \Psi_{P_{j<i}}(\boldsymbol{x}) = \Psi_P(\boldsymbol{x}) \\ s(K_P^{\mathrm{wt}}; \boldsymbol{x}) &= \Phi_{P_{i<j}}(\boldsymbol{x}) + \Phi_{P_{j<i}}(\boldsymbol{x}) = \Phi_P(\boldsymbol{x}) \end{aligned}$$

completing the proof. $\square$

Together with Proposition 2.2.14, one now has reason to believe that the cone and semigroup perspective of [9] and [20] will bear fruit in the signed poset case.

Corollary 3.6.2. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Both Ψ_P and Φ_P (resp. $\Psi_{P^\vee}^\vee$ and $\Phi_{P^\vee}^\vee$) vanish when $\widehat{G}(P)$ has an isotropic component or, equivalently, Σ_P

has a balanced component.

Corollary 3.6.3. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset such that K_P^{rt} and K_P^{wt} are pointed, full-dimensional and simplicial. Then

$$\Psi_P(\boldsymbol{x}) = \frac{1}{2}\Psi^\vee_{P^\vee}(\boldsymbol{x}) \overset{(1)}{=} |u_1 \wedge \cdots \wedge u_n| \prod_{i=1}^{n} \langle \boldsymbol{x}, u_u \rangle^{-1} \overset{(3)}{=} 2^k \prod_{i=1}^{n} \langle \boldsymbol{x}, u_u \rangle^{-1} \text{ and}$$
$$\Phi_P(\boldsymbol{x}) = 2\Psi^\vee_{P^\vee}(\boldsymbol{x}) \overset{(2)}{=} |J_1 \wedge \cdots \wedge J_n| \prod_{i=1}^{n} \langle \boldsymbol{x}, J_i \rangle^{-1} \overset{(4)}{=} 2^{n-k} \prod_{i=1}^{n} \langle \boldsymbol{x}, J_i \rangle^{-1},$$

where $u_1, \ldots, u_n$ are the extreme rays of K_P^{rt}, $J_1, \ldots, J_n$ are the extreme rays of K_P^{wt} and k is the number of (non-loop) cycles in the Hasse diagram of P.

Proof. Equalities (1) and (2) are consequences of Corollary 3.5.21. Equality (3) is the result of combining Theorem 3.6.1 with [62, Lemma 8A.2]. Recalling that K_P^{rt} and K_P^{wt} are dual cones and the definition of coroot (see Definitions 2.3.3 and 2.3.4), one sees that

$$(u_i)(J_i)^\top = 2I_n,$$

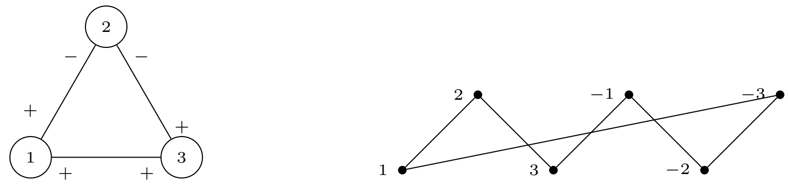
since $\langle u_i, J_k \rangle = 2\delta_{ik}$ (for some indexing of the u_i and J_k). Therefore,

$$|J_1 \wedge \cdots \wedge J_n| = \frac{2^n}{|u_1 \wedge \cdots \wedge u_n|},$$

giving equality (4). □

Corollary 3.6.3 is an analogue of [9, Proposition 3.2(1)], which shows that for a poset, the numerator of Ψ_P was 1 for a tree and 0 for a forest, covering the two cases where K_P^{rt} is simplicial, full-dimensional and pointed. However, instead of being an indicator of connectedness, it counts the number of cycles in the Hasse diagram.

For example, consider the poset P whose Hasse diagram and Fischer poset are shown in Figure 3.10. The extreme rays of K_P^{rt} are $(1, -1, 0), (0, -1, 1), (1, 0, 1)$ and the extreme rays



(a) Hasse diagram (b) $\widehat{G}(P)$

Figure 3.10: Hasse diagram and $\widehat{G}(P)$ for $P = \{e_1 + e_2, e_1 - e_2, e_1 + e_3, e_1, e_3)\}$

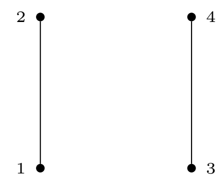
of K_P^{wt} are $(1,1,1), (1,-1,-1), (-1,-1,1)$. Applying Proposition 2.1.5, one has that

$$\Psi_P(\boldsymbol{x}) = \left|\det\begin{pmatrix} 1 & 0 & 1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\right| \frac{1}{(x_1 - x_2)(x_3 - x_2)(x_1 + x_3)} = \frac{2}{(x_1 - x_2)(x_3 - x_2)(x_1 + x_3)},$$

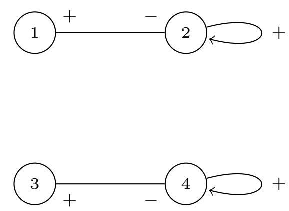
and

$$\Phi_P(\boldsymbol{x}) = \left|\det\begin{pmatrix} 1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\right| \frac{1}{(x_1 + x_2 + x_3)(x_1 - x_2 - x_3)(-x_1 - x_2 + x_3)}$$
$$= \frac{4}{(x_1 + x_2 + x_3)(x_1 - x_2 - x_3)(-x_1 - x_2 + x_3)}.$$

On the other hand, the second half of Corollary 3.6.3 is a generalization of Theorem 1.1.2, which computed Φ_P for forests. For example, consider the forest F in Figure 3.11(a) and its embedding as a signed poset in Figure 3.11(b). Embedding a poset as a signed poset preserves its ideals and linear extensions and therefore preserves Φ. In particular, this means that $\det(J_i)$, where J_i runs over the connected non-extensible ideals (i.e. all connected ideals



(a) A forest F



(b) F embedded as a signed poset, F_B

Figure 3.11: A forest and its embedding as a signed poset

in this case) will be ± 1, so

$$\Phi_F(\boldsymbol{x}) = \Phi_{F_B}(\boldsymbol{x}) = \left| \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \right| \frac{1}{x_1 x_3 (x_1 + x_2)(x_3 + x_4)} = \prod_{j \in F} \frac{1}{\sum_{i <_F j} x_i}.$$

Chapter 4

The Root Cone Semigroup

This chapter considers $K_P^{\mathrm{rt}} \cap L^{\mathrm{rt}}$, the semigroup associated to the root cone in types B and C. After discussion of the generators, Section 4.1 will consider the toric ideal of an oriented signed graph, but in a more general setting from which one can recover the signed poset, digraph and poset and graph cases. Section 4.2 will give a sufficient condition for the semigroup ring to be a complete intersection, enabling the computation of $\Psi_P(\boldsymbol{x})$ and $\Psi^{\vee}_{P^{\vee}}(\boldsymbol{x})$ via Proposition 2.2.14.

Recall that $\widehat{G}(P)$ is equipped with an involution, ι, sending i to $-i$. This involution will make repeated appearances throughout this chapter.

Proposition 4.0.1. *The semigroups $K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$ and $K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}$ are generated by the elements of P and $P^{\vee}$ corresponding to orbits of edges of $\widehat{G}_B(P)$ and $\widehat{G}_C(P^{\vee})$, respectively, under the involution.*

Proof. In both cases, it suffices to show that every element of P and $P^{\vee}$ lies in the semigroup generated by the orbits of edges of $\widehat{G}_B(P)$ and $\widehat{G}_C(P^{\vee})$. First, suppose $\pm e_i \pm e_j \in P$ with $i \neq j$. It is also an element of $P^{\vee}$. Then, by definition, $\pm i < \mp j$ in both $\widehat{G}_B(P)$ and $\widehat{G}_C(P^{\vee})$. Summing the elements corresponding to edges (see Table 3.1) along a chain from i to $-j$ gives $\pm e_i \pm e_j$ in each case. Next, suppose $\pm e_i \in P$. Then $\pm 2e_i \in P^{\vee}$. By definition, $\pm i < 0 < \mp i$

in $\widehat{G}_B(P)$. Summing the elements corresponding to edges along a chain from $\pm i$ to 0 gives $\pm e_i$. On the other hand, since $\pm 2e_i \in P^\vee$, by definition one has $\pm i < \mp i$ in $\widehat{G}_C(P^\vee)$ and adding the elements corresponding to edges along a chain from $\pm i$ to $\mp i$ gives $\pm 2e_i$. $\square$

This generating set is in fact minimal. All edges of $\widehat{G}_B(P)$ correspond to edges of the Hasse diagram of P, which in turn correspond to the extreme rays of the root cone. It may be that $\widehat{G}_C(P^\vee)$ has "extra" edges relative to the Hasse diagram, namely $\delta i \to -\epsilon j$ and $\epsilon j \to -\delta i$, should the Hasse diagram contain loops at i and j. However, $\delta e_i + \epsilon e_j$ is *not* in the semigroup generated by $\delta 2e_i$ and $\epsilon 2e_j$, so the extra edge in the Fischer poset really does correspond to a semigroup generator. Proposition 4.0.1 generalizes (via the embedding from Section 3.4) the result in type A (Propositions 5.1 and 7.1 of [9]), where $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}}$ was generated by the roots corresponding to the covering relations of a poset.

4.1 The Toric Ideal of an Oriented Signed Graph

Recall from Section 3.1 that one can view a signed poset as an oriented signed graph. This section will consider the generators of the toric ideal associated to an oriented signed graph and then use that viewpoint to understand the generators of the toric ideals associated to posets, directed graphs, graphs and signed posets. In other words, we will describe the toric ideals for *all* affine semigroup rings in $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ generated by monomials of the form $1, t_i^{\pm 1}, t_i^{\pm 2}, t_i^{\pm 1} t_j^{\pm 1}$.

Suppose Σ is a signed graph and τ a bidirection of Σ. (Here Σ is allowed to have multiple edges as well as self-loops and half edges.) The vectors $\tau(i,e)e_i + \tau(j,e)e_j$ as $e = (i,j)$ runs over the edges of Σ (with a half-edge $e = (i, -)$ giving $\tau(i,e)e_i$) generate a semigroup contained in $\mathbb{Z}^{|V(\Sigma)|}$.

Definition 4.1.1. Suppose Σ is a signed graph and τ a bidirection of Σ. Define a polynomial ring $S_\Sigma = k[U_e]$ where e runs over the (bidirected) edges of Σ. The *toric ideal* of Σ is the

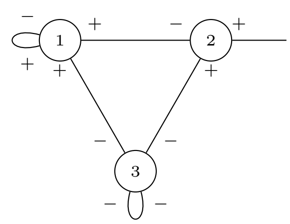


Figure 4.1: An oriented signed graph

kernel of the map $\varphi\colon k[U_e] \to k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ defined by

$$\varphi(U_e) = \begin{cases} t_i^{\tau(i,e)} t_j^{\tau(j,e)} & e = (i,j), \quad i \neq j \\ t_i^{\tau(i,e)} & e = (i,-) \text{ a half edge} \\ t_i^{2\tau(i,e)} & e = (i,i) \text{ a negative loop} \\ 1 & e = (i,i) \text{ a positive loop} \end{cases}$$

The convention will be to put a bar over negative coordinates to save space.

As an example, consider the oriented signed graph in Figure 4.1. In this case,

$$S_\Sigma = k[U_1, U_2, U_{\bar{3}}, U_{1\bar{2}}, U_{2\bar{3}}, U_{1\bar{3}}]$$

and φ is defined by

$$U_1 \mapsto 1$$
$$U_2 \mapsto t_2$$
$$U_{\bar{3}} \mapsto t_3^{-2}$$
$$U_{1\bar{2}} \mapsto t_1 t_2^{-1}$$
$$U_{2\bar{3}} \mapsto t_2 t_3^{-1}$$
$$U_{1\bar{3}} \mapsto t_1 t_3^{-1}$$

```
i1 : loadPackage "FourTiTwo";
A = transpose matrix
   {{1,0,0},{0,1,0},{0,0,-1},{1,-1,0},{0,1,-1},{1,0,-1}};
S = QQ[U1,U2,Um3,U1m2,U2m3,U1m3,Degrees=>entries transpose A];
I = toricGroebner(A,S)

             3        6
o2 : Matrix ZZ  <--- ZZ

i4 : using temporary file name /tmp/M2-22805-3
-------------------------------------------------
4ti2 version 1.3.2, Copyright (C) 2006 4ti2 team.
4ti2 comes with ABSOLUTELY NO WARRANTY.
This is free software, and you are welcome
to redistribute it under certain conditions.
For details, see the file COPYING.
-------------------------------------------------
Using 64 bit integers.
4ti2 Total Time:  0.00 secs.

o4 = ideal (- U1*Um3 + U1m3, - U1 + U2*U1m2, - U2*Um3 + U2m3)

o4 : Ideal of S
```

Figure 4.2: Computing a toric ideal with Macaulay2

The toric ideal I_Σ is then $(U_{1\bar{2}}U_{2\bar{3}} - U_{1\bar{3}}, U_2^2U_{\bar{3}} - U_{2\bar{3}}^2, U_1 - 1)$. (See Figure 4.2 for an example of the computation with Macaulay2 [29] and 4ti2 [24].)

Understanding the generators of the toric ideal of an oriented signed graphs requires the notion of the signed covering of an oriented signed graph.

Definition 4.1.2. Given a signed graph $\Sigma = (\Gamma, \sigma)$ oriented by τ, define a directed graph $\widetilde{\Sigma}$ whose vertices are $V(\Sigma) \times \{\pm\}$ and whose edges are determined by the edges of Σ according to Table 4.1. The graph $\widetilde{\Sigma}$ is the *signed covering* of Σ.

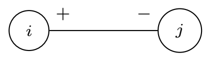
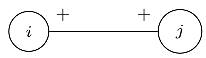
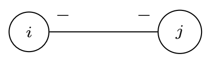
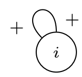
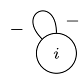
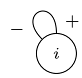
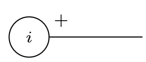
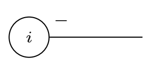
edge in Σ	edge(s) in $\widetilde{\Sigma}$
	$i \to j,\ -j \to -i$
	$i \to -j,\ j \to -i$
	$-i \to j,\ -j \to i$
	$i \to -i$
	$-i \to i$
	self-loops at i, $-i$
	half edges $i \to$, $\to -i$
	half edges $-i \to$, $\to i$

Table 4.1: Correspondence between edges in Σ and $\widetilde{\Sigma}$

When $\widetilde{\Sigma}$ contains half edges, it will be convenient to imagine an extra vertex 0 as the other end of the half edges. With this extra vertex, when $P \subset \Phi_{B_n}$ is a signed poset, $\widehat{G}_B(P)$ is the signed cover of a signed graph. $\widehat{G}_C(P^\vee)$ is also the signed cover of a signed graph, with or without the dummy 0, as there are no half edges. (See Definition 3.1.4 for the definitions of $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$.) Like $\widehat{G}(P)$, a signed covering $\widetilde{\Sigma}$ is equipped with an involutive digraph anti-automorphism sending i to $-i$ (and fixing 0). Figure 4.3 shows the signed covering of the oriented signed graph in Figure 4.1.

Definition 4.1.3. Suppose Σ is a signed graph and $\widetilde{\Sigma}$ is its signed covering. Consider a cycle C in $\widetilde{\Sigma}$ and orient it in some way (i.e. choose a direction in which to traverse C). This partitions the edges of C into $W \sqcup A$, where W is the set of edges such that the orientation is consistent with the direction of the edge in $\widetilde{\Sigma}$ and A consists of the edges oriented opposite their direction in $\widetilde{\Sigma}$. Say C is *fixed orientation-wise* by the involution if $\iota(C) = C$ as edges and $W_C = W_{\iota(C)}$ and $A_C = A_{\iota(C)}$.

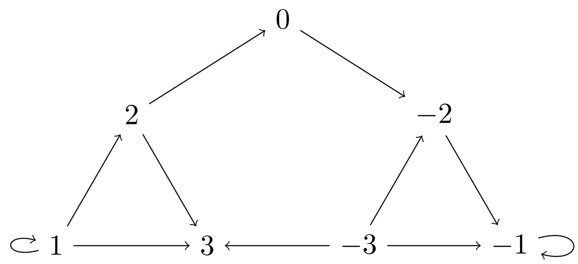


Figure 4.3: Signed covering of the oriented signed graph in Figure 4.1

For example, consider the cycle $3 \to 1 \to 2 \to 0 \to -2 \to -1 \to -3 \to 3$ in the graph in Figure 4.3. Then

$$W_C = \{(1,2),(2,0),(0,-2),(-2,-1),(-3,3)\} \quad \text{and} \quad A_C = \{(3,1),(-1,-3)\}.$$

While this cycle is fixed *edgewise* by the involution, $\iota(C)$ is oriented in the opposite direction: $-3 \to -1 \to -2 \to 0 \to 2 \to 1 \to 3 \to -3$. Thus $W_{\iota(C)} = A_C$ and $A_{\iota(C)} = W_C$, so C is *not* fixed orientation-wise. On the other hand, consider the signed covering in Figure 4.4 (which is actually $\widehat{G}_C(P^\vee)$ for $P^\vee = \{+e_1 - e_2, +e_1 + e_2, +2e_1\}$). It has a single cycle $1 \to 2 \to -1 \to -2 \to 1$, which *is* fixed orientation-wise by the involution, with $W_C = W_{\iota(C)} = \{(1,2),(2,-1)\}$ and $A_C = A_{\iota(C)} = \{(1,-2),(-2,-1)\}$.

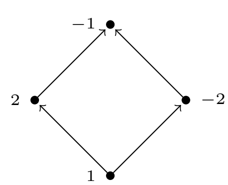


Figure 4.4: Another signed covering

Lemma 4.1.4. *Suppose $\widetilde{\Sigma}$ is the signed covering of an oriented signed graph Σ. Suppose C is a cycle in $\widetilde{\Sigma}$ fixed orientation-wise by the involution. Then if $e \in W_C$ is an edge, $\iota(e) \in A_C$.*

Proof. Suppose $e \in W_C$ and $\iota(e) \in W_C$. There are two cases, when e involves 0 and when it doesn't. First, suppose e does not have 0 as an endpoint. Then orienting C, one has

$$\cdots \to i \xrightarrow{e} j \to \cdots \to -j \xrightarrow{\iota(e)} -i \to \cdots$$

Applying the involution gives

$$\cdots \to -i \to -j \to \cdots \to j \to i \to \cdots$$

so $e, \iota(e) \in A_{\iota(C)}$, meaning C was not fixed orientation-wise by the involution.

On the other hand, suppose e does have 0 as an endpoint. Without loss of generality, one may assume one has

$$\cdots \to i \xrightarrow{e} 0 \xrightarrow{\iota(e)} -i \to \cdots$$

Applying the involution gives

$$\cdots \to -i \to 0 \to i \to \cdots,$$

so $e, \iota(e) \in A_{\iota(C)}$, meaning C was not fixed orientation-wise by the involution. □

If C is a cycle in $\widetilde{\Sigma}$, it gives rise to a relation in the semigroup:

$$\sum_{\alpha \in W(C)} \alpha = \sum_{\alpha \in A(C)} \alpha. \tag{4.1}$$

Recall that φ was the map from from $S_\Sigma = k[U_e]$ to the Laurent polynomial ring sending U_{ij} to $t_i^{\tau(i,e)} t_j^{\tau(j,e)}$ for the edge $e = (i, j)$ (see Definition 4.1.1).

Theorem 4.1.5. *Suppose Σ is an oriented signed graph. The toric ideal $I_\Sigma = \ker \varphi$ is generated by* cycle binomials

$$U(C) = \prod_{e \in W(C)} U_e - \prod_{e \in A(C)} U_e,$$

where C runs over the cycles of $\widetilde{\Sigma}$ not fixed by the involution.

Consider the oriented signed graph in Figure 4.1, whose signed covering is in Figure 4.3. Recall from the start of this section that the toric ideal is

$$I_\Sigma = (U_{1\bar{2}}U_{2\bar{3}} - U_{1\bar{3}}, U_2^2U_{\bar{3}} - U_{2\bar{3}}^2, U_1 - 1).$$

The signed covering $\widetilde{\Sigma}$ has five orbits of cycles not fixed orientation-wise by the involution. Their cycle binomials are:

$$U_1 - 1, \quad U_{1\bar{2}}U_{2\bar{3}} - U_{1\bar{3}}, \quad U_{1\bar{2}}^2U_2^2U_{\bar{3}} - U_{1\bar{3}}^2, \quad U_2^2U_{\bar{3}} - U_{2\bar{3}}^2, \quad U_{1\bar{2}}U_2^2U_{\bar{3}} - U_{2\bar{3}}U_{1\bar{3}}.$$

One needs to check that these five binomials generate I_Σ and not some larger ideal. One can use Macaulay2 to find the following relations which show that the five cycle binomials do generate I_Σ by writing the two "extra" generators in terms of the other three generators:

$$\begin{aligned} U_{2\bar{3}}(U_{1\bar{2}}U_{2\bar{3}} - U_{1\bar{3}}) + U_{1\bar{2}}(U_{\bar{3}}U_2^2 - U_{2\bar{3}}^2) &= U_{1\bar{2}}U_2^2U_{\bar{3}} - U_{2\bar{3}}U_{1\bar{3}} \\ (-U_{1\bar{2}}U_{2\bar{3}} - U_{1\bar{3}})(U_{1\bar{3}} - U_{1\bar{2}}U_{2\bar{3}}) + U_{1\bar{2}}^2(U_{\bar{3}}U_2^2 - U_{2\bar{3}}^2) &= U_{1\bar{2}}^2U_{\bar{3}}U_2^2 - U_{1\bar{3}}^2. \end{aligned}$$

Proposition 4.1.6. *Suppose C is a cycle in $\widetilde{\Sigma}$ fixed orientation-wise by the involution. Then its cycle binomial is zero.*

Proof. A priori, one has that $\iota(W_C) = A_{\iota(C)}$ and $\iota(A_C) = W_{\iota(C)}$. If a cycle C is fixed by the involution, one has that $W_C = W_{\iota(C)}$ by definition, but this means $W_{\iota(C)} = \iota(A_{\iota(C)})$ so the two sums in (4.1) are over the same set, meaning the cycle binomial is 0. $\square$

It will occasionally be convenient to think of *all* cycle binomials as generating the toric ideal. This is acceptable since the proposition shows the extra cycle binomials are 0.

Proof of Theorem 4.1.5. Toric ideals are generated by binomials $U^\alpha - U^\beta$ such that α and β have disjoint support. (See Sturmfels [57, Corollary 4.3].) One must then show that if such a $U^\alpha - U^\beta$ is in I_Σ, it is in the ideal generated by the cycle binomials of $\widetilde{\Sigma}$.

Suppose $U^\alpha - U^\beta$ is in I_Σ and U^α and U^β have disjoint support. The binomial corresponds to a relation

$$\sum a_{\delta i,\epsilon j}(\delta e_i - \epsilon e_j) = \sum b_{\delta i \epsilon j}(\delta e_i - \epsilon e_j),$$

where $\delta, \epsilon \in \{\pm\}$ and, for ease of notation, one takes $e_0 = 0$. One builds a digraph on vertex set $\pm[n]$ as follows. For each term $\delta e_i - \epsilon e_j$ in the left hand sum, take $a_{\delta i,\epsilon j}$ copies of the directed edge $\delta i \to \epsilon j$ and $a_{\delta i,\epsilon j}$ copies of $-\epsilon j \to -\delta i$. For each term $\delta e_i - \epsilon e_j$ in the right hand sum, take $b_{\delta i,\epsilon j}$ copies of $\delta i \leftarrow \epsilon j$ and $b_{\delta i,\epsilon j}$ copies of $-\epsilon j \leftarrow -\delta i$. In other words, one gets $a_{\delta i,\epsilon j}$ edges with orientation coinciding with that of $\widetilde{\Sigma}$ and $b_{\delta i,\epsilon j}$ edges with orientation opposite that of $\widetilde{\Sigma}$.

One has constructed a digraph where the in-degree equals the out-degree at each vertex, so one has a collection of cycles, possibly with multiplicity. By construction, each edge and its image under the involution are oriented the same way relative to their direction in $\widetilde{\Sigma}$. Consequently, by Lemma 4.1.4, none of these cycles can be fixed orientation-wise by the involution. Furthermore, if C is a cycle fixed *edgewise* by the involution, one will have an even number of copies of C.

Call these pairs of cycles (possibly with multiplicity) $C_1, \dots, C_k$. Induct on k. If $k = 1$, then $U^\alpha - U^\beta$ was itself a cycle binomial. Suppose that binomials with no common factors corresponding to $k-1$ pairs of cycles lie in the ideal generated by the cycle binomials. Then by induction one has

$$\begin{aligned} U^\alpha - U^\beta &= \prod_{i=1}^{k} U_{W(C_i)} - \prod_{i=1}^{k} U_{A(C_i)} \\ &= (U_{W(C_1)} - U_{A(C_1)}) \prod_{i=2}^{k} U_{W(C_i)} + U_{A(C_1)} \left(\prod_{i=2}^{k} U_{W(C_i)} - \prod_{i=2}^{k} U_{A(C_k)} \right), \end{aligned}$$

where $U_{W(C_i)} = \prod_{e \in W(C_i)} U_e$ and $U_{A(C_i)} = \prod_{e \in A(C_i)}$. Consequently, $U^\alpha - U^\beta$ lies in the ideal generated by the cycle binomials. Thus one has that the toric ideal is generated by the cycle binomials. □

4.1.1 The toric ideal of the root cone semigroup

As noted earlier, examining the definitions of $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ (Definition 3.1.4), one sees that $\widehat{G}_B(P)$ and $\widehat{G}_C(P)$ are both the signed coverings of some oriented signed graph (at least when one anchors the half edges to a 0 in the case of $\widehat{G}_B(P)$). Recall from Proposition 4.0.1 that the semigroups $K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$ and $K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}$ are generated by the edges of these oriented signed graphs. Denote the two semigroup rings $k[K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}]$ and $k[K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}]$ by R_P^{rt} and $R_{P^\vee}^{\mathrm{rt}}$, respectively. In this case, Theorem 4.1.5 gives the following.

Corollary 4.1.7. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Let $S_P^{\mathrm{rt}} = k[U_{\delta i,\epsilon j}]$ and $S_{P^\vee}^{\mathrm{rt}} = k[U_{\delta i,\epsilon j}]$, where the pairs $(\delta i, \epsilon j)$ run over orbits under the involution of edges $\delta i \to \epsilon j$ in $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$, respectively. Then the toric ideals I_P^{rt} and $I_{P^\vee}^{\mathrm{rt}}$ are generated by the cycle binomials corresponding to cycles in $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$, respectively, not fixed orientation-wise by the involution and

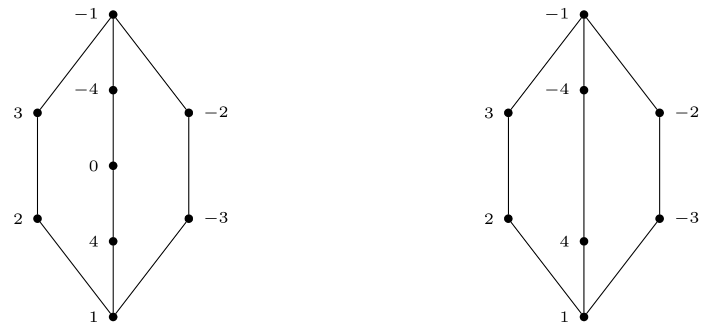
$$R_P^{\mathrm{rt}} = S_P^{\mathrm{rt}}/I_P^{\mathrm{rt}} \quad \text{and} \quad R_{P^\vee}^{\mathrm{rt}} = S_{P^\vee}^{\mathrm{rt}}/I_{P^\vee}^{\mathrm{rt}}.$$

Consider $P = \{+e_1 - e_2, +e_1 + e_3, +e_1 - e_3, +e_1 + e_2, +e_1 - e_4, +e_1 + e_4, +e_1, +e_4\}$. Figure 4.5 shows $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$. Then

$$S_P^{\mathrm{rt}} = k[U_{12}, U_{23}, U_{1\bar{3}}, U_{14}, U_{4\bar{4}}] \quad \text{and} \quad S_{P^\vee}^{\mathrm{rt}} = k[U_{12}, U_{23}, U_{1\bar{3}}, U_{14}, U_{40}].$$

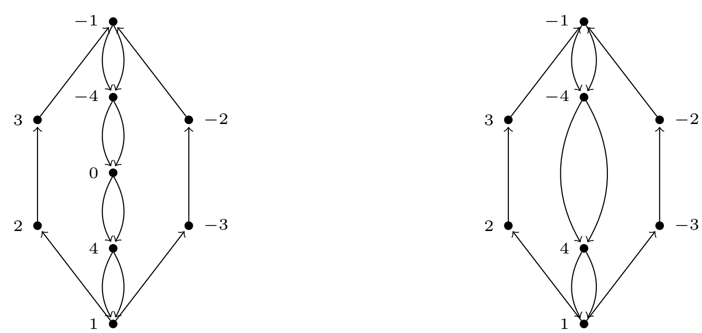
Both $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ have a cycle $1 \to 2 \to 3 \to -1 \to -2 \to -3 \to 1$ fixed orientation-wise by the involution, so its cycle binomial is 0. Figure 4.6 shows the cycles constructed in the proof of Theorem 4.1.5.

In both $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$, there is a single pair $(C, \iota(C))$, so the toric ideals are



(a) $\widehat{G}_B(P)$ (b) $\widehat{G}_C(P^\vee)$

Figure 4.5: $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ for $P = \{+e_1 - e_2, +e_1 + e_3, +e_1 - e_3, +e_1 + e_2, +e_1 - e_4, +e_1 + e_4, +e_1, +e_4\}$



(a) $\widehat{G}_B(P)$ (b) $\widehat{G}_C(P^\vee)$

Figure 4.6: The directed (multi-)graph from the proof of Theorem 4.1.5

principal. One then has

$$I_P^{\mathrm{rt}} = (U_{12}U_{23}U_{1\bar{3}} - U_{14}^2U_{40}^2) \quad \text{and} \quad I_{P^\vee}^{\mathrm{rt}} = (U_{12}U_{23}U_{1\bar{3}} - U_{14}^2U_4).$$

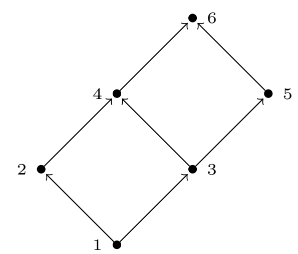
4.1.2 Toric ideals of posets and directed graphs

Another application of Theorem 4.1.5 is to posets and directed graphs. The semigroup associated to a directed graph is generated by $e_i - e_j$ for each edge $i \to j$. When the directed graph is the Hasse diagram of a poset, this is the root cone semigroup in type A of [9]. Using a similar argument to the proof of Theorem 4.1.5 (but simpler due to having the Hasse diagram rather than the signed covering), Boussicault, Féray, Lascoux and Reiner showed in [9, Proposition 8.1] that, given a poset, its toric ideal is generated by binomials corresponding to cycles in the Hasse diagram. Their argument easily generalizes to all directed graphs. Earlier, Gitler, Reyes and Villarreal obtained the same result for directed graphs [26, Proposition 4.3] using a result of Sturmfels [56, §5].

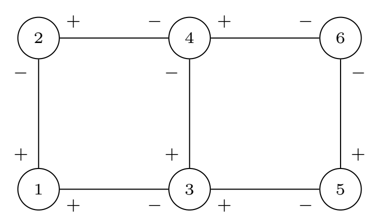
To see how Theorem 4.1.5 works for directed graphs, first observe that in a signed graph Σ with an orientation coming from a directed graph G, all edges are signed $+$, meaning all cycles are balanced. Furthermore, the signed covering of Σ is $G \sqcup \iota(G)$ and a cycle in G corresponds to a pair of cycles $(C, \iota(C))$ in $G \sqcup \iota(G)$, and no cycle in the signed covering is fixed orientation-wise by the involution. Consequently, the generating set from Theorem 4.1.5 is the same as the generating set from [9] and [26].

As an example, consider the digraph (and poset, depending on perspective) shown in Figure 4.7(a). The associated oriented signed graph is shown in Figure 4.7(b) and its signed covering in Figure 4.7(c). Then $S_G = k[U_{1\bar{2}}, U_{1\bar{3}}, U_{2\bar{4}}, U_{3\bar{4}}, U_{3\bar{5}}, U_{4\bar{6}}, U_{5\bar{6}}]$ and, according to Theorem 4.1.5, the toric ideal is

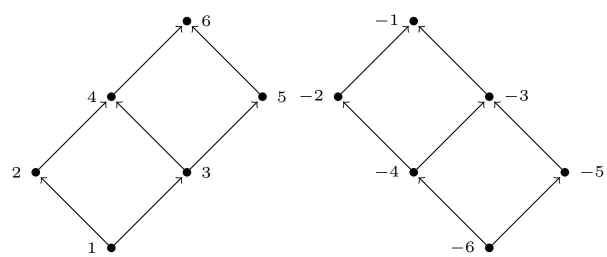
$$I_G = (U_{1\bar{2}}U_{2\bar{4}} - U_{1\bar{3}}U_{3\bar{4}}, U_{3\bar{4}}U_{4\bar{6}} - U_{3\bar{5}}U_{5\bar{6}}, U_{1\bar{2}}U_{2\bar{4}}U_{4\bar{6}} - U_{1\bar{3}}U_{3\bar{5}}U_{5\bar{6}}).$$



(a) digraph/poset



(b) associated oriented signed graph



(c) signed covering

Figure 4.7: A digraph, its associated oriented signed graph and signed covering

4.1.3 Toric ideals of graphs

Suppose G is a (unsigned) graph with no loops or multiple edges. Associated to G is the semigroup generated by the vectors $e_i + e_j$ where (i, j) runs over all edges of G. This semigroup then gives rise to a toric ideal I_G. Of course, this is the same semigroup as one obtains from an oriented signed graph Σ where G is Σ's underlying graph, all edges are signed $-$ and Σ is oriented with all incidences $+$. Consequently, Theorem 4.1.5 gives a set of generators for I_G. This generating set lies between two other known generating sets.

Definition 4.1.8. A sequence $v_0, \dots, v_n$ of (not necessarily distinct) vertices in a graph G is a *closed walk* if $v_0 = v_n$ and there is an edge between v_i and v_{i+1} for $i = 0, \dots, n-1$. It is an *even closed walk* if n is even. Let $S_G = k[U_e]$ where e runs over the edges of G. Associated to an even closed walk $w = \{v_0, \dots, v_{2k}\}$ is a binomial

$$T_w = U_{v_0v_1}U_{v_2v_3}\cdots U_{v_{2k-2}v_{2k-1}} - U_{v_1v_2}U_{v_3v_4}\cdots U_{v_{2k-1}v_{2k}}$$

Theorem 4.1.9 (Villarreal, [59, Proposition 3.1])**.** *Suppose G is a graph. Then its toric ideal has this description:*

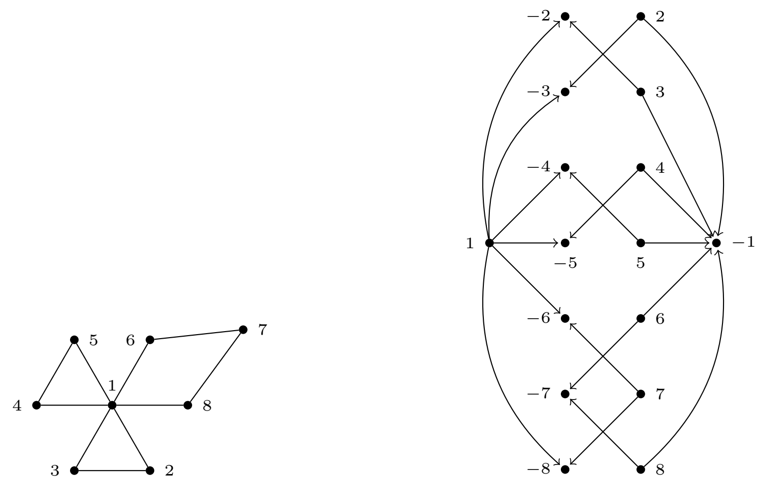
$$I_G = (T_w \colon w \text{ an even closed walk in } G).$$

This generating set and the generating set from Theorem 4.1.5 do not necessarily coincide. Consider the graph in Figure 4.8. According to Theorem 4.1.9, the toric ideal is

$$(U_{12}U_{13}U_{45} - U_{23}U_{14}U_{15}, U_{18}U_{67} - U_{16}U_{78}, U_{12}U_{13}U_{45}U_{18}U_{67} - U_{23}U_{14}U_{15}U_{78}U_{16}).$$

On the other hand, according to Theorem 4.1.5, it is

$$(U_{12}U_{13}U_{45} - U_{23}U_{14}U_{15}, U_{18}U_{67} - U_{16}U_{78}).$$



(a) A graph G (b) The signed covering of G

Figure 4.8: A graph and its signed covering

One has that

$$U_{12}U_{13}U_{45}(U_{67}U_{18} - U_{16}U_{78}) - U_{16}U_{78}(U_{23}U_{14}U_{15} - U_{12}U_{13}U_{45}) \\ = U_{12}U_{13}U_{45}U_{18}U_{67} - U_{23}U_{14}U_{15}U_{78}U_{16},$$

so the two ideals really are equal.

Since the signed covering is bipartite, any cycle will correspond to an even closed walk in G. Therefore, the generating set of Theorem 4.1.5 is contained in the generating set of Theorem 4.1.9.

Definition 4.1.10. If $I \subset k[x_1, \ldots, x_k]$ is a toric ideal, a *primitive binomial* is a binomial $\boldsymbol{x}^{u^+} - \boldsymbol{x}^{u^-} \in I$ such that there does not exist an $\boldsymbol{x}^{v^+} - \boldsymbol{x}^{v^-} \in I$ with $u \neq v$ and $\boldsymbol{x}^{v^+} | \boldsymbol{x}^{u^+}$ and $\boldsymbol{x}^{v^-} | \boldsymbol{x}^{u^-}$. The set of primitive binomials is called the *Graver basis* and generates the toric ideal.

See Sturmfels [57] for more on the Graver basis. Ohsugi and Hibi described the Graver basis of the toric ideal of a graph in [45].

Theorem 4.1.11 (Ohsugi and Hibi, [45, Lemma 3.2])**.** *Let G be a finite connected graph. If $f \in I_G$ is primitive, it is $f = T_\Gamma$ where Γ is one of the following even closed walks:*

(i) Γ *is an even cycle*

(ii) Γ *is a pair of odd cycles having exactly one common vertex*

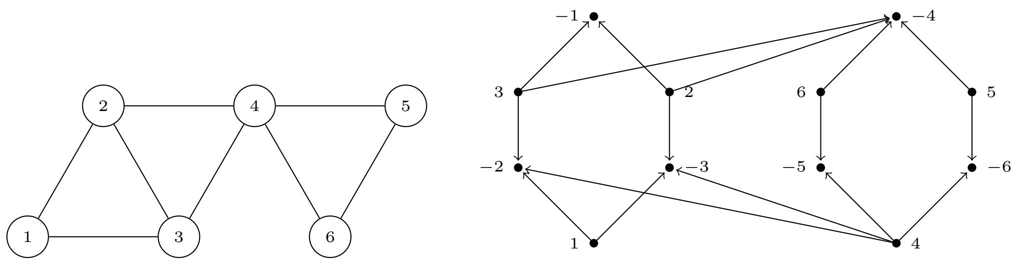
(iii) $\Gamma = (C_1, \Gamma_1, C_2, \Gamma_2)$ *where C_1 and C_2 are odd cycles having no common vertex and where Γ_1 and Γ_2 are walks of G, both of which connect a vertex v_1 in C_1 to a vertex v_2 of C_2.*

A moment's thought will show that an even closed walk Γ corresponding to a primitive binomial corresponds to a cycle not fixed orientation-wise by the involution. However, the converse is not true. Consider the graph in Figure 4.9. The cycle $1 \to -2 \to 3 \to -1 \to 2 \to -4 \to 5 \to -6 \to 4 \to -3 \to 1$ in the signed covering is not fixed orientation-wise by the involution, but does not project to an even closed walk of any of the forms in Theorem 4.1.11. The binomial corresponding to this cycle is $U_{12}U_{13}U_{24}U_{56}U_{34} - U_{23}U_{12}U_{45}U_{46}U_{13}$, which is clearly not primitive as the two terms are not relatively prime. Cancelling $U_{12}U_{13}$ from both terms gives the primitive binomial $U_{24}U_{56}U_{34} - U_{23}U_{45}U_{46}$, corresponding to the even closed walk $2 \to 4 \to 6 \to 5 \to 4 \to 3 \to 2$.

The following proposition summarizes the results of this section, linking the generating sets of Theorems 4.1.5, 4.1.9, and 4.1.11.

Proposition 4.1.12. *Suppose G is a finite connected graph. There are then three generating sets, $S_1 \subset S_2 \subset S_3$ for the toric ideal I_G:*

- $S_1 = \{T_w \colon w$ *an even closed walk in the forms of Theorem 4.1.11*$\}$
- $S_2 = \{U(C) \colon C$ *a cycle in the signed covering*$\}$
- $S_3 = \{T_w \colon w$ *an even closed walk*$\}$.



(a) G (b) Signed covering of G

Figure 4.9: A graph for which the generating sets from Theorem 4.1.5 and Theorem 4.1.11 differ

4.2 Some Complete Intersection Rings R_P^{rt}

Now that one understands the toric ideal, one can turn one's attention to the question of computing Ψ_P and $\Psi_P^\vee$ via complete intersection presentations. This section gives factorizations of

$$\Psi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)x_n}\right) \quad \text{and}$$
$$\Psi^\vee_{P^\vee}(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)\cdots(x_{n-1} - x_n)2x_n}\right)$$

for a certain class of signed posets using Proposition 2.2.14.

Recall Theorem 1.1.1, where Greene showed that, for a strongly planar poset,

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_\rho (x_{\min(\rho)} - x_{\max(\rho)})}{\prod_{i \lessdot j}(x_i - x_j)},$$

where ρ runs over all the regions enclosed by the Hasse diagram of P and $i \lessdot j$ over the covering relations of the poset. Boussicault and Féray [8] showed that a similar factorization occurs for posets that are "gluings of diamonds along chains" and, in particular, for strongly planar posets, recovering Greene's result. In [9], Boussicault, Féray, Lascoux and Reiner gave an algebraic explanation for the disconnecting chains of [8] by showing certain posets (including strongly planar posets) are complete intersections by constructing a regular sequence using the operation of opening/closing a notch.

The situation for signed posets is quite similar. We will construct a regular sequence by opening/closing signed notches, culminating in Theorem 4.2.12.

Theorem 4.2.12. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset such that $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has an embedding in $\mathbb{R}^2$ which is strongly planar. Then*

- *$\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has an embedding that is both centrally symmetric and strongly planar,*
- *R_P^{rt} (resp. $R_{P^\vee}^{\mathrm{rt}}$) is a complete intersection, with I_P (resp. $I_{P^\vee}$) generated by the cycle binomials of the cycles in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) defining the faces of the graph, and*
- *one has*

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon})} \quad \textit{and}$$
$$\mathrm{Hilb}(R_{P^\vee}^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon})},$$

where ρ runs over all regions enclosed by $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) not fixed by the involution, e runs over the orbits of edges $\delta a \to \epsilon b$ ($a < b$) in $\widehat{G}_C(P)$, x_0 is taken to be 1 and

$$\boldsymbol{x}^\rho = x_{\min(\rho)}^{\mathrm{sgn}(\min(\rho))} x_{\max(\rho)}^{-\mathrm{sgn}(\max(\rho))}.$$

The next several sections will build up to the proof of Theorem 4.2.12 in Section 4.2.3, where what it means for a signed poset to be strongly planar will be defined.

- Section 4.2.1 will reduce to the case of signed posets consisting of a single biconnected component (see Definition 4.2.1 and Proposition 4.2.2) and whose root cones are full-dimensional (see Propositions 4.2.3 and 4.2.4).

- Section 4.2.2 defines the notion of a signed notch (see Definition 4.2.5), explains what it means to open/close a signed notch and what impact that has on the root cone semigroup ring (see Proposition 4.2.9).

- Section 4.2.3 defines strongly planar posets (see Definition 4.2.10), proves a number of propositions and lemmas regarding strongly planar posets and then proves Theorem 4.2.12.

- After the proof, Section 4.2.4 will explain how Theorem 4.2.12 can be used to compute Ψ_P and $\Psi^\vee_{P^\vee}$.

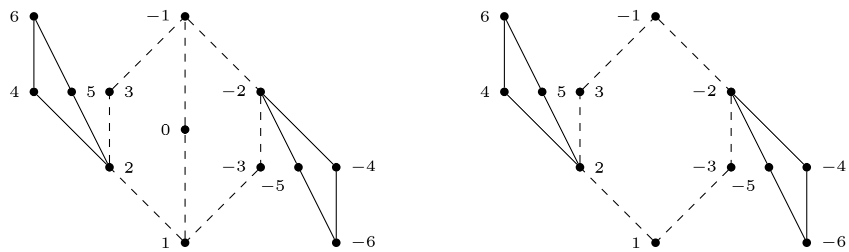
4.2.1 A few reductions

The first reduction will be to the case of signed posets consisting of a single biconnected component.

Definition 4.2.1. Let $P \subset \Phi_{B_n}$ and $P^\vee \subset \Phi_{C_n}$ be signed posets. Let $A \subset P$ (resp. $P^\vee$) be the elements corresponding to the edges of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$). Say two elements of A are *cycle equivalent* if there is a cycle in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) not fixed orientation-wise by the involution passing through an edge corresponding to each element. Taking the transitive closure gives an equivalence relation. Combine equivalence classes lying in the same orbit of the involution. This partitions the edges of $\widehat{G}_B(P)$ (rep. $\widehat{G}_C(P^\vee)$) into the *biconnected components* of P (resp. $P^\vee$). Each biconnected component corresponds to a signed poset, which will also be called the biconnected components of P (resp. $P^\vee$).

Consider $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_1 + e_3, +e_1 - e_4, +e_1 - e_5, +e_1 - e_6, +e_2 - e_3, +e_2 - e_4, +e_2 - e_5, +e_2 - e_6, +e_4 - e_6, +e_5 - e_6, +e_1\}$. Figure 4.10 shows $\widehat{G}(P)$ with the edges of one biconnected component solid and the edges of the other dashed.

Proposition 4.2.2. *Let $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) be a signed poset and $P_1, \ldots, P_k$ (resp.*



(a) $\widehat{G}_B(P)$ (b) $\widehat{G}_C(P^\vee)$

Figure 4.10: $\widehat{G}(P)$ for $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_1 + e_3, +e_1 - e_4, +e_1 - e_5, +e_1 - e_6, +e_2 - e_3, +e_2 - e_4, +e_2 - e_5, +e_2 - e_6, +e_4 - e_6, +e_5 - e_6, +e_1\}$ with biconnected components indicated by dashed and solid lines.

$P_1^\vee, \ldots, P_k^\vee$) its biconnected components. Then

$$R_P^{\mathrm{rt}} \cong R_{P_1}^{\mathrm{rt}} \otimes_k \cdots \otimes R_{P_k}^{\mathrm{rt}} \quad \textit{and} \quad R_{P^\vee}^{\mathrm{rt}} \cong R_{P_1^\vee}^{\mathrm{rt}} \otimes_k \cdots \otimes_k R_{P_k^\vee}^{\mathrm{rt}}$$

and, as a consequence,

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \prod_{i=1}^{k} \mathrm{Hilb}(R_{P_i}^{\mathrm{rt}}, \boldsymbol{x}) \quad \textit{and} \quad \mathrm{Hilb}(R_{P^\vee}^{\mathrm{rt}}, \boldsymbol{x}) = \prod_{i=1}^{k} \mathrm{Hilb}(R_{P_i^\vee}^{\mathrm{rt}}, \boldsymbol{x}).$$

Proof. Recall that $R_P^{\mathrm{rt}} = S_P/I_P$ and $R_{P^\vee}^{\mathrm{rt}} = S_{P^\vee}/I_{P^\vee}$, where I_P and $I_{P^\vee}$ are generated, respectively, by the cycle binomials of cycles of $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ not fixed by the involution. Every edge in $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ lies in a unique biconnected component and is thus associated to a unique P_i (resp. $P_i^\vee$). Consequently, $S_P \cong \bigotimes_{\ell=1}^k S_{P_\ell}$ and $S_{P^\vee} \cong \bigotimes_{\ell=1}^k S_{P_\ell^\vee}$. Furthermore, each cycle not fixed by the involution lies wholly in a single biconnected component. Consequently, $I_P = \bigoplus_{\ell=1}^k I_{P_k}$ and $I_{P^\vee} = \bigoplus_{\ell=1}^k I_{P_k^\vee}$. Then

$$R_P^{\mathrm{rt}} \cong S_P/I_P = S/\oplus_{\ell=1}^k I_{P_k} \cong \bigotimes_{\ell=1}^{k} S_{P_\ell}/I_{P_\ell} \cong R_{P_1}^{\mathrm{rt}} \otimes_k \cdots \otimes_k R_{P_k}^{\mathrm{rt}}$$

and

$$R^{\mathrm{rt}}_{P^\vee} \cong S_{P^\vee}/I_{P^\vee} = S/\oplus_{\ell=1}^{k} I_{P^\vee_k} \cong \bigotimes_{\ell=1}^{k} S_{P^\vee_\ell}/I_{P^\vee_\ell} \cong R^{\mathrm{rt}}_{P^\vee_1} \otimes_k \cdots \otimes_k R^{\mathrm{rt}}_{P^\vee_k},$$

as claimed. $\square$

For the remainder of this chapter, it will be assumed that all signed posets consist of a single biconnected component. In particular, one may assume that $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ have no vertices of degree one.

Furthermore, signed posets for which K^{rt}_P is not full-dimensional need not be considered here, as they reduce to work of Boussicault, Féray, Lascoux, and Reiner [9], as is now explained.

Suppose P is a signed poset such that K^{rt}_P is not full-dimensional. There are two possibilities:

- P can be relabeled so that $P \subset \Phi_{B_{n-1}}$ (resp. $P^\vee \subset \Phi_{C_{n-1}}$), i.e. one should think of $K^{\mathrm{rt}}_P \subset \mathbb{R}^{n-1}$ rather than $K^{\mathrm{rt}}_P \subset \mathbb{R}^n$.
- The signed graph Σ_P underlying the Hasse diagram of P is balanced. (See Definition 3.1.3 for the definition of the Hasse diagram of a signed poset and Definition 3.5.12 for the notion of a balanced signed graph.)

While the first case is straightforward, the second requires a little care.

Proposition 4.2.3. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset such that Σ_P is balanced. Then P (resp. $P^\vee$) is isomorphic to a poset $P' \subset \Phi_{A_{n-1}} \cap \Phi_{B_n}$ (resp. $P' \subset \Phi_{A_{n-1}} \cap \Phi_{C_n}$).*

Proof. One may assume $\widehat{G}(P)$ consists of a single biconnected component. Consequently, Σ_P must be connected. Since Σ_P is balanced, the vertices can be partitioned into sets V^+ and V^- such that edges labeled $+$ join a vertex in V^+ to one in V^- and edges labeled $-$ join two vertices in either V^+ or V^-. (This is Theorem 3.5.17.) Note that since Σ_P is balanced, it cannot contain any self-loops. Therefore, there is no $\pm i < \mp i$ in $\widehat{G}(P)$. (In particular, should one be considering $\widehat{G}_B(P)$, 0 must lie in its own connected component.) Then $\widehat{G}(P)$ consists of two isotropic components (plus 0 in $\widehat{G}_B(P)$). One can then read off a signed permutation sending P to some $P' \subset \Phi_{A_{n-1}}$ (it is the permutation flipping signs so that all vertices in each isotropic component have the same sign), as desired. □

Next, one checks that, given an $\Phi_{A_{n-1}}$-poset, its root cone semigroup does not change when one switches among the $\Phi_{A_{n-1}}$, Φ_{B_n} and Φ_{C_n} root lattices.

Proposition 4.2.4. *Suppose $P \subset \Phi_{A_{n-1}}$ is a poset. Then the semigroups $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}}$, $K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$ and $K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}$ are equal.*

Proof. First, begin by observing that $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}} \subset K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$ and $K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}$. Suppose $\alpha \in K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$. Then α can be written as a positive integer linear combination of elements of P corresponding to the covering relations in $\widehat{G}_B(P)$. However, these elements correspond precisely to the covering relations in the Hasse diagram of P, which are a minimal generating set for $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}}$ (see [9, Proposition 7.1]). Therefore, $\alpha \in K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}}$, so $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}} = K_P^{\mathrm{rt}} \cap L_B^{\mathrm{rt}}$. The same argument shows $K_P^{\mathrm{rt}} \cap L_A^{\mathrm{rt}} = K_P^{\mathrm{rt}} \cap L_C^{\mathrm{rt}}$. □

As a consequence of the last two propositions, if P is a signed poset and K_P^{rt} is not full-dimensional, its root cone semigroup ring can be understood using the results of [9]. If $\widehat{G}(P)$ is disconnected, it either has multiple biconnected components or consists of a single biconnected component made up of two isotropic connected components, meaning K_P^{rt} is not full-dimensional. Consequently, for the remainder of the chapter it will be assumed that $\widehat{G}(P)$ is connected and consists of a single biconnected component.

4.2.2 Signed Notches

In [9], a $\vee$-*shaped notch* (resp. a $\wedge$-*shaped notch*) in a poset was defined to be a triple of elements (a, b, c) such that $a \lessdot b, a \lessdot c$ (resp. $a \gtrdot b, a \gtrdot c$) and b and c are in different connected components of $P \setminus P_{\leq a}$ (resp. $P \setminus P_{\geq a}$). One defines a signed notch as a pair of notches in this sense.

Definition 4.2.5. Given a signed poset P, a *signed notch* in $\widehat{G}(P)$ is a pair (a, b, c) and $(-a, -b, -c)$ such that both (a, b, c) and $(-a, -b, -c)$ are notches in $\widehat{G}(P)$ and, if in $\widehat{G}_B(P)$, neither b nor c is 0.

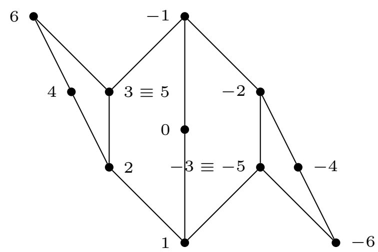
Consider again the signed poset P from Figure 4.10. Together $(2, 3, 5)$ and $(-2, -3, -5)$ form a signed notch.

If (a, b, c) is a notch in $\widehat{G}(P)$, it follows from the properties of the involution that $(-a, -b, -c)$ is also a notch in the type A sense, but facing the other way. Consequently, one may assume that (a, b, c) is a $\vee$-shaped notch and $(-a, -b, -c)$ is a $\wedge$-shaped notch. One may also always assume, without loss of generality, that $\mathrm{sgn}(b) = \mathrm{sgn}(c)$, by relabeling the elements of $\widehat{G}(P)$ as necessary.

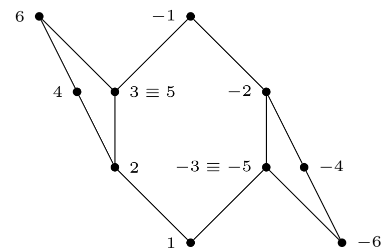
One next wants to show that closing a signed notch in either $\widehat{G}_B(P)$ or $\widehat{G}_C(P^\vee)$ gives the $\widehat{G}_B$ or $\widehat{G}_C$ associated to some other signed poset. If (a, b, c) and $(-a, -b, -c)$ form a signed notch in $\widehat{G}(P)$, the poset $\widehat{G}(P)/\{b \equiv c, -b \equiv -c\}$ is the poset obtained by *closing the notch*. It will often be useful to think of this operation as closing the type A notch (a, b, c) and then the notch $(-a, -b, -c)$ (or vice versa). To legitimize this point of view, one needs to show the following:

- $(-a, -b, -c)$ remains a notch in $\widehat{G}(P)/\{b \equiv c\}$ (Proposition 4.2.6)
- There is a signed poset P' such that $\widehat{G}(P') = \widehat{G}(P)/\{b \equiv c, -b \equiv -c\}$. Such a P' will be said to have been obtained from P by *closing the signed notch.* (Propositions 4.2.7 and 4.2.8)

The result of closing the signed notch formed by $(2, 3, 5)$ and $(-2, -3, -5)$ in P shown in Figure 4.11.



(a) $\widehat{G}_B(P)/\{3 \equiv 5, -3 \equiv -5\}$



(b) $\widehat{G}_C(P^\vee)/\{3 \equiv 5, -3 \equiv -5\}$

Figure 4.11: $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ from Figure 4.10 after closing the signed notch $(2, 3, 5)$ and $(-2, -3, -5)$.

Proposition 4.2.6. *If (a, b, c) and $(-a, -b, -c)$ form a signed notch in $\widehat{G}(P)$, $(-a, -b, -c)$ remains a notch in $\widehat{G}(P)/\{b \equiv c\}$.*

Proof. One knows from [9, Definition 8.5] that $\widehat{G}(P)/\{b \equiv c\}$ is a poset and thus so is $(\widehat{G}(P)/\{b \equiv c\})/\{-b \equiv -c\}$, as long as $(-a, -b, -c)$ remains a notch in $\widehat{G}(P)/\{b \equiv c\}$. The only way in which $(-a, -b, -c)$ can fail to be a notch in $\widehat{G}(P)/\{b \equiv c\}$ is if $-b$ and $-c$ lie in the same connected component of $(\widehat{G}(P)/\{b \equiv c\}) \smallsetminus (\widehat{G}(P)/\{b \equiv c\})_{\geq -a}$. In that case, there is a path from $-b$ to $-c$ avoiding $(\widehat{G}(P)/\{b \equiv c\})_{\geq -a}$. However, since $(-a, -b, -c)$ was a notch in $\widehat{G}(P)$, this path cannot lift to a path in $\widehat{G}(P)$, so it must pass through the vertex $\{b \equiv c\}$. Therefore at least one of b and c lies in $\widehat{G}(P)_{\geq -a}$. Without loss of generality, assume $b \in \widehat{G}(P)_{\geq -a}$. But then $b \equiv c \geq -a$ in $\widehat{G}(P)/\{b \equiv c\}$, a contradiction. Thus, one must have that $(-a, -b, -c)$ is a notch in $\widehat{G}(P)/\{b \equiv c\}$.

$\square$

With Proposition 4.2.6 in hand, it is easier to address $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ separately, as the next two propositions.

Proposition 4.2.7. *If (a, b, c) and $(-a, -b, -c)$ form a signed notch in $\widehat{G}_B(P)$, there is a signed poset $P' \subset \Phi_{B_n}$ such that $\widehat{G}_B(P') = \widehat{G}_B(P)/\{b \equiv c, -b \equiv -c\}$.*

Proof. From Proposition 4.2.6, one knows that $\widehat{G}_B(P)/\{b \equiv c\}$ has a notch $(-a, -b, -c)$. Closing the second notch $(-a, -b, -c)$ one obtains

$$\begin{aligned}\widehat{G}_B(P') = (\widehat{G}_B(P)/\{b \equiv c\})/\{-b \equiv -c\} \\ = (\widehat{G}_B(P)/\{-b \equiv -c\})/\{b \equiv c\} = \widehat{G}_B(P)/\{b \equiv c, -b \equiv -c\},\end{aligned}$$

as the resulting poset is independent of the order in which the notches are closed. Since $\mathrm{sgn}(b) = \mathrm{sgn}(c)$, this poset has an involution $\pm i \mapsto \mp i$ such that $i \to j$ is sent to $-j \to -i$, since $\widehat{G}_B(P)$ had an involution with this property.

To show that $\widehat{G}_B(P') = \widehat{G}_B(P)/\{b \equiv c, -b \equiv -c\}$ is really associated to a signed poset P', one must show that $\widehat{G}_B(P')$ has the property that if $i < -i$, then $i < 0 < -i$. Suppose $i < -i$ in $\widehat{G}_B(P')$. It suffices to consider only the case where $i \not< -i$ in $\widehat{G}_B(P)$. Then, without loss of generality, one may assume that $b < -i$ and $i < c$. First, consider the case where $a \neq 0$. One then has the situation depicted in Figure 4.12, where the solid lines are edges in the Hasse diagram of $\widehat{G}_B(P)$ and the dashed lines are chains. This gives a path from b to

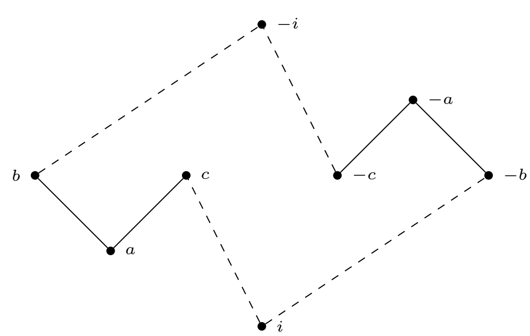


Figure 4.12: After closing two notches in the proof of Proposition 4.2.7

c, via $-i, -c, -a, -b$ and i. Since (a, b, c) is a notch, this path must intersect $\widehat{G}_B(P)_{\leq a}$ at some d. There are a few cases.

(a) **Suppose** $-c \leq d \leq -i$**.** Then $-c < c$ in $\widehat{G}_B(P)$. Therefore, $-c < 0 < c$. Then in $\widehat{G}_B(P')$, one has $b \equiv c < -i$ and $i < -b \equiv -c$, so since $-c < 0 < c$ one has $i < -b \equiv -c < 0 < b \equiv c < -i$, as desired.

(b) **Suppose $i \leq d \leq -b$.** Then in $\widehat{G}_B(P)$ one has $i \leq d < a < -i$, so $i < 0 < -i$ and this relation is preserved in $\widehat{G}_B(P')$.

(c) **Suppose $i \leq d < c$.** Then one has $i \leq d < c < -i$ in $\widehat{G}_B(P)$, so $i < 0 < -i$ and this relation is preserved in $\widehat{G}_B(P')$.

(d) **Suppose $-a = d < a$.** Then $i < -i$ in $\widehat{G}_B(P)$, so $i < 0 < -i$ and this relation is preserved in $\widehat{G}_B(P')$.

In the other case, suppose $a = 0$. Then $-b < 0 < b$. Since $-i > b$, $i < -b$, so $i < -i$ in $\widehat{G}_B(P)$, so $i < 0 < -i$ and this relation is preserved in $\widehat{G}_B(P')$. □

Proposition 4.2.8. *If (a, b, c) and $(-a, -b, -c)$ form a signed notch in $\widehat{G}_C(P^\vee)$, there is a signed poset $P'^\vee \subset \Phi_{C_n}$ such that $\widehat{G}_C(P'^\vee) = \widehat{G}_C(P^\vee)/\{b \equiv c, -b \equiv -c\}$.*

Proof. From Proposition 4.2.6, one knows that $\widehat{G}_C(P^\vee)/\{b \equiv c\}$ has a notch $(-a, -b, -c)$. Closing the second notch $(-a, -b, -c)$ one obtains

$$\begin{aligned}\widehat{G}_C(P'^\vee) &= (\widehat{G}_C(P^\vee)/\{b \equiv c\})/\{-b \equiv -c\} \\ &= (\widehat{G}_C(P^\vee)/\{-b \equiv -c\})/\{b \equiv c\} = \widehat{G}_C(P^\vee)/\{b \equiv c, -b \equiv -c\},\end{aligned}$$

as the resulting poset is independent of the order in which the notches are closed. Since $\operatorname{sgn}(b) = \operatorname{sgn}(c)$, this poset has an involution $\pm i \mapsto \mp i$ such that $i \to j$ is sent to $-j \to -i$, since $\widehat{G}_C(P^\vee)$ had an involution with this property.

It remains to show that $\widehat{G}_C(P'^\vee)$ has the property that if $i < -i$ and $j < -j$ (possibly after relabelling) that $i < -j$ and $j < -i$. Closing a notch introduces relations and does not remove any relations. There are then two cases:

1. **Suppose $i < -i$ in $\widehat{G}_C(P'^\vee)$, but not in $\widehat{G}_C(P^\vee)$ and $j < -j$ in both $\widehat{G}_C(P^\vee)$ and $\widehat{G}_C(P'^\vee)$.** Then, without loss of generality, suppose $i < b$ and $c < -i$ in $\widehat{G}_C(P^\vee)$. Then $-b < -i$ and $i < -c$. Then one has the situation in $\widehat{G}_C(P^\vee)$ depicted in Figure 4.13, where the dashed edges are chains and the solid edges are edges in the Hasse diagram of $\widehat{G}_C(P^\vee)$. Since (a, b, c) is a notch in $\widehat{G}_C(P^\vee)$, one must have that the path from b

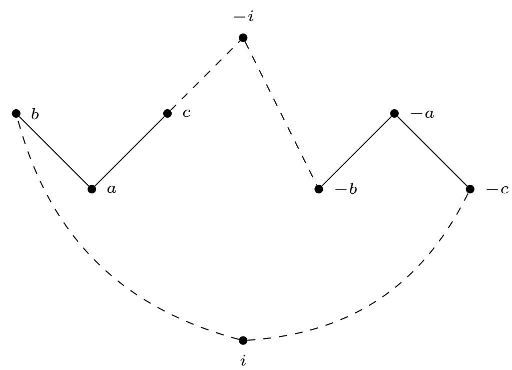


Figure 4.13: $\widehat{G}_C(P^\vee)$ in case 1 of the proof of Proposition 4.2.8

to c via i must pass through $\widehat{G}_C(P^\vee)_{\leq a}$ at some d. There are a few cases

(1.a) **Suppose d lies in the chain from b to i or the chain from i to $-c$.** Then $i < a$, so $i < -i$ in $\widehat{G}(P)$, a contradiction.

(1.b) **Suppose $d < a$ and $-b \leq d < -i$.** Then, $-b < d < a < b$. Consequently, if $j < -j$ in $\widehat{G}_C(P^\vee)$ and $b \neq j$, one has that $-b < -j$ and $j < b$ in both $\widehat{G}_C(P^\vee)$ and $\widehat{G}_C(P'^\vee)$. Then, in $\widehat{G}_C(P'^\vee)$, one has $j < b \equiv c < -i$ and $i < -b \equiv -c < -j$, as required.

(1.c) **Suppose $d = -a$ and $-a < a$.** Then in $\widehat{G}_C(P^\vee)$ one has that $i < -i$, a contradiction.

(1.d) **Suppose $d = -c$ and $-c < a$.** Then $\widehat{G}_C(P^\vee)$ is as in Figure 4.14. Then $i < -i$ in $\widehat{G}_C(P^\vee)$, a contradiction.

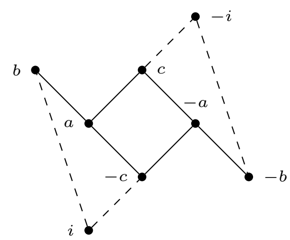


Figure 4.14: $\widehat{G}_C(P^\vee)$ in case 1(d) in the proof of Proposition 4.2.8

2. Suppose $i < -i$ and $j < -j$ in $\widehat{G}_C(P'^\vee)$, but neither relation exists in $\widehat{G}_C(P^\vee)$. There are again two cases (up to relabeling the vertices).

   (2.a) **Suppose** $i < b$**,** $c < -i$ **and** $j < c$**,** $-j > b$**.** Then in $\widehat{G}_C(P'^\vee)$, one has $i < b \equiv c < -j$ and $j < b \equiv c < -i$, as required.

   (2.b) **Suppose** $i, j < b$ **and** $c < -i, -j$**.** Without loss of generality, suppose $i < j$. Consequently $-j < -i$. Then in $\widehat{G}_C(P'^\vee)$, one has $i < j < -j < -i$, giving the required relations.

Thus, this new poset is $\widehat{G}_C(P'^\vee)$ for some signed poset $P'^\vee$. $\square$

Now that one understands what it means to close a notch in a signed poset, the natural question is what effect this maneuver has on the semigroup ring.

Proposition 4.2.9. *Let $P \subset \Phi_{B_n}$ be a signed poset and (a, b, c) and $(-a, -b, -c)$ form a signed notch in $\widehat{G}_B(P)$. Let P' be the signed poset obtained from P by closing this notch. Then*

$$R^{\mathrm{rt}}_{P'} \cong \begin{cases} R^{\mathrm{rt}}_P/(t_a^\delta t_b^\epsilon - t_a^\delta t_c^\epsilon) & \text{if } a \neq 0 \\ R^{\mathrm{rt}}_P/(t_b^\epsilon - t_c^\epsilon) & \text{if } a = 0 \end{cases}$$

where $\delta = \mathrm{sgn}(a)$ and $\epsilon = -\mathrm{sgn}(b)$.

Let $P^\vee \subset \Phi_{C_n}$ be a signed poset and (a, b, c) and $(-a, -b, -c)$ form a signed notch in

$\widehat{G}_C(P^\vee)$. *Let* $P'^\vee$ *be the signed poset obtained from* $P^\vee$ *by closing this notch. Then*

$$R^{\mathrm{rt}}_{P'^\vee} \cong R^{\mathrm{rt}}_{P^\vee}/(t_a^\delta t_b^\epsilon - t_a^\delta t_c^\epsilon),$$

where $\delta = \operatorname{sgn}(a)$ *and* $\epsilon = -\operatorname{sgn}(b)$.

To see how this works, consider once again $P = \{+e_1 - e_2, +e_1 + e_2, +e_1 - e_3, +e_1 + e_3, +e_1 - e_4, +e_1 - e_5, +e_1 - e_6, +e_2 - e_3, +e_2 - e_4, +e_2 - e_5, +e_2 - e_6, +e_4 - e_6, +e_5 - e_6, +e_1\}$ shown in Figure 4.10 (page 83). Closing the notch $(2,3,5)$ and $(-2,-3,-5)$ gives the poset shown in Figure 4.11 (page 87), $Q = \{+e_1 - e_2, +e_1 + e_2, +e_1 + e_3, +e_1 - e_3, +e_1 - e_4, +e_1 - e_6, +e_2 - e_3, +e_2 - e_4, +e_3 - e_6, +e_4 - e_6\}$ (with $3 \equiv 5$ and $3 \equiv -5$ being renamed 3 and -3, respectively). In type B, one has $S^{\mathrm{rt}}_P = k[U_{12}, U_{23}, U_{\bar{1}3}, U_{24}, U_{25}, U_{46}, U_{56}, U_{10}]$. Define a map $\psi\colon S^{\mathrm{rt}}_P \to R^{\mathrm{rt}}_Q$ by

$$\begin{array}{llll} U_{12} \mapsto t_1 t_2^{-1} & U_{23} \mapsto t_2 t_3^{-1} & U_{\bar{1}3} \mapsto t_1 t_3 & U_{24} \mapsto t_2 t_4^{-1} \\ U_{25} \mapsto t_2 t_3^{-1} & U_{46} \mapsto t_4 t_6^{-1} & U_{56} \mapsto t_3 t_6^{-1} & U_{10} \mapsto t_1 \end{array}$$

Then

$$\begin{aligned} R^{\mathrm{rt}}_Q &\cong S^{\mathrm{rt}}_P / \ker\psi \\ &\cong S^{\mathrm{rt}}_P/(U_{24}U_{46} - U_{25}U_{56}, U_{12}U_{\bar{1}3}U_{25} - U_{10}^2, U_{23} - U_{25}) \\ &= S^{\mathrm{rt}}_P/(I^{\mathrm{rt}}_P + (U_{23} - U_{25})) \\ &\cong (S^{\mathrm{rt}}_P/I^{\mathrm{rt}}_P)/(\overline{U}_{23} - \overline{U}_{25}) \\ &\cong R^{\mathrm{rt}}_P/(t_2 t_3^{-1} - t_2 t_5^{-1}) \end{aligned}$$

Proof of Proposition 4.2.9. Recall the definitions of S^{rt}_P, $S^{\mathrm{rt}}_{P^\vee}$ and of the toric ideals I^{rt}_P and $I^{\mathrm{rt}}_{P^\vee}$ from Section 4.1.1. Let P' and $P'^\vee$ be the signed posets obtained by closing notches in $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$, respectively. One defines maps from S^{rt}_P and $S^{\mathrm{rt}}_{P^\vee}$ to the semigroup rings $R^{\mathrm{rt}}_{P'} = k[K^{\mathrm{rt}}_{P'} \cap L^{\mathrm{rt}}_{B}]$ and $R^{\mathrm{rt}}_{P'^\vee} = k[K^{\mathrm{rt}}_{P'^\vee} \cap L^{\mathrm{rt}}_{C}]$.

Define a map $\psi\colon S^{\mathrm{rt}}_P \to R^{\mathrm{rt}}_{P'}$ by

$$U_{\delta i,\epsilon j} \mapsto \begin{cases} t_i^\delta t_j^{-\epsilon} & i,j \neq c,0 \\ t_i^\delta t_b^{-\epsilon} & j = c, i \neq 0 \\ t_b^\delta t_j^{-\epsilon} & i = c \\ t_j^{-\epsilon} & i = 0, j \neq c \\ t_b^{-\epsilon} & i = 0, j = c \end{cases}$$

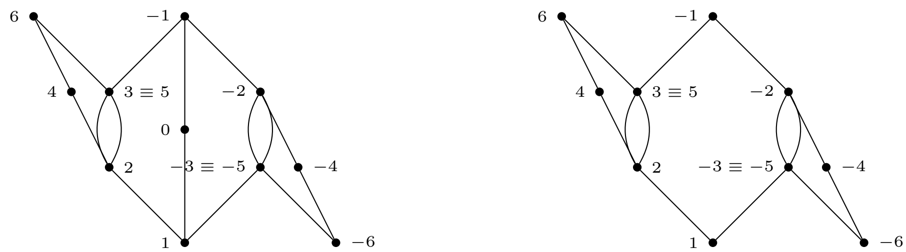
and define a map $\psi^\vee\colon S^{\mathrm{rt}}_{P^\vee} \to R^{\mathrm{rt}}_{P'^\vee}$ by

$$U_{\delta i,\epsilon j} \mapsto \begin{cases} t_i^\delta t_j^{-\epsilon} & i,j \neq c \\ t_b^\delta t_j^{-\epsilon} & i = c \\ t_i^\delta t_b^{-\epsilon} & j = c \end{cases}$$

One shows that $\ker\psi = I_P + (U_{ab} - U_{ac})$ and $\ker\psi^\vee = I_{P^\vee} + (U_{ab} - U_{ac})$. The argument is the same for P and $P^\vee$. To simplify notation, in the remainder of the proof, write J for the kernel, I for the toric ideal, let the variables of the polynomial ring S be U_{ij} and use $\widehat{G}(P)$ to denote $\widehat{G}_B(P)$ or $\widehat{G}_C(P)$, as applicable. Thus, one needs to show that $J = I + (U_{ab} - U_{ac})$.

Recall that I is generated by the cycle binomials of cycles in $\widehat{G}(P)$ not fixed by the involution. Let $U(C)$ be such a cycle binomial. The definition of φ (resp. ψ) ensures that $U(C) \in J$, so $I \subset J$. Both U_{ab} and U_{ac} have the same image in $R^{\mathrm{rt}}_{P'}$, so $(U_{ab} - U_{ac}) \subset J$. Thus $I + (U_{ab} - U_{ac}) \subset J$.

Let P^+ be the directed graph that has the same vertices and edges as $\widehat{G}(P')$, but with the edges $-b \equiv -c \to -a$ and $a \to b \equiv c$ doubled. The argument in the proof of Theorem 4.1.5 shows that J is generated by the cycle binomials corresponding to cycles of P^+ not fixed by



(a) P_C^+ (b) P_C^+

Figure 4.15: P_B^+ and P_C^+ for $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ from Figure 4.11

the involution. (See Figure 4.15 for P^+ for the $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ from Figure 4.11.)

One then needs to show that for every cycle of P^+ not fixed by the involution, its circuit binomial lies in $I + (U_{ab} - U_{ac})$. There are a number of cases.

1. **Suppose C is a cycle in P^+ not fixed by the involution passing through neither $b \equiv c$ nor $-b \equiv -c$.** Then C lifts to a cycle in $\widehat{G}(P)$, meaning $U(C) \in I_P$.

2. **Suppose C is a cycle in P^+ not fixed by the involution, but passing through at least one of $b \equiv c$ and $-b \equiv -c$.** One can partition the edges incident to $b \equiv c$ and $-b \equiv c$ into $E_b \sqcup E_c$ according to whether they lift to edges incident to either b or $-b$ or to either c or $-c$ in $\widehat{G}(P)$.

   (2.a) **Suppose the edges of C incident to $b \equiv c$ or $-b \equiv -c$ all lie in one of E_b and E_c.** Then C lifts to a cycle in $\widehat{G}(P)$, so $U(C) \in I$.

   (2.b) **Suppose C contains an edge from both E_b and E_c.** Assume C has an edge from each of E_b and E_c incident to $b \equiv c$. The other case is symmetric.

   Since (a, b, c) was a notch in $\widehat{G}(P)$, one has that b and c lie in different connected components of $\widehat{G}(P) \setminus \widehat{G}(P)_{\leq a}$, so C must pass through at least one vertex $d \leq a$. Let π_{da} be a saturated chain between d and a. Let C_b be the cycle in P^+ that follows C from $b \equiv c$ to d, then π_{da} from d to a and finishes along the edge in E_b

between a and $b \equiv c$. Let C_c be the cycle in P^+ that follows C from $b \equiv c$ to d, then π_{da} from d to a and finishes along the edge in E_c from a to $b \equiv c$.

(2.b.i) **Suppose C_b and C_c both lift to cycles in $\widehat{G}(P)$ and, with C_b and C_c are oriented so they traverse π_{da} in opposite directions.** One has

$$U(C) = U(C_b)\left(\prod_{e \in W(\pi_{dc})} U_e\right) + U(C_c)\left(\prod_{e \in A(\pi_{bd})} U_e\right) + (U_{ab} - U_{ac})\left(\prod_{e \in W(\pi_{dc})} U_e\right)\left(\prod_{e \in A(\pi_{bd})} U_e\right)\left(\prod_{e \in W(\pi_{da})} U_e\right),$$

where π_{dc} is the portion of C_c between d and c and π_{bd} is the portion of C_b between d and b. Note that it is possible one of C_b and C_c is fixed by the involution, in which case its cycle binomial is zero since $W(C) = A(C)$. Thus, $U(C) \in I + (U_{ab} - U_{ac})$, as desired.

(2.b.ii) **Now suppose at least one of C_b and C_c does not lift to a cycle in $\widehat{G}(P)$.** One can use the previous argument to show both $U(C_b)$ and $U(C_c)$ lie in $I_P + (U_{ab} - U_{ac})$. Without loss of generality, suppose C_b does not lift to a cycle in $\widehat{G}(P)$. (If neither lifts, one repeats the following argument with each cycle.) Since C_b does not lift to a cycle in $\widehat{G}(P)$, it must pass through $-b \equiv -c$, along one edge in E_b and one in E_c. Since $(-a, -b, -c)$ was a notch in $\widehat{G}(P)$, there is $d' \geq -a \in P^+$ which C_b passes through. Then choose a saturated chain between d' and $-a$, call it π'_{da}, with the proviso that if π'_{da} includes a and $b \equiv c$, it includes the edge in C. One can then construct C_{bb} and C_{bc} as in the previous case and the same argument shows that $U(C_b) \in I_P + (U_{ab} - U_{ac})$. (However, if π'_{da} coincides with C, it will be the case that one of C_{bb} and C_{cc} is not genuinely a cycle, rather one will have a chain from (say) b to d' and then return to b down the same chain, in which case $U(C_{bb}) = 0$.)

Thus, $J = I + (U_{ab} - U_{ac})$.

To finish the proof, one sees that

$$\begin{aligned}
R^{\mathrm{rt}}_{P'} &\cong S_P / \ker\psi \\
&\cong S_P/(I_P + (U_{ab} - U_{ac})) \\
&\cong (S_P/I_P)/(\bar{U}_{ab} - \bar{U}_{ac}) \\
&\cong R^{\mathrm{rt}}_P/(t_a^{\mathrm{sgn}(a)} t_b^{-\mathrm{sgn}(b)} - t_a^{\mathrm{sgn}(a)} t_c^{-\mathrm{sgn}(c)}),
\end{aligned}$$

where $t_a = 1$ if $a = 0$, and

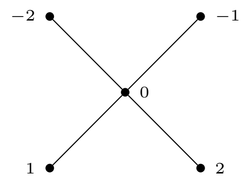
$$\begin{aligned}
R^{\mathrm{rt}}_{P'^\vee} &\cong S_{P^\vee} / \ker\psi^\vee \\
&\cong S_{P^\vee}/(I_{P^\vee} + (U_{ab} - U_{ac})) \\
&\cong (S_{P^\vee}/I_{P^\vee})/(\bar{U}_{ab} - \bar{U}_{ac}) \\
&\cong R^{\mathrm{rt}}_{P^\vee}/(t_a^{\mathrm{sgn}(a)} t_b^{-\mathrm{sgn}(b)} - t_a^{\mathrm{sgn}(a)} t_c^{-\mathrm{sgn}(c)}).
\end{aligned}$$

$\square$

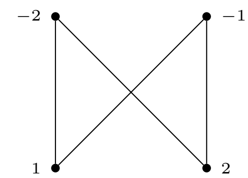
4.2.3 Strongly Planar Signed Posets

Having looked at the notion of opening notches and its impact on R^{rt}_P, one is ready to show that a certain class of signed posets have R^{rt}_P a complete intersection, implying that the numerator of Ψ_P factors.

Definition 4.2.10. A poset is said to be *strongly planar* if, after the addition of a maximal element $\hat{1}$ and a minimal element $\hat{0}$, there is an embedding of its Hasse diagram in $\mathbb{R}^2$ that is planar and has the property that if $a <_P b$, the y-coordinate of a is smaller than that of b. A signed poset $P \subset \Phi_{B_n}$ (resp. $P \subset \Phi_{C_n}$) will be said to be *strongly planar* if $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is strongly planar.



(a) $\widehat{G}_B(P)$



(b) $\widehat{G}_C(P^\vee)$

Figure 4.16: $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ for $P = \{+e_1 + e_2, +e_1, +e_2\}$

As an example, consider $P = \{+e_1+e_2, +e_1, +e_2\}$. Figure 4.16 shows $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$. One sees that $\widehat{G}_B(P)$ is strongly planar, but $\widehat{G}_C(P^\vee)$ is not.

Definition 4.2.11. An embedding of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) in $\mathbb{R}^n$ is said to be *centrally symmetric* if it is fixed by the map $(x, y) \mapsto (-x, -y)$.

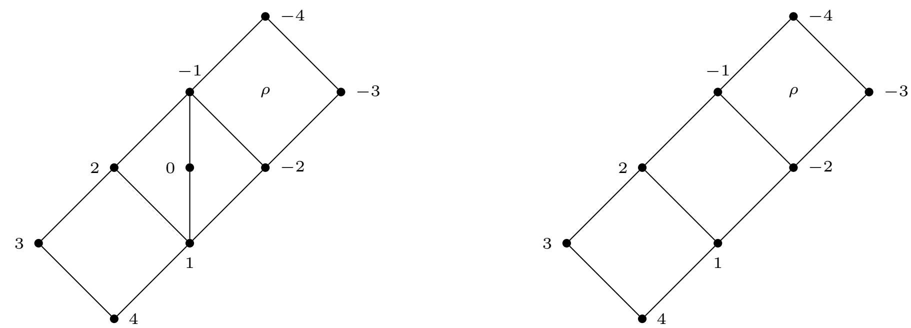
The main result of this section will be the following.

Theorem 4.2.12. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset such that $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has an embedding in $\mathbb{R}^2$ which is strongly planar. Then*

- *$\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has an embedding that is both centrally symmetric and strongly planar,*
- *R_P^{rt} (resp. $R_{P^\vee}^{\mathrm{rt}}$) is a complete intersection, with I_P (resp. $I_{P^\vee}$) generated by the cycle binomials of the cycles in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) defining the faces of the graph, and*
- *one has*

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon})} \quad \text{and}$$
$$\mathrm{Hilb}(R_{P^\vee}^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon})},$$

where ρ runs over all regions enclosed by $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) not fixed by the involution, e runs over the orbits of edges $\delta a \to \epsilon b$ ($a < b$) in $\widehat{G}_C(P)$, x_0 is taken to be



(a) $\widehat{G}_B(P)$ (b) $\widehat{G}_C(P^\vee)$

Figure 4.17: A signed poset with $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ both strongly planar

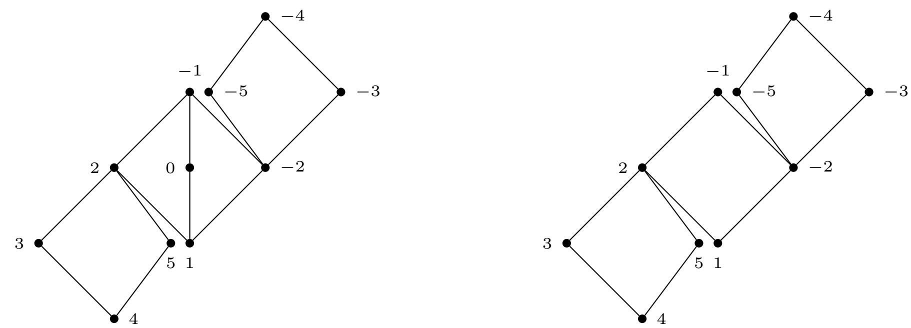
1 *and*

$$\boldsymbol{x}^\rho = x_{\min(\rho)}^{\mathrm{sgn}(\min(\rho))} x_{\max(\rho)}^{-\mathrm{sgn}(\max(\rho))}.$$

The proof of Theorem 4.2.12 is quite involved and requires a number of propositions. First, the idea of the proof is illustrated using the posets in Figure 4.17. The proof is by induction on the number of orbits of regions enclosed by $\widehat{G}(P)$ under the involution. Proposition 4.2.14 will show that one can find at least one "rightmost" region in $\widehat{G}(P)$, such as the region marked by ρ in Figure 4.17. Lemma 4.2.17 gives that the region ρ is not fixed by the involution. Lemma 4.2.19 guarantees that $\widehat{G}(P)$ can be obtained by closing a notch along the left border of ρ. Figure 4.18 shows the $\widehat{G}(P')$ obtained by opening a notch along $(-2,-1)$ and $(1,2)$ in $\widehat{G}(P)$. Proposition 4.2.9 gives that

$$R_P^{\mathrm{rt}} = R_{P'}^{\mathrm{rt}}/(x_1x_2^{-1} - x_5x_2^{-1}) \quad \text{and} \quad R_{P'^\vee}^{\mathrm{rt}} = R_{P'^\vee}^{\mathrm{rt}}/(x_1x_2^{-1} - x_5x_2^{-1}).$$

Typically, one thinks of $R_{P'}^{\mathrm{rt}}$ as graded by $\mathbb{Z}^5$ and R_P^{rt} as graded by $\mathbb{Z}^4$. One can alter the grading of $R_{P'}^{\mathrm{rt}}$ so that $\deg x_1 = \deg x_5 = (1,0,0,0)$, so that $R_{P'}^{\mathrm{rt}}$ is also graded by $\mathbb{Z}^4$ and R_P^{rt} is a quotient of $R_{P'}^{\mathrm{rt}}$ by a homogeneous ideal. Keeping the altered grading in mind, one



(a) $\widehat{G}_B(P')$ (b) $\widehat{G}_C(P^{\vee\prime})$

Figure 4.18: Having opened one notch to obtain P'

then has from Proposition 2.2.12 that

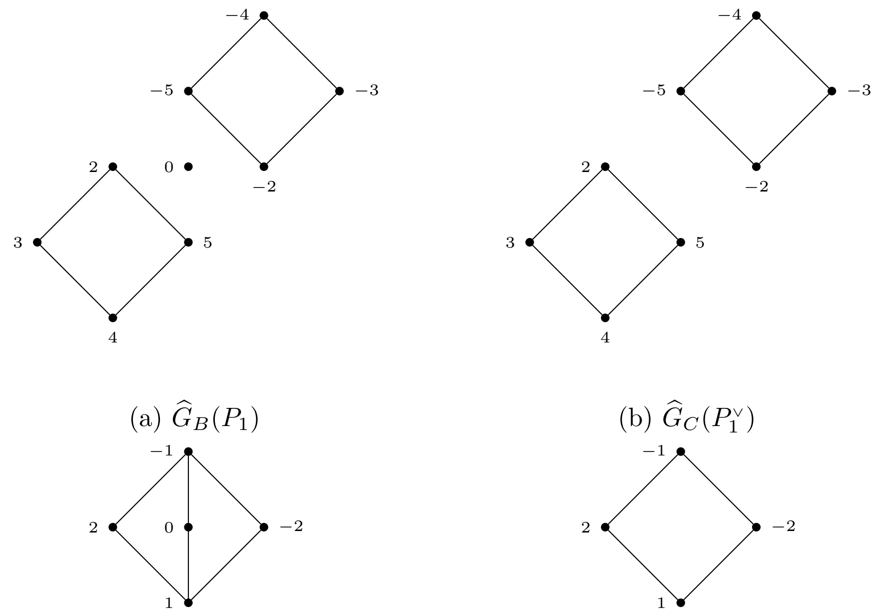
$$\begin{aligned} \mathrm{Hilb}(R^{\mathrm{rt}}_{P}, \boldsymbol{x}) &= (1 - x_1 x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P'}, \boldsymbol{x})|_{x_5 = x_1} \quad \text{and} \\ \mathrm{Hilb}(R^{\mathrm{rt}}_{P^{\vee}}, \boldsymbol{x}) &= (1 - x_1 x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P'^{\vee}}, \boldsymbol{x})|_{x_5 = x_1} \end{aligned} \tag{4.2}$$

$\widehat{G}_B(P')$ and $\widehat{G}_C(P'^{\vee})$ each have two biconnected components, associated to signed posets P_1 and P_2, shown in Figure 4.19. Both P_1 and $P_1^{\vee}$ have a single biconnected component and principal toric ideals generated by $U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}U_{\bar{2}5}$. Then one has

$$\begin{aligned} \mathrm{Hilb}(R^{\mathrm{rt}}_{P_1}, \boldsymbol{x}) &= \frac{(1 - x_4 x_2^{-1})}{(1 - x_4 x_3^{-1})(1 - x_3 x_2^{-1})(1 - x_4 x_5^{-1})(1 - x_5 x_2^{-1})} \quad \text{and} \\ \mathrm{Hilb}(R^{\mathrm{rt}}_{P_1^{\vee}}, \boldsymbol{x}) &= \frac{(1 - x_4 x_2^{-1})}{(1 - x_4 x_3^{-1})(1 - x_3 x_2^{-1})(1 - x_4 x_5^{-1})(1 - x_5 x_2^{-1})}. \end{aligned} \tag{4.3}$$

Looking at Figures 4.19(c) and 4.19(d), one sees that an additional notch can be opened in $\widehat{G}_B(P_2)$, but not in $\widehat{G}_C(P_2^{\vee})$. Opening the notch in $\widehat{G}_B(P_2)$ results in a signed poset P_3 shown in Figure 4.20. Both $\widehat{G}_B(P_3)$ and $\widehat{G}_C(P_2^{\vee})$ each have only one biconnected component and a single cycle. This gives

$$\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2}, \boldsymbol{x}) = (1 - x_1)\mathrm{Hilb}(R^{\mathrm{rt}}_{P_3}, \boldsymbol{x})|_{x_3 = x_1}$$



(a) $\widehat{G}_B(P_1)$ (b) $\widehat{G}_C(P_1^\vee)$

(c) $\widehat{G}_B(P_2)$ (d) $\widehat{G}_C(P_2^\vee)$

Figure 4.19: P' broken into biconnected components

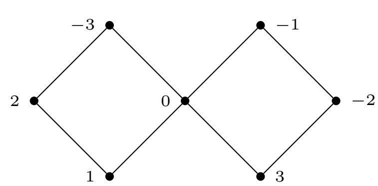


Figure 4.20: $\widehat{G}_B(P_3)$

$$= \left.\frac{(1-x_1)(1-x_1x_3)}{(1-x_1x_2^{-1})(1-x_2x_3)(1-x_3)(1-x_1)}\right|_{x_3=x_1} \tag{4.4}$$
$$= \frac{1-x_1^2}{(1-x_1x_2^{-1})(1-x_1x_2)(1-x_1)}$$

and

$$\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2^\vee}, \boldsymbol{x}) = \frac{(1-x_1^2)}{(1-x_1x_2^{-1})(1-x_1x_2)} \tag{4.5}$$

Since P_1 and P_2 are the biconnected components of P', from Proposition 4.2.2, one has that $R^{\mathrm{rt}}_{P'} = R^{\mathrm{rt}}_{P_1} \otimes R^{\mathrm{rt}}_{P_2}$ and $R^{\mathrm{rt}}_{P^{\vee\prime}} = R^{\mathrm{rt}}_{P_1^\vee} \otimes R^{\mathrm{rt}}_{P_2^\vee}$, meaning

$$\mathrm{Hilb}(R^{\mathrm{rt}}_{P'}, \boldsymbol{x}) = \mathrm{Hilb}(R^{\mathrm{rt}}_{P_1}, \boldsymbol{x})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2}, \boldsymbol{x}) \quad \text{and}$$
$$\mathrm{Hilb}(R^{\mathrm{rt}}_{P^{\vee\prime}}, \boldsymbol{x}) = \mathrm{Hilb}(R^{\mathrm{rt}}_{P_1^\vee}, \boldsymbol{x})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2^\vee}, \boldsymbol{x}).$$

Combining (4.2), (4.3), (4.4) and (4.5) gives

$$\begin{aligned}
\mathrm{Hilb}(R^{\mathrm{rt}}_{P}, \boldsymbol{x}) &= (1-x_1x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P}, \boldsymbol{x})|_{x_5=x_1} \\
&=(1-x_1x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_1}, \boldsymbol{x})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2}, \boldsymbol{x})|_{x_5=x_1} \\
&= \left.\frac{(1-x_1x_2^{-1})(1-x_4x_2^{-1})(1-x_1^2)}{(1-x_4x_3^{-1})(1-x_3x_2^{-1})(1-x_4x_5^{-1})(1-x_5x_2^{-1})(1-x_1x_2^{-1})(1-x_1x_2)(1-x_1)}\right|_{x_5=x_1} \\
&=\frac{(1-x_4x_2^{-1})(1-x_1^2)}{(1-x_4x_3^{-1})(1-x_3x_2^{-1})(1-x_4x_1^{-1})(1-x_1x_2^{-1})(1-x_2x_4)(1-x_1x_2)(1-x_1)}.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Hilb}(R^{\mathrm{rt}}_{P^\vee},\boldsymbol{x}) &= (1-x_1x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P^{\vee\prime}},\boldsymbol{x})|_{x_5=x_1}\\
&=(1-x_1x_2^{-1})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_1^\vee},\boldsymbol{x})\mathrm{Hilb}(R^{\mathrm{rt}}_{P_2^\vee},\boldsymbol{x})|_{x_5=x_1}\\
&= \left.\frac{(1-x_1x_2^{-1})(1-x_4x_2^{-1})(1-x_1^2)}{(1-x_4x_3^{-1})(1-x_3x_2^{-1})(1-x_4x_5^{-1})(1-x_5x_2^{-1})(1-x_1x_2^{-1})(1-x_1x_2)}\right|_{x_5=x_1}\\
&=\frac{(1-x_1x_2^{-1})(1-x_4x_2^{-1})(1-x_1^2)}{(1-x_4x_3^{-1})(1-x_3x_2^{-1})(1-x_4x_1^{-1})(1-x_1x_2^{-1})(1-x_1x_2^{-1})(1-x_1x_2)}\\
&=\frac{(1-x_4x_2^{-1})(1-x_1^2)}{(1-x_4x_3^{-1})(1-x_3x_2^{-1})(1-x_4x_1^{-1})(1-x_1x_2^{-1})(1-x_1x_2^{-1})(1-x_1x_2)}.
\end{aligned}$$

The edges which correspond to the notches that are opened form a set of *disconnecting chains* splitting $\widehat{G}(P)$ into biconnected components.

Definition 4.2.13. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. A *disconnecting chain* of $\widehat{G}(P)$ is a chain $c_1 \lessdot c_2 \lessdot \cdots \lessdot c_k$ such that removing $c_1 \lessdot \cdots \lessdot c_k$ and $-c_k \lessdot \cdots \lessdot -c_1$ breaks $\widehat{G}(P)$ into three connected components.

The first step in the proof is to locate a region to work with.

Proposition 4.2.14. *Suppose Q is a strongly planar poset whose Hasse diagram is connected. Then Q has a region ρ such that any vertex other than the maximum and minimum element of ρ in the right border of ρ is in the border of no other region. Such a region is called a* rightmost region.

By considering a strongly planar poset, one has the luxury of assuming a given strongly planar embedding, allowing sensible notions of “left” and “right”.

Definition 4.2.15. Suppose Q is a strongly planar poset. The *rightmost cover* of $x \in Q$ is the y that covers x such that if one traversed a small circle around x counterclockwise (starting from the bottom, say) one passes the edge leading to y last. The *rightmost lower cover* is the $z \lessdot x$ such that the edge leading to z is encountered first when traveling clockwise from the top of the circle.

Proof of Proposition 4.2.14. Without loss of generality, one may assume that every vertex in Q has degree at least two, since one makes this assumption of $\widehat{G}(P)$. Since Q is strongly planar, the poset $\widehat{Q}$ obtained by adding a $\hat{0}$ and $\hat{1}$ to Q is also planar. Construct a saturated chain $\hat{0} \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot \hat{1}$ such that c_{i+1} is the rightmost cover of c_i for all i and c_1 is the rightmost cover of $\hat{0}$.

Let $a = c_i$, where i is maximal such that c_i is covered by at least two elements. (Since c_1 is a minimal element of Q, and Q has been assumed to have no vertices of degree 1, it has at least two covers, so such an a exists.) Construct a chain $a \lessdot b_1 \lessdot \cdots \lessdot b_\ell \lessdot \hat{1}$ where b_1 is the cover of a that is rightmost but one and b_{i+1} is the rightmost cover of b_i.

Claim: c_k is the rightmost lower cover of $\hat{1}$ in $\widehat{Q}$.

Suppose not and $d \lessdot \hat{1}$ is to the right of c_k. Then d must lie above some minimal element of Q, say f. Since Q is strongly planar, a maximal chain from f to d must intersect $c_1 \lessdot \cdots \lessdot c_k$. Let j be maximal such that c_j is in this chain from f to d. Since $d \neq c_k$, one must have that $j \neq k$. Since d is to the right of c_k, the chain must continue along the rightmost cover of c_j. However, this is c_{j+1}, contradicting the maximality of j.

Claim: $b_\ell = c_k$.

Suppose not. Then c_k has degree ≥ 2 and it was maximal in Q, so it must cover some $d \neq c_{k-1}$. Then Q has a minimal element a' such that $a' < d$. As Q is strongly planar, a saturated chain from a' to d must include either some b_i or c_j, $j < k$. If the chain includes b_i, then there must be some m where b_{m+1} is not the rightmost cover of b_m, a contradiction. If c_j is in the chain, but no b_i, then, since $d \neq c_k$, there must be an i such that $c_n > a$ and c_n has more than two covers, contradicting that $a = c_i$ was maximal with this property.

Let ρ be the region enclosed by the b_i and c_j. Note that $b_\ell = c_k$ need not be the maximal element of ρ. However, by construction, a will be the minimal element of ρ.

Claim: The c_j with $c_j > a$ are rightmost in Q, i.e. they are not in the left border of any region.

Suppose c_j is on the right border of ρ (and is not $\max(\rho)$ or $\min(\rho)$), that c_j is on the left border of some region σ and that j is minimal such that this is the case. Then, since c_{j+1} is the rightmost cover of c_j, the edge (c_j, c_{j+1}) must be in the left border of σ. Consequently, c_j cannot be the minimal element of σ. Then, by the minimality of j, there is some $d \lessdot c_j$ such that d is to the right of c_{j-1}, as depicted in Figure 4.21 The region σ must have a

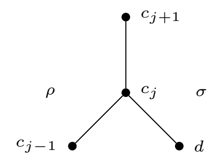


Figure 4.21: The situation in the last claim of the proof of Proposition 4.2.14

minimum element, call it f. Then $f \geq g$ for some g that is minimal in Q. Then, since Q is strongly planar and c_1 is the rightmost minimal element of Q, a saturated chain from g to f must pass through some c_i. But then, since c_{i+1} is the rightmost cover of c_i, one must have that f is one of the c's, a contradiction. $\square$

One can now use the previous result to prove a few lemmas specific to $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$.

Lemma 4.2.16. *Suppose $P^\vee \subset \Phi_{C_n}$ is a Φ_{C_n}-signed poset such that $\widehat{G}_C(P^\vee)$ is strongly planar. Then there is at most one i such that $\pm 2e_i$ corresponds to an edge in $\widehat{G}_C(P^\vee)$.*

Proof. Suppose $P^\vee \subset \Phi_{C_n}$ is a signed poset such that $\widehat{G}_C(P^\vee)$ is strongly planar and $i \lessdot -i$ and $j \lessdot -j$. Then one must have $i < -j$ and $j < -i$. Then if $\widehat{G}_C(P^\vee)$ is embedded in the plane (so that if $a < b$, then $y_a < y_b$), there must be a path from i to $-j$ and j to $-i$, but these paths must intersect. For $\widehat{G}_C(P^\vee)$ to be planar, they must intersect at a vertex, say k. But then $\widehat{G}_C(P^\vee)$ would cease to be the Hasse diagram of a poset, as the relation $i < -i$ would follow by transitivity from $i < k$ and $k < -i$. Thus, such a $\widehat{G}_C(P^\vee)$ cannot be strongly planar. $\square$

Lemma 4.2.17. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset such that $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is strongly planar. Then, if $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) encloses more than one region, a cycle defining a rightmost region of $\widehat{G}(P)$ is not fixed by the involution.*

Proof. Suppose C is a cycle in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) that encloses a rightmost region, ρ. Suppose C is fixed by the involution. Let σ be a region to the immediate left of ρ. The type B and type C cases are slightly different.

First, suppose $P \subset \Phi_{B_n}$. There are two cases.

1. **Suppose 0 is not in the left border of ρ.** Let e be an edge that lies in the left border of ρ and the right border of σ. Since 0 is not in the left border of ρ, an edge and its image under the involution cannot both be in the left border. Therefore, since ρ is fixed by the involution, $\iota(e)$ must lie in the right border of ρ. But $\iota(e)$ is also in the border of $\iota(\sigma)$, contradicting that ρ is a rightmost region.

2. **Suppose 0 is in the left border of ρ.** Then if j is in the left border, $-j$ must also be in the left border, since ρ is fixed by the involution and $\pm j < 0 < \mp j$. Then the left border is fixed by the involution. Since ρ itself is fixed by the involution, this means that the right border must also be fixed and thus symmetric, a contradiction, since 0 is in the left border, not the right.

Next, consider $P^\vee \subset \Phi_{C_n}$. There are two cases.

1. **Suppose there is no edge of the form $(i, -i)$ in the left border of ρ.** Since σ is to the left of ρ, there is some edge e that lies in the left border of ρ and the right border of σ. Since ρ is fixed by the involution, $\iota(e)$ lies in the border of ρ. Since there is no edge of the form $(i, -i)$ in the left border, it is not symmetric and $\iota(e)$ cannot be in the left border, so it is in the right border. But, $\iota(e)$ is in the border of $\iota(\sigma)$, meaning ρ cannot be a rightmost region, a contradiction.

2. **Suppose there is an edge $(i, -i)$ in the left border of ρ.** Then, by the definition of $\widehat{G}_C(P^\vee)$, if j is in the left border of ρ, then $-j$ is also in the left border of ρ. From Lemma 4.2.16, one knows that the right border of ρ does not include an edge $(k, -k)$, so there must be some k in the right border such that $-k$ is not in the right border. (In fact, the absence of such an edge means only one of k and $-k$ is in the right border for any k that is not maximal or minimal in C.) Then $\iota(k) = -k$ is a vertex in $\iota(C)$, which cannot be C, so C is not fixed by the involution.

□

Proposition 4.2.18. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. Then $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has a strongly planar embedding if and only if it has a centrally symmetric strongly planar embedding.*

Proof. One needs to show only that if $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is strongly planar, it has a centrally symmetric strongly planar embedding. Instead, prove a slightly stronger statement, namely that if $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has a strongly planar embedding, it also has a centrally symmetric strongly planar embedding such that the outer border of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is the same in both the first strongly planar embedding and the centrally symmetric strongly planar embedding.

Induct on the number of regions in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$). There are two base cases.

First, suppose $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) consists of a single region, bounded by a single cycle. One needs to address type B and type C separately.

- In type C, consider $\widehat{G}_C(P^\vee)$. Since it is strongly planar, it has a single maximum and a single minimum, which must be i and $-i$.

  Claim: There is no $j \neq i$ such that $j < -j$ or $-j < j$ in $\widehat{G}_C(P^\vee)$.

  Suppose not. Since $\widehat{G}_C(P)$ consists of a single cycle, one has that either $i < j < -j < -i$, in which case one border of $\widehat{G}_C(P^\vee)$ is fixed by the involution, so $\widehat{G}_C(P^\vee)$ must have more than one region, a contradiction, or j lies in the left border of $\widehat{G}_C(P)$ and $-j$ lies in the right border of $\widehat{G}_C(P^\vee)$. In the latter case, one must have that the left border of $\widehat{G}_C(P^\vee)$ is sent to the right border under the involution, i.e. if $i \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot -i$ is the left border, then $i \lessdot -c_k \lessdot \cdots \lessdot -c_1 \lessdot -i$ is the right border. However, it is then impossible that $j < -j$ since $\widehat{G}_C(P^\vee)$ encloses a single region.

Thus, when $\widehat{G}_C(P^\vee)$ is strongly planar and encloses a single region, $\widehat{G}_C(P^\vee)$ can be embedded centrally symmetrically by evenly spacing the vertices around a circle centered at the origin with $-i$ at the top and $-i$ at the bottom.

- In type B, consider $\widehat{G}_B(P)$. Since $i < -i$, one must have $i < 0 < -i$, meaning either the left or right border of $\widehat{G}_B(P)$ is symmetric about 0, so it is fixed by the involution. But then $\widehat{G}_B(P)$ must have more than one region, contradicting the assumption.

Next, consider the second base case where $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) consists of two regions. From Lemma 4.2.17, one knows that the rightmost region ρ is not fixed by the involution. Therefore, the border between the two regions consists of a chain fixed by the involution and the two regions are exchanged by the involution. One then has the scenario depicted in Figure 4.22. One sees that $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) (when translated appropriately) is

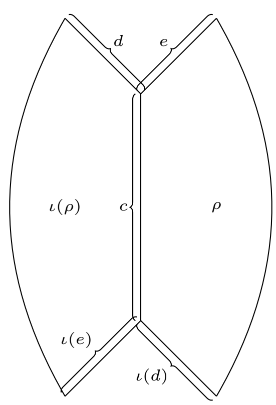


Figure 4.22: Two regions exchanged by the involution separated by a chain, c, fixed by the involution

centrally symmetric and strongly planar.

Now suppose $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) encloses more than two regions. Then, by Proposition 4.2.14, it has a rightmost region, call it ρ. The right border of ρ is defined by some chain $\min(\rho) \lessdot c_1 \lessdot \cdots \lessdot c_k \lessdot \max(\rho)$. Since $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is a poset and ρ is enclosed by a cycle, $k \geq 1$. Since ρ is rightmost, by definition each edge in its right border lies in no cycle defining a region other than ρ. The same then must be true of the image of the right border of ρ in $\iota(\rho)$. Deleting $c_1, \ldots, c_k, -c_1, \ldots, -c_k$ and the incident edges gives $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P'^\vee)$) for a signed poset P'

Since $\widehat{G}(P)$ was strongly planar, $\widehat{G}(P')$ is strongly planar. By induction, $\widehat{G}(P')$ has a centrally symmetric strongly planar embedding. The left border of ρ is now part of the right border of $\widehat{G}(P')$, meaning the chain $c_1 \lessdot \cdots \lessdot c_k$ and be attached to form a new region on the right of $\widehat{G}(P')$ and similarly for $-c_k \lessdot \cdots \lessdot -c_1$ on the left. Since $\pm\min(\rho)$ and $\pm\max(\rho)$ are positioned so as to be centrally symmetric, it is possible to place the chains in such a way as to preserve central symmetry. Replacing the chains in $\widehat{G}(P')$ gives a centrally symmetric strongly planar embedding of $\widehat{G}(P)$. $\square$

Lemma 4.2.19. *Suppose $P \subset \Phi_{B_n}$ (resp. $P \subset \Phi_{C_n}$) is a signed poset such that $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is centrally symmetric and strongly planar. Suppose ρ is a rightmost region of $\widehat{G}(P)$ and $c_1 \lessdot \cdots \lessdot c_k$ is the portion of the left border of ρ which is part of the right border of some other region(s), with $k \geq 2$ and $c_k \neq -c_{k-1}$ (or $c_1 \neq -c_2$). Then $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) is obtained from $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P')$) for some signed poset P' by closing a signed notch (c_{k-1}, c_k, c'_k) and $(-c_{k-1}, -c_k, -c'_k)$ or a signed notch (c_2, c_1, c'_1) and $(-c_2, -c_1, -c'_1)$.*

Proof. Begin by observing that in the type B case, one of c_1 and c_k is nonzero, and in the type C case, at least one of $c_k \neq -c_{k-1}$ and $c_1 \neq -c_2$ holds. Without loss of generality, assume $c_k \neq 0$ in $\widehat{G}_B(P)$ and $c_k \neq -c_{k-1}$ in $\widehat{G}_C(P^\vee)$. (The argument in the other cases is symmetric.)

Construct $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P')$) by replacing c_k by c_k and c'_k, with an edge between c_{k-1}

and each of c_k and c'_k, with the edges incident to c_k in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) that are not in the border of ρ moved to be incident to c'_k. Replace $-c_k$ by $-c_k$ and $-c'_k$ and partition the edges incident to $-c_k$ in $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) in the same way. By construction, $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P'c)$) will be centrally symmetric and strongly planar and have an involution sending $i < j$ to $-j < -i$ (and fixing 0 in the type B case). Then to show P' is really a signed poset, one needs to check the conditions in Proposition 3.1.5 and Proposition 3.1.6.

First, consider the type B case. Suppose $i < -i$ in $\widehat{G}_B(P')$. Since opening a notch does not introduce relations, one must have that $i < -i$ in $\widehat{G}_B(P)$, so $i < 0 < -i$ in $\widehat{G}_B(P)$. Suppose one does not have $-i > 0$ in $\widehat{G}_B(P')$. There are two cases where $0 < -i$ in $\widehat{G}_B(P)$ could fail to lift to $\widehat{G}_B(P')$.

- **Suppose $-i > c_k$ and $0 < c'_k$, but $0 \not< -i$.**

  By construction (keeping in mind that $\widehat{G}_B(P)$ is centrally symmetric and strongly planar), 0 must be in the right border of ρ and $c'_k = \max(\rho)$. However 0 being in the right border contradicts central symmetry.

- **Suppose $-i > -c_k$ and $0 < -c'_k$, but $0 \not< -i$.**

  By construction, one has that 0 is in the part of the border of $\iota(\rho)$ in both $\widehat{G}_B(P)$ and in $\widehat{G}_B(P')$ not shared by any other region. Thus, since $i < 0$ in $\widehat{G}_B(P)$, one must have $i < 0$ in $\widehat{G}_B(P')$, contradicting that $i < -i$ in $\widehat{G}_B(P')$.

In the type C case, one must check that if $i < -i$ and $j < -j$ in $\widehat{G}_C(P')$, then $i < -j$ and $j < -i$.

First, observe that if $i < -i$ in $\widehat{G}_C(P')$, then $i < -i$ in $\widehat{G}_C(P^\vee)$. (If $i = c'_k$ in $\widehat{G}_C(P')$, then $c_k < -c_k$ in $\widehat{G}_C(P^\vee)$.) Now suppose that $i < -i$ and $j < -j$ in $\widehat{G}_C(P')$. Then $i < -j$ and $j < -i$ in $\widehat{G}_C(P^\vee)$ (possibly vacuously if $i = j$ in $\widehat{G}_C(P^\vee)$). Should these relations fail to lift to $\widehat{G}_C(P')$, one is in one of the following two cases.

1. **Suppose $i \leq c_k$ and $-j \geq c'_k$.**

There are several subcases.

(1.a) **Suppose $i < c_{k-1}$.** Then $i < -j$ and one is done.

(1.b) **Suppose $i < c_k$, but $i \not< c_{k-1}$.** In other words, i is on the right border of ρ and $c_k = \max(\rho)$. Since ρ is not fixed by the involution and is a right region, $-i$ is not on the right border of ρ. (If it were, either ρ would contain an edge $(k, -k)$ and thus not be a right region, or $i = \min(\rho)$, $-i = \max(\rho)$ and ρ would be cut out by a cycle fixed by the involution, since $\max(\rho)$ has degree 2, by construction.) Therefore, one cannot have that $i < -i$ in $\widehat{G}_C(P^\vee)$, a contradiction.

(1.c) **Suppose $i = c_k$.** Then $c_k < -c_k$, but, by construction of $\widehat{G}_C(P'^\vee)$, elements in the border of ρ other than $c_1 \lessdot \cdots \lessdot c_{k-1}$ are not in the border of any other region in $\widehat{G}_C(P')$ and $-c_k$ is in the border of $\iota(\rho)$, a contradiction, a contradiction.

2. **Suppose $i \leq c'_k$ and $-j \geq c_k$.** In this case, $-j$ must be on the outer border of ρ, as it is not one of $c_1 \lessdot \cdots \lessdot c_{k-1}$. Then, if $j < -j$, one has that either $-c_k < c_k$ or $-c_1 < c_1$. In the latter case, it follows that $-c_k < c_k$. Therefore, without loss of generality, suppose $-j = c_k$. Consider two cases.

(2.a) **There is a chain from $-c_k$ to c_k which is not fixed by the involution.** This chain and its image form a cycle fixed by the involution, having c_k as its maximum and $-c_k$ as its minimum. But then c_k, which has degree two, must cover two elements in this cycle, meaning $c_k = \max(\rho)$ and this cycle is the one that cuts out ρ. This is a contradiction, since ρ is not fixed by the involution.

(2.b) **There is a unique chain from $-c_k$ to c_k.** By construction of $\widehat{G}_C(P')$, this chain passes through c_{k-1} and $-c_{k-1}$ and extends uniquely to a maximal chain from $\min(\iota(\rho))$ to $\max(\rho)$. On the other hand, since $i < -i$, there is a chain from $-i$ to i. Since $\widehat{G}_C(P^\vee)$ is centrally symmetric and strongly planar, this chain must pass through the chain from $\min(i\rho))$ to $\max(\rho)$, meaning $-c_{k-1} < -i$ and

$i < c_{k-1}$, so $-c_k < -i$ and $i < c_k$, as desired.

Lastly, one must show that (c_{k-1}, c_k, c'_k) and $(-c_{k-1}, -c_k, -c'_k)$ forms a signed notch in $\widehat{G}(P')$. However, this follows from the fact $c_1 \lessdot \cdots \lessdot c_{k-1}$ and $-c_{k-1} \lessdot \cdots \lessdot -c_1$ are disconnecting chains in $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P')$).

$\square$

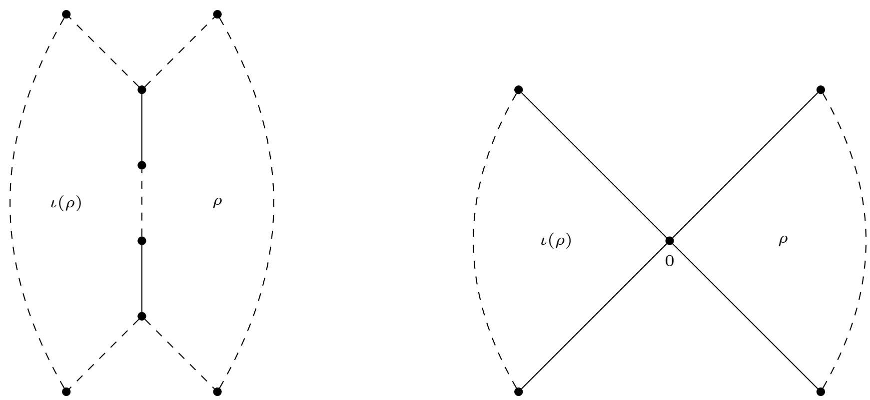
One is finally ready to prove Theorem 4.2.12.

Proof of Theorem 4.2.12. Begin by realizing one may assume courtesy of Proposition 4.2.18 that $\widehat{G}(P)$ is embedded in $\mathbb{R}^n$ so as to be centrally symmetric and strongly planar. Next, recall that a centrally symmetric strongly planar $\widehat{G}_B(P)$ contains at least two regions. If a centrally symmetric strongly planar $\widehat{G}_C(P^\vee)$ has no region not fixed by the involution, central symmetry guarantees it consists of one region, cut out by a cycle fixed by the involution. This fixed cycle is the only cycle of $\widehat{G}_C(P^\vee)$, so the toric ideal $I^{\mathrm{rt}}_{P^\vee}$ is the zero ideal and $R^{\mathrm{rt}}_{P^\vee}$ is a complete intersection.

Thus, one may assume $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has at least one region not fixed by the involution. Induct on the number of two-element orbits of regions under the involution of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$). Since $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P)$) is centrally symmetric and strongly planar, it has a rightmost region ρ and ρ is not fixed by the involution.

As a base case, consider the case where $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has only one orbit. The case where $\widehat{G}_C(P^\vee)$ consists of a single orbit of one region has already been addressed. Thus, one may assume in both the $\widehat{G}_B(P)$ and $\widehat{G}_C(P^\vee)$ cases that there are precisely two regions, which are exchanged by the involution. Call them ρ and $\iota(\rho)$, with ρ being the right region. There are two cases.

1. **The regions ρ and $\iota(\rho)$ share an edge and so intersect along a chain.** (See Figure 4.23(a).)



(a) Case 1 (b) Case 2

Figure 4.23: The first two cases of the proof of Theorem 4.2.12

Then $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$) has three cycles: the cycle defining ρ, its image under the involution, which defines $\iota(\rho)$, and the cycle consisting of the "outer border" of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$), i.e. those edges surrounding only one of ρ and $\iota(\rho)$. This last cycle must be fixed by the involution, as the involution exchanges the cycles defining ρ and $\iota(\rho)$ while fixing the border between them. Then I_P^{rt} (resp. $I_{P^\vee}^{\mathrm{rt}}$) is generated by the cycle defining ρ and, as the toric ideal is then principal, R_P^{rt} (resp. $R_{P^\vee}^{\mathrm{rt}}$) is a complete intersection.

2. **The regions ρ and $\iota(\rho)$ do not share an edge and instead share a single vertex.** (See Figure 4.23(b). Note that this case only arises in $\widehat{G}_B(P)$ (where the shared vertex is 0).

   Then the cycle defining ρ and its imagine under the involution, which defines $\iota(\rho)$, are the only cycles in $\widehat{G}_B(P)$, so I_P^{rt} is a principal ideal and R_P^{rt} is a complete intersection.

Suppose $\widehat{G}(P)$ has $n > 1$ orbits of regions and the result holds for posets P' such that $\widehat{G}(P')$ has k orbits of regions not fixed by the involution. Let ρ be a rightmost region of $\widehat{G}(P)$. Let $c_1 \lessdot \cdots \lessdot c_k$ be the chain defining the left border of ρ. Note that the removal of

$c_1 \lessdot \cdots \lessdot c_k$ would disconnect $\widehat{G}(P)$.

Let $c_{j_1} \lessdot \cdots \lessdot c_{j_\ell}$ be the portion of the border of ρ such that c_{j_1} is the minimal vertex in the border that lies in the border of more than one region and c_{j_ℓ} is the maximal vertex in the border that lies in the border of more than one region. (Equivalently, c_{j_1} and c_{j_ℓ} are minimal and maximal, respectively, elements in the chain having degree ≥ 3. See Figure 4.24.) Induct on ℓ.

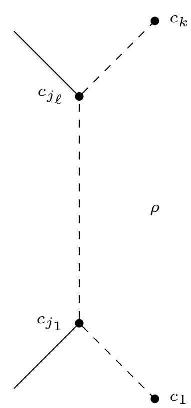


Figure 4.24: The chain $c_{j_1} \lessdot \cdots \lessdot c_{j_\ell}$ in the proof of Theorem 4.2.12.

Postpone discussing the base case and suppose $\ell \geq 3$.

Claim: P is obtained by closing a notch in some signed poset P' such that $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P'^{\vee})$) has a right region whose left border is $c_1 \lessdot \cdots \lessdot c_k$ such that $c_{j_{\ell-1}}$ is the maximal element along the left border that is in the border of more than one region.

There are three cases.

1. **Suppose** $c_{j_\ell - 1} \neq -c_{j_\ell}$ **and** $c_{j_\ell - 1} \neq 0$**.**

   By Lemma 4.2.19, there is a signed poset P' such that P is obtained from P' by closing a signed notch $(c_{j_\ell - 1}, c'_{j_\ell}, c_{j_\ell})$ and $(-c_{j_\ell - 1}, -c'_{j_\ell}, -c_{j_\ell})$.

   By construction, $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P')$) has a right region whose left border is $c_1 \lessdot \cdots \lessdot c_k$, such that $c_{j_\ell - 1}$ is the maximal element along the left border that is in the border of more than one region. Then $\widehat{G}_B(P')$ (resp. $\widehat{G}_C(P')$) also has a right region whose right border is $-c_k \lessdot \cdots \lessdot -c_1$, such that $-c_{j_\ell - 1}$ is the minimal element along the right border that is in the border of more than one region.

The result then follows by induction.

2. **Suppose $c_{\ell-1} = -c_\ell$. This case can only arise in type C. Then since $\ell \geq 3$, from Lemma 4.2.16, one knows that $c_1 \neq -c_2$.**

   Once again, from Lemma 4.2.19, there is a signed poset P' such that $\widehat{G}_C(P^\vee)$ is obtained from $\widehat{G}_C(P')$ by closing the signed notch comprised of $(-c_{j_2}, -c_{j_1}, -c'_{j_1})$ and $(c_{j_2}, c_{j_1}, c'_{j_1})$. By construction, $\widehat{G}_C(P')$ has a right region whose left border is $c_1 \lessdot \cdots \lessdot \cdot c_k$, such that $c_{j_2} \lessdot \cdots \lessdot c_{j_\ell}$ is the shortest saturated chain including all vertices in the left border of this region that lie in the right border of another region.

   The result then follows by induction.

3. **Suppose $c_{j_\ell} = 0$. This case only arises in type B. Since $\ell \geq 3$, one knows that $c_{j_1} \neq 0$.**

   Then, from Lemma 4.2.19, one has a signed poset $P' \subset \Phi_{B_n}$ such that $\widehat{G}_B(P)$ is obtained from $\widehat{G}_B(P')$ by closing a signed notch $(-c_{j_2}, -c_{j_1}, -c'_{j_1})$ and $(c_{j_2}, c_{j_1}, c'_{j_1})$ By construction, $\widehat{G}_B(P')$ has a right region whose right left border is $c_1 \lessdot \cdots \lessdot \cdot c_k$, such that $c_{j_2} \lessdot \cdots \lessdot c_{j_\ell}$ is the shortest saturated chain including all the vertices in the left border of this region that lie in the right border of another region.

   The result then follows by induction.

For the base case $\ell = 2$, address types B and C separately. In type B, there are two cases.

1. **Neither c_1 nor c_2 is 0.**

   From Lemma 4.2.19, one has a signed poset P' such that $\widehat{G}_B(P)$ is obtained from $\widehat{G}_B(P')$ by closing a signed notch (c_1, c_2, c'_2) and $(-c_1, -c_2, -c'_2)$. Then $\widehat{G}_B(P')$ has a rightmost region ρ such that the left border of ρ has precisely one vertex in the border of another region and similarly for the right border of $\iota(\rho)$. Then ρ and $\iota(\rho)$ form a biconnected component in $\widehat{G}(P')$.

2. **One of c_1 and c_2 is 0. Without loss of generality, assume $c_1 = 0$.**

   One applies Lemma 4.2.19 once again, and one has a signed poset P' such that $\widehat{G}_B(P)$ is obtained from $\widehat{G}_B(P')$ by closing a signed notch (c_1, c_2, c_2') and $(-c_1, -c_2, -c_2')$. Then $\widehat{G}_B(P')$ has a rightmost region ρ such that the left border of ρ has precisely one vertex in the border of another region and similarly for the right border of $\iota(\rho)$. Then ρ and $\iota(\rho)$ form a biconnected component in $\widehat{G}(P')$.

In type C, there are two cases.

1. **Suppose $c_1 \neq -c_2$.**

   In this case, Lemma 4.2.19 applies and there is a signed poset P' such that $\widehat{G}_C(P^\vee)$ is obtained from $\widehat{G}_C(P')$ by closing a signed notch (c_1, c_2, c_2') and $(-c_1, -c_2, -c_2')$. Then $\widehat{G}_C(P')$ has a rightmost region ρ such that the left border of ρ has precisely one vertex in the border of another region and similarly for the right border of $\iota(\rho)$. Then ρ and $\iota(\rho)$ form a biconnected component in $\widehat{G}_C(P')$.

2. **Suppose $c_1 = -c_2$.**

   In this case, one has that (c_1, c_2) is the only edge of either ρ or $\iota(\rho)$ shared by another region. Since (c_1, c_2) is of the form $(i, -i)$, it must be shared by ρ and $\iota(\rho)$ (and, since $\widehat{G}(P)$ is strongly planar, by no other region). Consequently, ρ and $\iota(\rho)$ must form an entire biconnected component of $\widehat{G}_C(P^\vee)$.

In each of these cases, the rightmost region ρ and its image under the involution form a biconnected component of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$). Let P_1 be this biconnected component and P_2 correspond to the rest of $\widehat{G}_B(P)$ (resp. $\widehat{G}_C(P^\vee)$). Then, by the same argument as in the proof of Proposition 4.2.2, one has

$$R_P^{\mathrm{rt}} = R_{P_1}^{\mathrm{rt}} \otimes_k R_{P_2}^{\mathrm{rt}} \quad \text{and} \quad R_{P^\vee}^{\mathrm{rt}} = R_{P_1}^{\mathrm{rt}} \otimes_k R_{P_2}^{\mathrm{rt}}.$$

One has that $\widehat{G}(P_2)$ is centrally symmetric and strongly planar since $\widehat{G}(P)$ was and $\widehat{G}(P_2)$ has one fewer region not fixed by the involution than $\widehat{G}(P)$, so the result follows by induction for $R_{P_2}^{\mathrm{rt}}$. One then has

$$I_P^{\mathrm{rt}} = (U(C_\rho)) \oplus I_{P_2}^{\mathrm{rt}} = (U(C_\rho), U(C_{\sigma_1}), \dots, U(C_{\sigma_{n-1}}))$$

and

$$I_{P^\vee}^{\mathrm{rt}} = (U(C_\rho)) \oplus I_{P_2}^{\mathrm{rt}} = (U(C_\rho), U(C_{\sigma_1}), \dots, U(C_{\sigma_{n-1}})),$$

where $\sigma_1, \dots, \sigma_{n-1}$ are the orbits of regions of $\widehat{G}(P_2)$ not fixed by the involution. Then

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \mathrm{Hilb}(R_{P_1}^{\mathrm{rt}}, \boldsymbol{x})\mathrm{Hilb}(R_{P_2}^{\mathrm{rt}}, \boldsymbol{x}) \quad \text{and} \quad \mathrm{Hilb}(R_{P^\vee}^{\mathrm{rt}}, \boldsymbol{x}) = \mathrm{Hilb}(R_{P_1}^{\mathrm{rt}}, \boldsymbol{x})\mathrm{Hilb}(R_{P_2}^{\mathrm{rt}}, \boldsymbol{x}).$$

Since, in each case, $I_{P_1}^{\mathrm{rt}}$ is a principal ideal and ρ has a single maximum and a single minimum, one knows that

$$\mathrm{Hilb}(R_{P_1}^{\mathrm{rt}}, \boldsymbol{x}) = \frac{(1 - \boldsymbol{x}^\rho)}{\prod_{e \in C_\rho}(1 - x_a^\delta x_b^{-\epsilon})},$$

where the product in the denominator runs over edges $e = \delta a \to \epsilon b$ in the cycle defining ρ, and taking $x_0 = 1$ in type B. By induction,

$$\mathrm{Hilb}(R_{P_2}^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\sigma (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon})},$$

where σ runs over orbits of regions enclosed by $\widehat{G}(P_2)$ not fixed by the involution and e runs over the orbits of edges $\delta a \to \epsilon b$ $(a < b)$ in $\widehat{G}(P_2)$.

Then

$$\mathrm{Hilb}(R_P^{\mathrm{rt}}, \boldsymbol{x}) = \frac{\prod_\rho (1 - \boldsymbol{x}^\rho)}{\prod_e (1 - x_a^\delta x_b^{-\epsilon}),}$$

where ρ runs over all orbits of regions enclosed by $\widehat{G}(P)$, as desired. □

4.2.4 Computing Ψ_P

While the proof of Theorem 4.2.12 proceeds as above, using Proposition 2.2.14 to compute Ψ_P requires having a regular sequence generating the toric ideal. This regular sequence is implicit in Theorem 4.2.12—the cycle binomials of the cycles cutting out the regions of $\widehat{G}(P)$ form a regular sequence—though one needs to work in the opposite order to the proof to see the regular sequence, building the poset up from cycles rather than breaking it apart into cycles.

Recall the posets P_1 (Figure 4.19(a) and 4.19(b)), P_2 (Figures 4.19(c) and 4.19(d)) and P_3 (Figure 4.20) from the previous section. To find a regular sequence generating I_P^{rt}, one starts by noting that

$$R_{P_3}^{\mathrm{rt}} \cong k[U_1, U_{1\bar{2}}, U_{23}, U_3]/(U_{1\bar{2}}U_{23} - U_1U_3).$$

One then closes the notch to obtain $\widehat{G}_B(P_2)$, which has the following impact on the semigroup ring courtesy of Proposition 4.2.9:

$$\begin{aligned} R_{P_2}^{\mathrm{rt}} &\cong k[U_1, U_{1\bar{2}}U_{23}, U_3]/(U_{1\bar{2}}U_{23} - U_1U_3, U_1 - U_3) \\ &\cong k[U_1, U_{1\bar{2}}, U_{12}]/(U_{1\bar{2}}U_{12} - U_1^2) \end{aligned}$$

One sees that $\widehat{G}_B(P')$ has two biconnected components, one coming from P_1 and the other from P_2.

$$R_{P_1}^{\mathrm{rt}} \cong k[U_{\bar{3}4}, U_{4\bar{5}}, U_{\bar{2}3}, U_{\bar{2}5}]/(U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}U_{\bar{2}5})$$

Then

$$\begin{aligned} R^{\mathrm{rt}}_{P'} &\cong R^{\mathrm{rt}}_{P_1} \otimes R^{\mathrm{rt}}_{P_2} \\ &\cong k[U_1, U_{1\bar{2}}, U_{12}, U_{\bar{2}3}, U_{\bar{3}4}, U_{4\bar{5}}, U_{\bar{2}5}]/(U_{1\bar{2}}U_{12} - U_1^2, U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}U_{\bar{2}5}). \end{aligned}$$

Closing the last notch to obtain $\widehat{G}_B(P)$ means

$$\begin{aligned} R^{\mathrm{rt}}_{P} &\cong k[U_1, U_{1\bar{2}}, U_{12}, U_{\bar{2}3}, U_{\bar{3}4}, U_{4\bar{5}}, U_{\bar{2}5}]/(U_{1\bar{2}}U_{12} - U_1^2, U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}U_{\bar{2}5}, U_{\bar{2}5} - U_{1\bar{2}}) \\ &\cong k[U_1, U_{1\bar{2}}, U_{12}, U_{\bar{2}3}, U_{\bar{3}4}, U_{\bar{1}4}]/(U_{12}U_{1\bar{2}} - U_1^2, U_{1\bar{2}}U_{\bar{1}4} - U_{\bar{3}4}U_{\bar{2}3}). \end{aligned}$$

Each step in this process results in a regular sequence, so one finishes with a regular sequence generating I^{rt}_P. One has

$$\mathrm{Hilb}(R^{\mathrm{rt}}_P, \boldsymbol{x}) = \frac{(1 - x_1^2)(1 - x_2^{-1}x_4)}{(1 - x_1)(1 - x_1x_2)(1 - x_1x_2^{-1})(1 - x_1^{-1}x_4)(1 - x_3^{-1}x_4)(1 - x_2^{-1}x_3)}.$$

Applying Proposition 2.2.14 gives

$$\Psi_P(\boldsymbol{x}) = \frac{2x_1(x_4 - x_2)}{x_1(x_1 + x_2)(x_1 - x_2)(x_4 - x_1)(x_4 - x_3)(x_3 - x_2)}. \tag{4.6}$$

On the other hand, Figure 4.25 shows the poset of order ideals of P, from which the linear extensions can be read off. The linear extensions are:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & -2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & -2 & -3 \end{pmatrix}.$$

Then

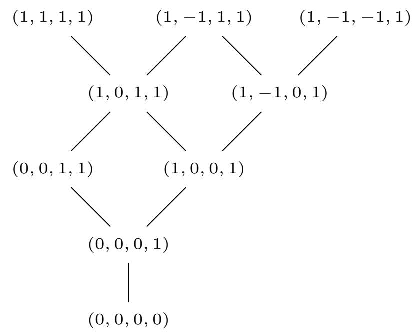


Figure 4.25: $J(P)$ for P in Figure 4.17

$$
\begin{aligned}
\Psi_P(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1-x_2)(x_2-x_3)(x_3-x_4)x_4}\right) = \\
\frac{1}{(x_4-x_3)(x_3-x_1)(x_1-x_2)x_2} - \frac{1}{(x_4-x_3)(x_3-x_1)(x_1+x_2)x_2} \\
+ \frac{1}{(x_4-x_1)(x_1-x_3)(x_3-x_2)x_2)} - \frac{1}{(x_4-x_1)(x_1-x_3)(x_2+x_3)x_2} \\
- \frac{1}{(x_4-x_1)(x_1+x_2)(x_2+x_3)x_3} - \frac{1}{(x_4-x_1)(x_1+x_2)(x_3-x_2)x_3} \\
= \frac{2(x_4-x_2)}{(x_4-x_3)(x_3-x_2)(x_4-x_1)(x_1-x_2)(x_1+x_2)} \cdot \frac{x_1}{x_1},
\end{aligned}
$$

agreeing with (4.6).

One builds $\widehat{G}_C(P^\vee)$ (see Figure 4.17) similarly, except one actually arrives at a principal ideal.

$$R^{\mathrm{rt}}_{P'^\vee} \cong R^{\mathrm{rt}}_{P_1^\vee} \otimes R^{\mathrm{rt}}_{P_2^\vee}.$$

However, note that $\widehat{G}_C(P_2^\vee)$ (see Figure 4.19(d)) consists of a single cycle fixed orientation-wise by the involution, so the only cycle binomial is 0. Consequently,

$$R^{\mathrm{rt}}_{P'^\vee} \cong k[U_{1\bar{2}}, U_{12}, U_{\bar{3}4}, U_{\bar{2}3}, U_{\bar{2}5}, U_{4\bar{5}}]/(U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}\bar{2}5),$$

and closing the notch gives

$$\begin{aligned}R^{\mathrm{rt}}_{P^\vee} &\cong k[U_{1\bar{2}}, U_{12}, U_{\bar{3}4}, U_{\bar{2}3}, U_{\bar{2}5}, U_{4\bar{5}}]/(U_{\bar{3}4}U_{\bar{2}3} - U_{4\bar{5}}\bar{2}5, U_{\bar{2}5} - U_{\bar{1}5}) \\ &\cong k[U_{1\bar{2}}, U_{12}, U_{\bar{3}4}, U_{\bar{2}3}, U_{\bar{1}4}]/(U_{\bar{3}4}U_{\bar{2}3} - U_{\bar{1}4}U_{\bar{2}5}).\end{aligned}$$

Then the Hilbert series is

$$\mathrm{Hilb}(R^{\mathrm{rt}}_{P^\vee}, \boldsymbol{x}) = \frac{1 - x_2^{-1}x_4}{(1 - x_1^{-1}x_4)(1 - x_3^{-1}x_4)(1 - x_2^{-1}x_3)(1 - x_1x_2^{-1})(1 - x_1x_2)}.$$

Applying Proposition 2.2.14 gives

$$\Psi^\vee_{P^\vee}(\boldsymbol{x}) = \frac{x_4 - x_2}{(x_4 - x_1)(x_4 - x_3)(x_3 - x_2)(x_1 - x_2)(x_1 + x_2)}.$$

On the other hand,

$$\begin{aligned}\Psi^\vee_{P^\vee}(\boldsymbol{x}) = \sum_{w \in \mathcal{L}(P)} w\left(\frac{1}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_4)2x_4}\right) \\ = \frac{x_4 - x_2}{(x_4 - x_3)(x_3 - x_2)(x_4 - x_1)(x_1 - x_2)(x_1 + x_2)},\end{aligned}$$

agreeing with the computation via the Hilbert series.

With the regular sequence in hand, one can apply Proposition 2.2.14 to obtain the following corollary.

Corollary 4.2.20. Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a strongly planar signed poset. Then

$$\Psi_P(\boldsymbol{x}) = \frac{\prod_\rho \mathrm{sgn}(\min(\rho))x_{\min(\rho)} - \mathrm{sgn}(\max(\rho))x_{\max(\rho)}}{\prod_{(i,j)\in\Sigma_P} \mathrm{sgn}(i)x_i - \mathrm{sgn}(j)x_j},$$

where ρ runs over the regions of $\widehat{G}(P)$ and (i, j) runs over the edges of the Hasse diagram.

Chapter 5

The Weight Cone Semigroup

Associated to the weight cone, K_P^{wt}, is a semigroup, called the *weight cone semigroup*, defined by the intersection of K_P^{wt} with the coweight lattice. In this chapter, only Φ_{B_n}-signed posets will be considered. Section 5.1 discusses the generators of the semigroup, Section 5.2 gives a generating set for the toric ideal. Finally, Section 5.3 considers the question of computing $\Phi_P(\boldsymbol{x})$ and

$$\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}.$$

Theorem 5.3.10 characterizes signed posets which are so-called initial complete intersections, for which the aforementioned sums can be readily computed, and Theorem 5.3.21 explains how the initial complete intersections are constructed.

Before beginning, recall some definitions and facts about the weight cone:

- an ideal I is said to be *extensible* if there is a (nonempty) $J \subset \pm[n]$ such that $I \cup J$ and $I \cup -J$ are ideals (see Definition 3.5.4)
- the extreme rays of K_P^{wt} correspond to the connected, nonextensible ideals of P (see Proposition 3.5.5)
- the elements of the weight cone semigroup are the P-partitions (see Section 3.2)

- when considering the ideals of a signed poset, it suffices to look at $\widehat{G}_C(P^\vee)$ instead of $\widehat{G}_B(P)$. To that end, $\widehat{G}(P)$ in this chapter denotes $\widehat{G}_C(P^\vee)$, even though signed posets are $P \subset \Phi_{B_n}$. (see Definition 3.2.5)

- the *signed support* of a P-partition f is an ideal

$$\mathrm{supp}_{\pm}(f) = \{\mathrm{sgn}(f_i)i \colon i \in [n], |f_i| \geq 1\}$$

  (see Definition 3.5.8).

- the set of connected ideals is denoted $J_{\mathrm{conn}}(P)$.

5.1 Generators of the Semigroup

Lemma 3.5.7 has a consequence that is key to the discussion of the weight cone semigroup, which is stated now.

Proposition 5.1.1. *Suppose $P \subset \Phi_{B_n}$ is a signed poset and f is a P-partition. Then there are ideals $J_1 \supset J_2 \supset \cdots \supset J_k$ such that $f = \chi_{J_1} + \chi_{J_2} + \cdots \chi_{J_k}$.*

The J_i are precisely those from Lemma 3.5.7.

Proposition 5.1.2. *The semigroup $K_P^{\mathrm{wt}} \cap L_B^{\mathrm{cowt}}$ is generated by the connected ideals of P, though not necessarily minimally.*

Proof. Suppose f is a P-partition. Then there are ideals $I_1 \subseteq \cdots \subseteq I_k$ such that

$$f = \sum_j \chi_{I_j}.$$

Let $I_j^{(i)}$ be the connected components of I_j. Then f is in the semigroup generated by the $I_j^{(i)}$, which lies in the semigroup generated by the connected ideals. $\square$

It is simple to see that this is not necessarily a minimal generating set. Suppose $P = \{e_1 + e_2, +e_2\}$. Then $J_{\mathrm{conn}}(P) = \{\{1\}, \{2\}, \{-1, 2\}\}$, but $\{1\}$ and $\{-1, 2\}$ suffice to generate the semigroup.

5.2 The Toric Ideal

To discuss the toric ideal of the weight cone semigroup, one needs a few additional definitions involving ideals of a signed poset.

Definition 5.2.1. Suppose J is an ideal of a signed poset. The *support* of J is the set $\operatorname{supp} J = \{i \colon \pm i \in J\} \subset [n]$. Two ideals, J and K, *intersect nontrivially* if $\operatorname{supp} J \cap \operatorname{supp} K \neq \emptyset$ and neither $J \subset K$ nor $K \subset J$. Two ideals which do not intersect nontrivially will be said to *intersect trivially*.

If J and K are ideals, let $J + K$ denote the P-partition $\chi_J + \chi_K$. In the sequel, it will be pairs of nontrivially intersecting connected ideals that are paramount. Denote the set of such pairs by $\Pi(P)$.

With this setup one can now give a presentation for the toric ideal of the weight cone semigroup. The result closely resembles that of Féray and Reiner [20, Theorem 1.2], save that one must account for the fact that the union of two ideals of a Φ_{B_n}-poset is not necessarily an ideal, since the union of two isotropic order ideals of $\widehat{G}(P)$ is not necessarily isotropic.

Definition 5.2.2. Suppose $P \subset \Phi_{B_n}$ is a signed poset. Let $S_P^{\mathrm{wt}} = k[U_{J_1}, \ldots, U_{J_k}]$, where the J_i are the connected ideals of P. Define a map

$$\begin{aligned} \varphi \colon S_P^{\mathrm{wt}} &\to k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \\ U_J &\mapsto \boldsymbol{x}^{\chi_J}. \end{aligned}$$

Suppose J_1, J_2 are two ideals which intersect nontrivially. Define

$$\operatorname{syz}(U_{J_1}, U_{J_2}) = U_{J_1} U_{J_2} - \prod U_{J^{(i)}} \prod U_{K^{(j)}},$$

where the $J^{(i)}$ are the connected components of $J_1 \cap J_2$ and the $K^{(j)}$ are the connected components of $\operatorname{supp}_{\pm}(J_1 + J_2)$.

Theorem 5.2.3. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. One has an exact sequence*

$$0 \to I_P^{\mathrm{wt}} \to S_P^{\mathrm{wt}} \to R_P^{\mathrm{wt}} \to 0,$$

with

$$I_P^{\mathrm{wt}} = (\operatorname{syz}(U_{J_1}, U_{J_2})),$$

where $\{J_1, J_2\}$ runs over $\Pi(P)$, the set of pairs of nontrivially intersecting connected ideals.

Theorem 5.2.3 is the generalization of the type A result [20, Theorem 1.2].

Consider the signed poset in Figure 5.1. The connected ideals are

$$\{1\}, \{-3\}, \{-4\}, \{1,2\}, \{1,4\}, \{-2,-3\}, \{1,2,4\}, \{1,2,3\}, \{1,2,3,4\},$$

so one has $S_P^{\mathrm{wt}} = k[U_1, U_{\bar{3}}, U_{\bar{4}}, U_{12}, U_{14}, U_{\bar{2}\bar{3}}, U_{124}, U_{123}, U_{1234}]$. Then

$$\begin{aligned} I_P^{\mathrm{wt}} = (&U_{\bar{3}}U_{123} - U_{12}, U_{\bar{3}}U_{1234} - U_{124}, U_{\bar{4}}U_{14} - U_1, U_{\bar{4}}U_{124} - U_{12}, U_{\bar{4}}U_{1234} - U_{123}, \\ &U_{12}U_{14} - U_1U_{124}, U_{12}U_{\bar{2}\bar{3}} - U_1U_{\bar{3}}, U_{\bar{2}\bar{3}}U_{124} - U_{14}U_{\bar{3}}, U_{124}U_{123} - U_{12}U_{1234}) \end{aligned}$$

The proof of Theorem 5.2.3 requires a number of lemmas and a few facts from the theory of Gröbner bases. Section 5.2.1 discusses Gröbner bases and Section 5.2.2 contains the lemmas and proof of the theorem.

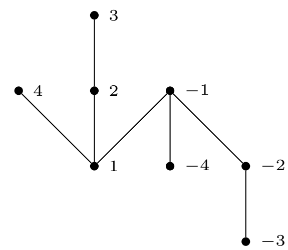


Figure 5.1: $P = \{+e_1, +e_1 - e_2, +e_1 - e_3, +e_1 - e_4, +e_2 - e_3\}$

5.2.1 Term Orders and Gröbner Bases

In this section, we pause for a moment to review a few facts about terms orders and Gröbner bases before the proof of Theorem 5.2.3 in the next section. This material is standard and can be found in many places, including [17, 19, 57].

Definition 5.2.4. Suppose $S = k[x_1, \ldots, x_n]$ is a polynomial ring. A *term order*, $<$, is a total order on the monomials of S such that

(a) there are no infinite descending chains

(b) $1 = \boldsymbol{x}^0$ is the minimal element

(c) if $\boldsymbol{x}^\alpha \leq \boldsymbol{x}^\beta$ and $\boldsymbol{x}^\gamma$ is any monomial, then $\boldsymbol{x}^\alpha \boldsymbol{x}^\gamma \leq \boldsymbol{x}^\beta \boldsymbol{x}^\gamma$.

Suppose $f = a_1\boldsymbol{x}^{\alpha_1} + \cdots + a_k\boldsymbol{x}^{\alpha_k}$ is a polynomial in R. The *multidegree* of f is $\alpha_{\deg f}$ which is maximal among $\alpha_1, \ldots, \alpha_k$ with respect to the term order $<$. The *initial term* of f is then $a_{\deg f}\boldsymbol{x}^{\alpha_{\deg f}}$, denoted $\mathrm{in}_< f$.

Definition 5.2.5. Given an ideal $I \subset R$, the *initial ideal* with respect to $<$ is

$$\mathrm{in}_< I = (\mathrm{in}_< f \colon f \in I).$$

An ideal and its initial ideal are connected by the notion of a Gröbner basis. A set $\mathcal{G} =$

$\{g_1, \ldots, g_m\} \subset I$ is a *Gröbner basis* of I if

$$\mathrm{in}_{<} I = (\mathrm{in}_{<} g_1, \ldots, \mathrm{in}_{<} g_m).$$

A priori, it is not clear that an arbitrary ideal in a polynomial ring should have a finite Gröbner basis. However, this is a consequence of Dickson's Lemma (see Theorem 1.9, Corollary 1.10 and page 20 of Ene and Herzog [19]). The following facts will underpin the proof of Theorem 5.2.3.

Theorem 5.2.6 (Macaulay [37])**.** *Suppose S is a polynomial ring over the field K, there is an ideal $I \subset S$ and a term order $<$. The monomials of S which do not belong to* $\mathrm{in}_{<} I$ *form a k-basis of S/I.*

Corollary 5.2.7. Suppose I is a homogeneous ideal in a polynomial ring S and $<$ is a term order. Then

$$\mathrm{Hilb}_{S/\mathrm{in}_{<} I}(\boldsymbol{x}) = \mathrm{Hilb}_{S/I}(\boldsymbol{x}).$$

Proposition 5.2.8. *If I is an ideal in a polynomial ring S and $\mathcal{G}$ is a Gröbner basis of I, then $\mathcal{G}$ generates I.*

As an aside, Gordan observed in [28] that this last proposition, when combined with the existence of finite Gröbner bases gives the Hilbert Basis Theorem, though it is common to rely on the Hilbert Basis Theorem to obtain the existence of finite Gröbner bases.

5.2.2 The Proof of Theorem 5.2.3

The basic idea of the proof of Theorem 5.2.3 is to show that the generating set given in the theorem is a Gröbner basis for I_P^{wt} in a particular term order. This is accomplished by showing that the rings $S_P^{\mathrm{wt}}/I_P^{\mathrm{wt}}$ and $S_P^{\mathrm{wt}}/I_P^{\mathrm{in}}$ share the same Hilbert series. (See Lemma 5.2.11 for the definition of I_P^{in}.)

Lemma 5.2.9. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. A P-partition f can be written uniquely as the sum of trivially intersecting connected ideals.*

Lemma 5.2.9 generalizes [20, Proposition 2.5(ii)], which has the same statement when P is a poset.

Proof. From Proposition 5.1.1, any P-partition can be written as $f = \sum \chi_{I_k}$ with ideals $I_1 \subset I_2 \subset \cdots \subset I_\ell$. Thus, the lemma is a matter of transforming this collection of ideals into a set of trivially intersecting connected ideals. Proceed by induction on $|f| = |f_1| + \cdots + |f_n|$. The base case $|f| = 0$ is trivial.

Suppose $|f| \geq 1$. Let $J = \operatorname{supp}_\pm(f)$. Recall that the signed support is an ideal and $J = I_k$. Let $J^{(1)}, J^{(2)}, \cdots, J^{(c)}$ be the connected components of J. Suppose $c > 1$. Consider $f|_{J^{(i)}}$. One has

$$f|_{J^{(i)}} = \sum_{i=1}^{k} I_k|_{J^{(i)}},$$

so $f|_{J^{(i)}}$ is a $J^{(i)}$-partition. One then has that $|J^{(i)}| < |J| \leq |f|$, so, by induction, $f|_{J^{(i)}}$ can be written uniquely as a sum of trivially intersecting connected ideals. Consequently, f can be written uniquely as a sum of trivially intersecting connected ideals.

In the other case suppose $c = 1$. Then $J = \operatorname{supp}_\pm(f)$ is connected. Let $\hat{f} = \sum_{i=i}^{k-1} I_i$. Then $f = \chi_J + \hat{f}$ and one has that $\hat{f}$ is a (possibly zero) P-partition with $|\hat{f}| < |f|$. Then, by induction, $\hat{f}$ can be written uniquely as a sum of trivially intersecting connected ideals. Further, each of these ideals is contained in $\chi_J = I_k$, so f can be written uniquely as a sum of trivially intersecting connected order ideals. □

Definition 5.2.10. Suppose $P \subset \Phi_{B_n}$ is a signed poset. Let

$$I_P^{\mathrm{in}} = (U_{J_1} U_{J_2})$$

with $\{J_1, J_2\}$ running over $\Pi(P)$.

Lemma 5.2.11. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. Then R_P^{wt} and $S_P^{\mathrm{wt}}/I_P^{\mathrm{in}}$ have the same $\mathbb{Z}^n$-graded Hilbert series, namely*

$$\sum_{f\in\mathcal{A}(P)} \boldsymbol{x}^f.$$

Proof. Since the P-partitions are the elements of the semigroup, R_P^{wt} has $\sum_{f\in\mathcal{A}(P)} \boldsymbol{x}^f$ as its $\mathbb{Z}^n$-graded Hilbert series. The monomials killed by I_P^{in} are precisely those mapped to P-partitions expressed as a sum of (at least) two nontrivially intersecting ideals. By Lemma 5.2.9, every P-partition can be written uniquely as a sum of nontrivially intersecting connected order ideals, i.e. each P-partition corresponds to a unique monomial surviving in $S_P^{\mathrm{wt}}/I_P^{\mathrm{in}}$. □

The notation I_P^{in} is not a coincidence, as Lemma 5.2.13 will show. One defines a term order for $S_P^{\mathrm{wt}} = k[U_J]$ that specializes to the term order given by Féray and Reiner in [20]. First, recall their term order. Place a total order $<$ on the U_J such that $U_J < U_K$ whenever $|J| < |K|$. (This amounts to choosing a linear extension of the poset of nonempty connected order ideals ordered by inclusion.) Suppose $U^J = U_{J_1}\cdots U_{J_r}$ and $U^K = U_{K_1}\cdots U_{K_s}$ are two monomials and assume without loss of generality that $r < s$ and the U_{J_i} and U_{K_ℓ} are ordered by $<$. Find the first i such that $J_i \neq K_i$. If $J_i < K_i$, then $U^J < U^K$, and if $K_i < J_i$, then $U^J < U^K$. If no such i exists, then U^J divides U^K, so $U^J < U^K$.

In type A, the initial term of $\mathrm{syz}(U_J, U_K)$ is always $U_J U_K$ with respect to this term order. However, in type B, this is not always the case. Consider the poset in Figure 5.1. $S_P^{\mathrm{wt}} = k[U_1, U_{\bar{3}}, U_{\bar{4}}, U_{12}, U_{14}, U_{\bar{2}\bar{3}}, U_{124}, U_{123}, U_{1234}]$. Order the ideals as the corresponding variables appear from left to right. Then, using the term order from [20], the leading term of $\mathrm{syz}(U_{123}, U_{\bar{3}}) = U_{123}U_{\bar{3}} - U_{12}$ is U_{12} and not $U_{123}U_{\bar{3}}$.

To resolve this issue, define a new term order.

Definition 5.2.12. Let $w = (|J|)_{J\in J_{\mathrm{conn}}(P)}$ be a weight vector and define a term order as follows. Consider monomials U^α and U^β. If $\langle w, \alpha\rangle > \langle w, \beta\rangle$, then $U^\alpha > U^\beta$. If $\langle w, \alpha\rangle = \langle w, \beta\rangle$, break the tie using the term order from [20] described above. Denote this new term order $\preceq$.

In the example from Figure 5.1, the weight vector is $(1,1,1,2,2,2,3,3,4)$, and $U_{123}U_{\bar{3}}$ has weight four while U_{12} has weight two. Consequently, using the $\preceq$ order, the initial term of $\mathrm{syz}(U_{123}, U_{\bar{3}})$ is $U_{123}U_{\bar{3}}$.

Lemma 5.2.13. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. Suppose J_1 and J_2 are connected ideals that intersect nontrivially. Then* $\mathrm{in}_{\preceq}(\mathrm{syz}(U_{J_1}, U_{J_2})) = U_{J_1}U_{J_2}$.

Proof. There are two cases to consider. First, suppose $\mathrm{supp}(J_1 + J_2) = \mathrm{supp}(J_1) \cup \mathrm{supp}(J_2)$. In other words, no cancellation occurs in $J_1 + J_2$. Then

$$|J_1| + |J_2| = |\,\mathrm{supp}_{\pm}(J_1 + J_2)| + |J_1 \cap J_2|.$$

In this case, the weight vector w has the same inner product with the exponent vectors of both monomials of $\mathrm{syz}(U_{J_1}, U_{J_2})$. Using the term order $>$ from [20] to break the tie, one will always have that $U_{J_1}U_{J_2} > \prod U_{J^i} \prod U_{K(j)}$, since any connected component of $J_1 \cap J_2$ is a (proper) subset of J_1 and J_2. Therefore, $\mathrm{in}_{\preceq}(\mathrm{syz}(U_{J_1}, U_{J_2})) = U_{J_1}U_{J_2}$.

In the other case, cancellation occurs in $J_1 + J_2$. In this case,

$$|J_1| + |J_2| > |\,\mathrm{supp}_{\pm}(J_1 + J_2)| + |J_1 \cap J_2|,$$

so $\mathrm{in}_{\preceq}(\mathrm{syz}(U_{J_1}, U_{J_2})) = U_{J_1}U_{J_2}$. $\square$

The proof of Theorem 5.2.3 now proceeds much as the proof does for type A given in [20].

Proof of Theorem 5.2.3. For simplicity, let $K = \ker(\varphi\colon S \to k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}])$ and, as before, let $I_P^{\mathrm{in}} = (U_{J_1}U_{J_2})$, where $\{J_1, J_2\}$ runs over $\Pi(P)$. Observe that by definition, $I_P \subset K$. One then has $\mathrm{in}_{\preceq}(I_P) \subset \mathrm{in}_{\preceq}(K)$. On the other hand, recall from Lemma 5.2.13 that

$U_{J_1}U_{J_2} = \mathrm{in}_\preceq(\mathrm{syz}(U_{J_1}, U_{J_2}))$, so $I_P^{\mathrm{in}} \subset \mathrm{in}_\preceq(I_P)$. Therefore, one has surjections

$$S/I_P^{\mathrm{in}} \to S/\mathrm{in}_\preceq(I_P) \to S/\mathrm{in}_\preceq(K).$$

By the definition of φ, the ideals K and I_P are homogeneous in the $\mathbb{Z}^n$-grading, so S/K and S/I_P each share Hilbert series with $S/\mathrm{in}_\preceq(K)$ and $S/\mathrm{in}_\preceq(I_P)$, respectively, by Corollary 5.2.7. Furthermore, one knows from Lemma 5.2.11 that S/K and S/I_P^{in} share the same $\mathbb{Z}^n$-Hilbert series. Hence, one has

$$\mathrm{Hilb}(S/I_P^{\mathrm{in}}, \boldsymbol{x}) = \mathrm{Hilb}(S/I, \boldsymbol{x}) = \mathrm{Hilb}(S/K, \boldsymbol{x}) = \mathrm{Hilb}(S/\mathrm{in}_\preceq(I), \boldsymbol{x}) = \mathrm{Hilb}(S/\mathrm{in}_\preceq(K), \boldsymbol{x}).$$

Consequently, the surjections must be isomorphisms and $S/I_P^{\mathrm{in}} \cong S/\mathrm{in}_\preceq(I_P) \cong S/\mathrm{in}_\preceq(K)$, meaning $I_P^{\mathrm{in}} = \mathrm{in}_\preceq(I_P) = \mathrm{in}_\preceq(K)$. One then has that the $\mathrm{syz}(U_{J_1}, U_{J_2})$ form a Gröbner basis for both I_P and K, meaning $I_P = K$, since an ideal is always generated by a Gröbner basis. $\square$

5.3 Complete intersections and the sum over linear extensions

In [46], Reiner proved the following result about the P-partition generating function for signed posets. Recall the definition of *major index* of a signed permutation w from Definition 2.3.7:

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i.$$

Proposition 5.3.1. *Suppose P is a signed poset. Then*

$$\sum_{f \in \mathcal{A}(P)} q^{|f|} = \frac{\sum_{w \in \mathcal{L}(P)} q^{\mathrm{maj}(w)}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

This parallels the result in type A due to Stanley (see [53] or [51]). Féray and Reiner observed (again for type A) that the left hand side is the Hilbert series of R_P^{wt} with the grading $\deg \boldsymbol{x}^f = |f|$. However, this is *not* the case when P is a signed poset—$\deg x^f = |f|$ is not a grading of R_P^{wt}, never mind a specialization of the $\mathbb{Z}^n$ grading! Additionally, while Theorem 5.2.3 gives a presentation of R_P^{wt} as $S_P^{\mathrm{wt}}/I_P^{\mathrm{wt}}$, the toric ideal I_P^{wt} is not homogeneous in this $\mathbb{N}$-grading.

All is not lost, however, as, using the same logic as in Lemma 5.2.11, one observes that when S_P^{wt} is graded by $\deg U_J = |J|$, one has

$$\mathrm{Hilb}(S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}, q) = \sum_{f \in \mathcal{A}(P)} q^{|f|}.$$

Additionally, if one grades S_P^{wt} by $\deg U_J = 1$, one has

$$\mathrm{Hilb}(S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}, t) = \sum_{f \in \mathcal{A}(P)} t^{\nu(f)},$$

where $\nu(f)$ is the number of ideals used in the unique expression of the P-partition f as a sum of nontrivially intersecting connected order ideals.

This section is concerned with computing these sums in the case where $S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}} P$ is a complete intersection. Section 5.3.1 will reduce the case of signed posets which are not full-dimensional to the work of Féray and Reiner in [20]. Section 5.3.2 will define the notion of *initial complete intersection* and characterize in two ways the signed posets which are initial complete intersections:

- they are the signed posets avoiding a certain list of induced subposets (Theorem 5.3.10);
- they are the signed posets constructed using certain moves (Theorem 5.3.21).

By shifting one's focus to $S/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}$ from S/I_P^{wt}, one sees immediate benefit in that one now has a minimal generating set for the ideal of interest.

Proposition 5.3.2. *Suppose P is a signed poset. Then* $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$ *is minimally generated by $U_J U_K$ where $\{J, K\}$ runs over all nontrivially intersecting pairs of connected order ideals of P.*

Proof. Since the $\mathrm{syz}(U_J, U_K)$ form a Gröbner basis of I, their initial terms, the $U_J U_K$, certainly generate $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$. A generating set of a monomial ideal is a minimal generating set precisely when none of the monomials divides another. (See [35, Proposition 1.1.6], for instance.) Since the $U_J U_K$ are all distinct and all quadratic, none can divide any of the others. □

5.3.1 Reducing to connected $\widehat{G}(P)$

As with the root cone, one is able to reduce discussion of the weight cone semigroup to a more convenient case: that where $\widehat{G}(P)$ is connected. First, one uses a similar logic to the biconnected component reduction of Proposition 4.2.2.

Definition 5.3.3. Suppose $P \subset \Phi_{B_n}$ is a signed poset. Let a *signed component* of $\widehat{G}(P)$ be

(a) a connected component P_1 of $\widehat{G}(P)$ such that if $i \in P_1$, then $-i \in P_1$, or

(b) a pair of connected components P_1, P_2 such that $P_1 = -P_2$.

Observe that each signed component of $\widehat{G}(P)$ is the Fischer poset of some smaller signed poset, call them $P_1, \ldots, P_k$. One then has the following.

Proposition 5.3.4. *Let P be a signed poset and let $P_1, \ldots, P_k$ be the signed posets corresponding to its signed components. Then*

$$R_P^{\mathrm{wt}} \cong R_{P_1}^{\mathrm{wt}} \otimes \cdots \otimes R_{P_k}^{\mathrm{wt}}.$$

Proof. Begin by observing that each connected order ideal of P lies entirely in a signed component. Therefore,

$$S_P^{\mathrm{wt}} \cong S_{P_1}^{\mathrm{wt}} \otimes \cdots \otimes S_{P_k}^{\mathrm{wt}},$$

and

$$I_P^{\mathrm{wt}} \cong I_{P_1}^{\mathrm{wt}} \oplus \cdots \oplus I_{P_k}^{\mathrm{wt}}.$$

Consequently, one has

$$R_P^{\mathrm{wt}} = R_{P_1}^{\mathrm{wt}} \otimes \cdots \otimes R_{P_k}^{\mathrm{wt}},$$

as desired. $\square$

As a result of Proposition 5.3.4 it suffices to consider signed posets having only one signed component. This assumption will hold for the remainder of the chapter.

As in Section 4.2.1, one may assume that each signed poset under consideration is not contained in $\mathrm{span}\{e_i \colon i \in A\}$ for any $A \subsetneq [n]$. We explain here why it suffices to consider signed posets P for which K_P^{rt} is full-dimensional and K_P^{wt} is pointed.

Since it was assumed that $\widehat{G}(P)$ has only one signed component, if K_P^{wt} is not pointed, one must have that $\widehat{G}(P)$ consists of two isotropic connected components, $P_1 = -P_2$. Without loss of generality, one may assume $P_1 = [n]$, meaning both $[n]$ and $-[n]$ are both ideals.

Proposition 5.3.5. *Suppose P is a signed poset such that K_P^{wt} is not pointed. Then there is a poset Q on $[n]$ such that*

$$R_P^{\mathrm{wt}} \cong R_Q^{\mathrm{wt}}[(x_1 x_2 \cdots x_n)^{-1}],$$

the localization of R_Q^{wt} at the multiplicatively closed set $\{(x_1 \cdots x_n)^k \colon k \geq 0\}$.

Proof. One may assume that $\widehat{G}(P)$ consists of two connected components K_1 and K_2 such that $K_1 = -K_2$ and $[n]$ are the vertices of K_1. Let Q be the poset whose Hasse diagram coincides with K_1. One defines a map

$$\varphi \colon R_P^{\mathrm{wt}} \to R_Q^{\mathrm{wt}}[(x_1 x_2 \cdots x_n)^{-1}]$$

as follows. Given any $f \in \mathcal{A}(P)$, the P-partition f can be written uniquely as a sum of nontrivially intersecting connected ideals: $f = J_1 + \cdots + J_k$. Without loss of generality, one may assume $J_1, \ldots, J_i \subset [n]$ and $J_{i+1}, \ldots, J_k \subset -[n]$. Then define

$$\varphi(\boldsymbol{x}^f) = \frac{\boldsymbol{x}^{J_1}}{1} \cdots \frac{\boldsymbol{x}^{J_i}}{1} \cdot \frac{\boldsymbol{x}^{([n] \smallsetminus -J_{i+1})}}{x_1 \cdots x_n} \cdots \frac{\boldsymbol{x}^{([n] \smallsetminus -J_k)}}{x_1 \cdots x_n}.$$

Since $[n]$ and $-[n]$ are both ideals of P, for any ideal $J \subset -[n]$, one has that $[n] \smallsetminus -J$ is also an ideal of P, meaning it is an ideal of Q. (It may not be a *connected* ideal of Q, but it is an Q-partition, which is all that is needed.)

Certainly φ is an injection. To complete the proof, one must now show that φ is surjective. A prototypical monomial of $R_Q^{\mathrm{wt}}[(x_1 x_2 \cdots x_n)^{-1}]$ is

$$\frac{\boldsymbol{x}^f}{(x_1 \cdots x_n)^k},$$

with k minimal (i.e. having cancelled as many powers of $x_1 \cdots x_n$ from the numerator as possible). Since f is an Q-partition, it is also a P-partition. Then $\boldsymbol{x}^f (x_1 \cdots x_n)^{-k} \in R_P^{\mathrm{wt}}$ and, since φ is a ring homomorphism,

$$\varphi(\boldsymbol{x}^f (x_1 \cdots x - n)^{-k}) = \frac{\boldsymbol{x}^f}{(x_1 \cdots x_n)^k},$$

meaning φ is surjective, completing the proof. $\square$

5.3.2 Characterizing the initial complete intersections

Proposition 5.3.6. *Suppose P is a signed poset. One has that*

$$0 \to (U_J U_K) \xrightarrow{\varphi} S_P^{\mathrm{wt}} \to S_P^{\mathrm{wt}} / \operatorname{in}_{\preceq} I_P^{\mathrm{wt}} \to 0, \tag{5.1}$$

where $\{J, K\}$ runs over the set $\Pi(P)$ of all pairs of nontrivially intersecting connected ideals, is a complete intersection presentation if and only if no connected order ideal of P intersects *two or more other connected ideals nontrivially.*

Proposition 5.3.6 is a special case of an easy fact about Stanley-Reisner rings.

Definition 5.3.7. Let V be a finite set and suppose Δ is a *simplicial complex* on V, i.e. a collection of subsets of V such that if $F \in \Delta$ and $F' \subset F$, then $F' \in \Delta$. Let S be a polynomial ring whose variables are indexed by elements of V. The *Stanley-Reisner ideal* of Δ is the ideal

$$I_\Delta = \left(\prod_{v \in F} x_v \colon F \notin \Delta \right).$$

The quotient ring S/I is then called a *Stanley-Reisner ring.*

Proposition 5.3.8. *Suppose I is a Stanley-Reisner ideal. The Stanley-Reisner ring S/I is a complete intersection if and only if*

$$I = (x_{11} \cdots x_{1n_1}, \ldots, x_{m1} \cdots x_{mn_m}),$$

i.e. I is generated by a collection of square-free monomials with pairwise disjoint support.

See Duval [16, Theorem 4.1.1] for proof of a more general statement.

Proposition 5.3.6 follows from the observation that I_P^{wt} is the Stanley-Reisner ideal of the simplicial complex on $J_{\mathrm{conn}}(P)$ defined by $F \subset J_{\mathrm{conn}}(P)$ if all ideals in F pairwise intersect trivially.

This characterization of posets such that $S/\operatorname{in}_\preceq I_P^{\mathrm{wt}}$ is a complete intersection allows one to characterize such posets as those avoiding a list of induced subposets, similar to that of [20, Theorem 10.5].

Definition 5.3.9. Suppose P is a signed poset such that

$$0 \to (U_J U_K) \xrightarrow{\varphi} S_P^{\mathrm{wt}} \to S_P^{\mathrm{wt}} / \operatorname{in}_\preceq I_P^{\mathrm{wt}} \to 0$$

is a complete intersection presentation. Then P is said to be an *initial complete intersection.*

Theorem 5.3.10. *Suppose P is a signed poset. P is an initial complete intersection if and only if P does not contain a signed poset isomorphic to any of those shown in Figure 5.2 as an induced subposet.*

Key to the proof will be the following lemma.

Lemma 5.3.11. *Suppose P' is a signed poset that contains an induced subposet P, such that P has a connected order ideal J which intersects at least two other connected ideals of P nontrivially. Then P' also has such an ideal.*

Proof. It suffices to examine the case where P is a signed poset on n and P' is a signed poset on $n+1$. Recall that an ideal is determined by the antichain of its maximal elements. Consequently, every connected ideal J of P corresponds to an ideal J' in $\widehat{G}(P')$ having the same determining antichain.

Claim: J' is an isotropic order ideal of $\widehat{G}(P')$ and thus an ideal of P'.

Suppose not. There are two cases.

- First suppose j_1 and j_2 are maximal elements of J (and thus of J') with $(n+1) < j_1$ and $-(n+1) < j_2$. Then one must have $-j_2 < j_1$ and $-j_1 < j_2$. Since P is an induced subposet of P', the same relation holds in P, which means J was not isotropic, a contradiction.

- Suppose $j \in J'$ such that $(n+1) < j$ and $-(n+1) < j$. Then $-j < j$, which contradicts that J was isotropic.

Thus, J' must be an ideal of P'.

If J and K intersect nontrivially in P and J' and K' are the corresponding ideals of P'. Certainly $J' \cap K' \neq \emptyset$. Suppose $J' \subset K'$, i.e. J' and K' do not intersect nontrivially. Since P is an induced subposet of P', that would force $J \subset K$, contradicting that J and K intersect nontrivially.

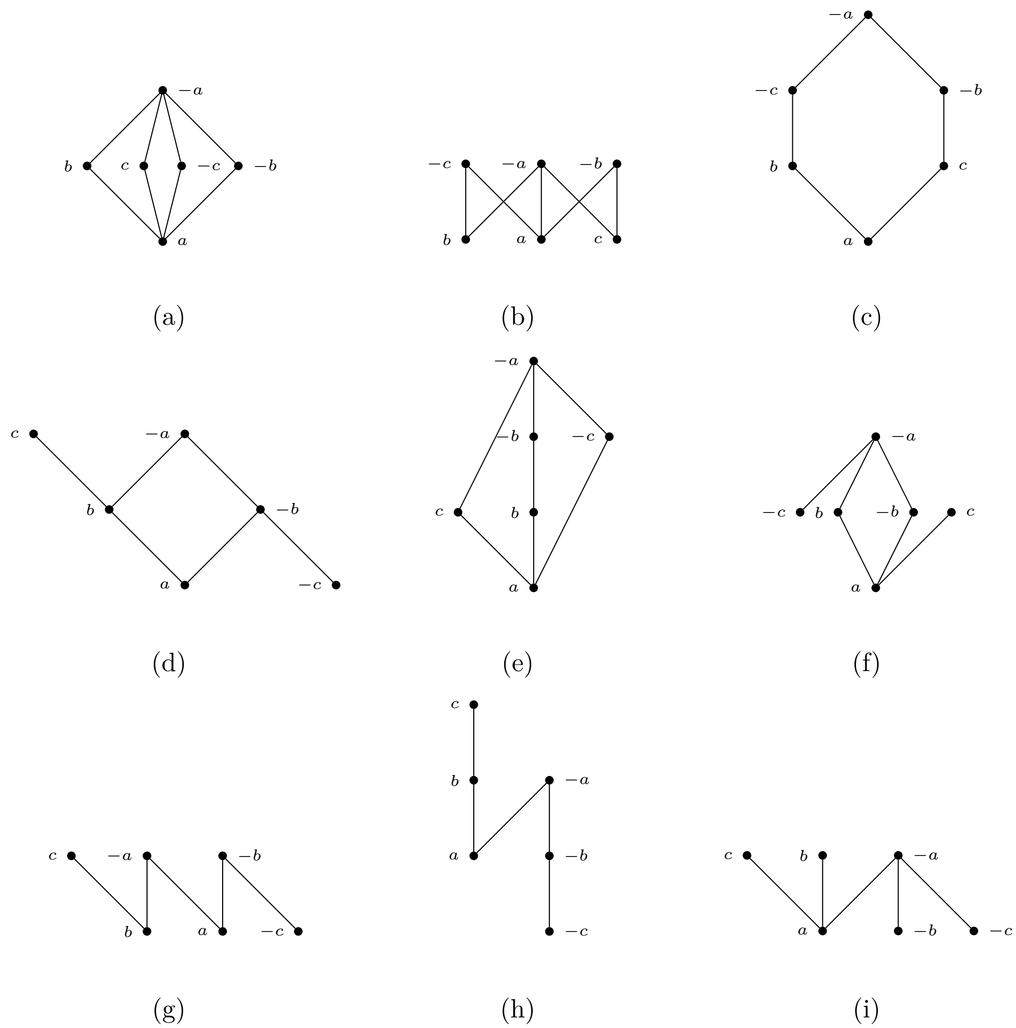
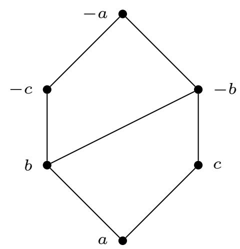
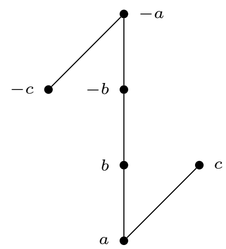


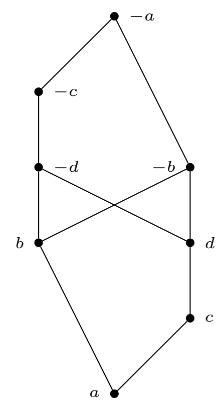
Figure 5.2: The excluded posets, part one



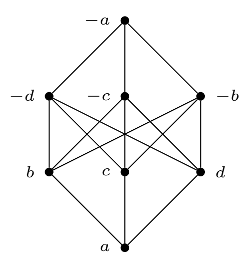
(j)



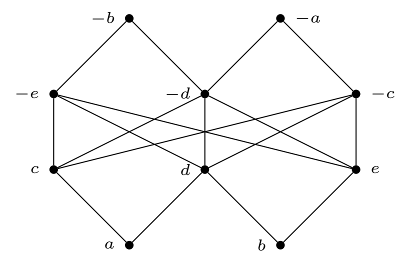
(k)



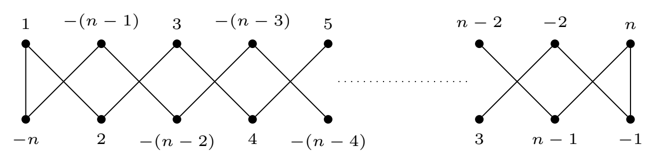
(l)



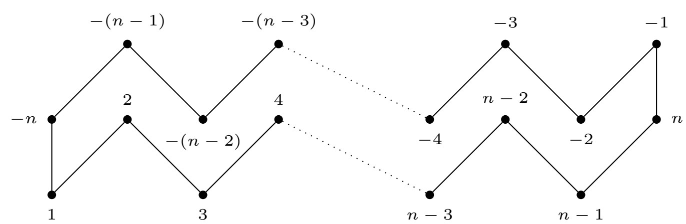
(m)



(n)



(o) for $n \geq 3$ odd



(p) for $n \geq 4$ even

Figure 5.2: The excluded posets, part two

Consequently, if P has a connected ideal J that intersects connected ideals K_1 and K_2 nontrivially, J' intersects K_1' and K_2' nontrivially, completing the proof. $\square$

While the above lemma makes the sufficiency of Theorem 5.3.10 straightforward, the necessity argument is made easier by two further lemmas.

Lemma 5.3.12. *Suppose $n \geq 3$ and $P \subset \Phi_{B_n}$ is a signed poset such that no $i < -i$. Then P is not an initial complete intersection.*

Proof. One may assume P is connected. Begin by observing that a member of the infinite families in Figures 5.2(o) and 5.2(p) cannot be an initial complete intersection as a consequence of Proposition 5.3.6. Since P has no $i < -i$, for $i \in \pm[x]$, there must be an $i \in [n]$ such that there is an isotropic path from i to $-i$. Since i and $-i$ are not comparable, this path must have at least two intermediate vertices and have a maximum and minimum other than i and $-i$. Then the image of this path under the involution will form a cycle C which is an induced subposet of P. Taking the smallest induced subposet of C which remains a cycle gives a member of one of the infinite families in Figures 5.2(o) and 5.2(p), so by Lemma 5.3.11, P is not an initial complete intersection. $\square$

Féray and Reiner characterized posets in type A such that R_P^{wt} is a complete intersection as being those posets avoiding the intersection of the three type A posets Figures 5.2(l),(m),(n) with $\Phi_{A_{n-1}}$. While $S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}$ being a complete intersection implies $S_P^{\mathrm{wt}}/I_P^{\mathrm{wt}}$ is a complete intersection (see [35, Exercise 3.3]), the converse is not, in general, true. For example, consider the signed poset in Figure 5.3. The semigroup ring R_P^{wt} has a complete intersection presentation:

$$0 \to (U_{\bar{4}}U_{14} - U_1, U_{\bar{3}}U_{134} - U_{14}, U_{\bar{4}}U_{134} - U_{13}) \to \\ k[U_1, U_2, U_{\bar{3}}, U_{\bar{4}}, U_{13}, U_{14}, U_{134}] \to R_P^{\mathrm{wt}} \to 0.$$

However, $\operatorname{in}_{\preceq} I_P^{\mathrm{wt}} = (U_{\bar{4}}U_{14}, U_{\bar{3}}U_{134}, U_{\bar{4}}U_{134})$, so $S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}$ is not a complete intersection.

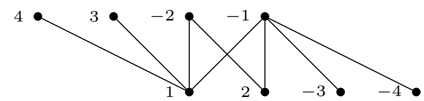


Figure 5.3: A signed poset P where R_P^{wt} is a complete intersection, but P is not an initial complete intersection.

The following lemma is an immediate consequence of Proposition 10.2 and Theorem 10.5 of [20].

Lemma 5.3.13. *Suppose $P \subset \Phi_{B_n}$ is a signed poset such that $i < -i$ for all i. Then P is an initial complete intersection if and only if P does not contain any of the signed posets in Figures 5.2(l),(m),(n) as an induced subposet.*

One is now ready to prove Theorem 5.3.10.

Proof of Theorem 5.3.10. The necessity follows from checking that all the posets in the list have at least one connected isotropic order ideal that intersects at least two other connected ideals nontrivially and applying Lemma 5.3.11.

For the sufficiency, Lemmas 5.3.12 and 5.3.13 allow one to immediately reduce to the case where there is $i \in \pm[n]$ such that $i < -i$ in P and there is some j such that j and $-j$ are incomparable. There are a number of cases.

(a) **Suppose $i \in \widehat{G}(P)$ is such that i and $-i$ are incomparable and i is not comparable to any $a < -a$.** Since $\widehat{G}(P)$ is connected, there must be some path from i to an a such that $a < -a$ such that no intermediate vertex is less than its negative. Call the vertices on this path $i, v_1, v_2, \ldots, v_k, a$. Let m be minimal such that $v_m > a$. Then one must have $v_{m-1} < v_m$. Taking the induced subposet on v_{m-1}, v_m, a gives a signed poset isomorphic to Figure 5.2(g).

(b) **Suppose every $i \in \widehat{G}(P)$ such that i and $-i$ are incomparable is comparable to some a such that $a < -a$.**

(b.i) **Suppose $\widehat{G}(P)$ has a connected isotropic order ideal J containing two elements i, j which are not comparable to their negatives and there is no a such that $a < -a$ and $i, j > a$.** Since J is connected, there must be a path from i to j contained entirely in J. If $j < i$, then the induced subposet on i, j, a gives a signed poset isomorphic to Figure 5.2(g).

If i and j are not comparable, one may assume without loss of generality that the path between them in J passes through only elements which are less than their negatives. Since i and j are not both comparable to any a with $a < -a$, the path between them must have at least three intermediate vertices, a_1, a_2, a_3, as shown in Figure 5.4. Since $a_1 < -a_1$ and i and $-i$ are incomparable, one knows the first

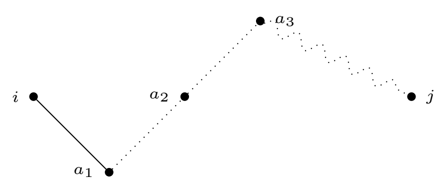


Figure 5.4: The path described in the second half of case (b)(i) in the proof of Theorem 5.3.10

vertex along the path must be below i. Then taking the induced subposet on i, a_2, a_3 gives a signed poset isomorphic to that in Figure 5.2(k).

(b.ii) **Assume instead that J contains i, j not comparable to their negatives and that there is an a with $a < -a$ with $i, j > a$.** Taking the induced subposet on i, j, a gives one of seven possibilities, which are all forbidden posets, namely Figures 5.2(a),(b),(c),(d),(f),(h),(i).

(b.iii) **Suppose J is a connected ideal containing precisely one element not less than its negative, call it i.** One knows that $I_{\leq}(-i)$ is isotropic, so J and $I_{\leq}(-i)$ intersect nontrivially.

Claim: i is a maximal element of J.

Suppose not and $j \in J$ with $j > i$. Since J contains only one element not less than its negative, i.e. i, $j < -j$, but the symmetry in $\widehat{G}(P)$ then forces $i < -i$, a contradiction.

Having proved the claim, there are two final cases to check.

(b.iii.1) **Suppose $J \cap K = \emptyset$.** Then one must have $-i \in K$. Since $K \neq I_{\leq}(-i)$, there must be $a, b \in K$ with $b, -i > a$ and $a < -a, b < -b$. However, the induced subposet on $a, b, -i$ is isomorphic to the signed poset of Figure 5.2(k).

(b.iii.2) **Suppose $J \cap K \neq \emptyset$.** Therefore, there must be j, ℓ in $J + K$ with $j < i, \ell$. Since $j, \ell \in J + K$, one must have that $j < -j$ and $\ell < -\ell$. One then has that the induced subposet on j, k, ℓ is isomorphic to one of the signed posets shown in Figures 5.2(j) and (k).

□

5.3.3 Constructing the complete intersection posets

In [20], Féray and Reiner also characterized the posets for which R_P^{wt} is a complete intersection as being the *forests with duplication*, those posets which could be constructed via operations they called *disjoint union*, *duplication of a hanger* and *hanging*. An analogous construction exists for signed posets which are initial complete intersections. Due to the reductions in Section 5.3.1, one can dispense with the disjoint union operation in the construction.

Recall that if P is a signed poset, $I_{<}(a) = \{b \in P \colon b < a\}$ and $I_{\leq}(a) = \{b \in P \colon b \leq a\}$, with the comparisons being made in $\widehat{G}(P)$. Let $P_{<a}$ denote the subposet of P induced by $\{i \colon \pm i \in I_{<}(a)\}$ and let $P_{\leq a}$ denote the subposet of P induced by $\{i \colon \pm i \in I_{\leq}(a)\}$.

Definition 5.3.14. Suppose P is a signed poset and $i \in \widehat{G}(P)$ is such that $i < -i$. Then i is said to be a *hanger* if, for each $a \in I_{<}(i)$, the induced subset on those vertices $\pm j$ such that $j < i$ (which may be empty) and for each $b \in G(P) \setminus G(P_{<i})$, any path from a to b must pass through i.

Definition 5.3.15. A signed poset P is said to have been *obtained by hanging P_1 below a in P_2* if

- a is a hanger in P,
- $P_1 = P_<(a)$, and
- P_2 is the induced subposet of P on i such that $\pm i \not< a$ in $\widehat{G}(P)$.

See Figure 5.5 for an example of hanging. When P is a type A signed poset, this definition

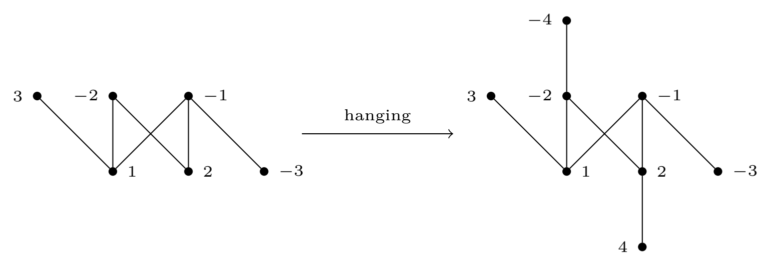


Figure 5.5: Obtaining a new poset by hanging a single element 4 below 2

of hanging coincides with the hanging of [20] in the sense that if P' is obtained from P by hanging, $P' \cap \Phi_{A_{n-1}}$ gives the poset that is the result of the same hanging in $P \cap \Phi_{A_{n-1}}$.

Lemma 5.3.16. *Suppose P is obtained by hanging P_1 below a in P_2 and both P_1 and P_2 are initial complete intersections. Then P is also an initial complete intersection.*

Proof. Suppose P is obtained by hanging P_1 below a in P_2 and both P_1 and P_2 are initial complete intersections. The connected (isotropic) order ideals of P are of three types:

(I) J for $J \in J_{\text{conn}}(P_1)$

(II) $J \cup P_1$ for $J \in J_{\text{conn}}(P_2)$ with $a \in J$.

(III) J for $J \in J_{\text{conn}}(P_2)$ with $a \notin J$.

Ideals of type I can only intersect other ideals of type I nontrivially and can therefore only intersect at most one other ideal nontrivially. A nontrivially intersecting pair involving ideals of type II or III corresponds to a nontrivially intersecting pair in P_2, so these ideals can also only be involved in at most nontrivially intersecting pair. Consequently, one has that P is an initial complete intersection. □

The next move also closely parallels [20].

Definition 5.3.17. The signed poset P' is said to have been obtained from P by *duplicating the hanger* a if $\widehat{G}(P') = \widehat{G}(P) \cup \{a', -a'\}$ with $i < a'$ or $a' < i$ whenever $i < a$ or $a < i$ in $\widehat{G}(P)$.

Figure 5.6 gives an example of duplicating a hanger.

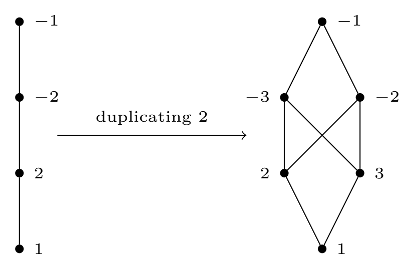


Figure 5.6: Obtaining a new poset by duplicating the hanger 2

Lemma 5.3.18. *Suppose P' is obtained from P by duplicating a hanger and P is an initial complete intersection. Then P' is also an initial complete intersection.*

Proof. Begin by noting that if a is a hanger in P, it must be that $I_{\leq a}$ intersects no other ideal nontrivially. Then the connected isotropic order ideals of P' are

- J for $J \in J_{\text{conn}}(P)$ with $a \notin J$
- $J \cup \{a'\}$ if $J \in J_{\text{conn}}(P)$ with $a \in J$
- $I_{\leq}(a')$.

The only nontrivially intersecting pair in P' that does not correspond to a nontrivially intersecting pair in P (i.e. the only pair created by the duplication) is $\{I_{\leq}(a), I_{\leq}(a')\}$, meaning P' must also have $S^{\mathrm{wt}}_{P'}/\mathrm{in} I^{\mathrm{wt}}_{P'}$ a complete intersection. $\square$

While hanging and duplication are translations of the type A definitions, an additional move is required to construct all the posets that are initial complete intersections.

Definition 5.3.19. The signed poset P' is said to have been obtained from P by a *type B hanging* of $n+1$ from i if $P' = \overline{P \cup \{e_i + \epsilon e_{n+1}\}}^{PLC}$, where $i < -i$ and $\epsilon \in \{\pm 1\}$, and $I_{\leq}(i)$ is a maximal ideal with respect to inclusion.

Figure 5.7 gives an example of type B hanging. The important thing to notice is that the requirement $I_{\leq}(i)$ be a maximal ideal severely limits the new ideals in P' compared to P.

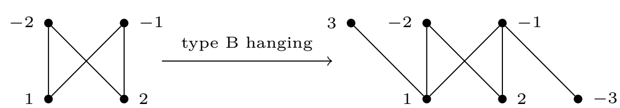


Figure 5.7: Obtaining a signed poset by the type B hanging of 3 above 1

Lemma 5.3.20. *Suppose P' is a signed poset obtained from P by a type B hanging. If P is an initial complete intersection, then P' is also.*

Proof. Suppose P' is obtained from P by a type B hanging. Then the connected isotropic order ideals are

- J for $J \in J_{\mathrm{conn}}(P)$
- $I_{\leq}(\epsilon(n+1))$
- $\{-\epsilon(n+1)\}$.

It is clear that $I_{\leq}(\epsilon(n+1))$ and $\{-\epsilon(n+1)\}$ intersect nontrivially and each intersect no other ideal nontrivially. The other ideals cannot be involved in more than one nontrivial intersection because they were not in P. Thus P' must be an initial complete intersection. $\square$

Theorem 5.3.21. *Up to isomorphism, the signed posets with $\widehat{G}(P)$ connected which are initial complete intersections are precisely those which can be constructed by hanging, duplication of a hanger and type B hanging from the two posets in Figure 5.8.*

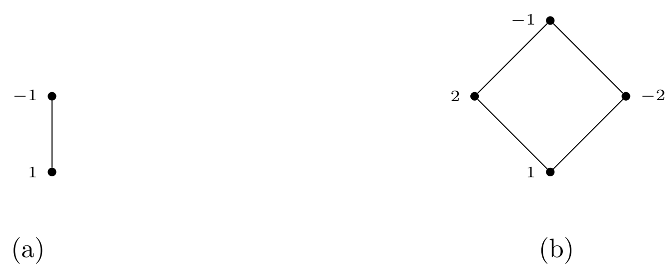


Figure 5.8: The posets from which the initial complete intersections are built

Proof. The proof proceeds by induction, with base cases of $n = 1$ and $n = 2$. Up to isomorphism, there is only one signed poset for $n = 1$, namely $P_1 = \{+e_1\}$ (see Figure 5.8a) and its toric ideal is trivial, so it is certainly an initial complete intersection. For $n = 2$, there are four signed posets which are initial complete intersections: those obtained from P_1 by hanging, duplication of a hanger and type B hanging, and the signed poset in Figure 5.8b, whose toric ideal is $(U_{12}U_{1\bar{2}} - U_1^2)$, which is principal, so the signed poset is an initial complete intersection.

Suppose P is a signed poset on $n \geq 3$ and $\widehat{G}(P)$ is connected and P is an initial complete intersection. Consider $a \in \widehat{G}(P)$ and let $I(a) = \{i \in \pm[n] : i < a\}$. From Lemma 5.3.12, one knows that one may assume $a < -a$. There are three cases.

(a) **Suppose that a is not minimal and for all $a' \in \widehat{G}(P)$ not comparable to a with $I_<(a')$ isotropic, $I_<(a) \cap I_<(a') = \emptyset$.** Then define two induced subposets of P: $P_{<a}$, the induced subposet on i such that $\pm i < a$, and $P \setminus P_{<a}$, the induced subposet on i such that $\pm i \not< a$. Furthermore, P is obtained by hanging $P_{<a}$ below $P \setminus P_{<a}$.

(b) **Suppose there exists $a' \in \widehat{G}(P)$ not comparable to a with $I_<(a')$ isotropic and $I_<(a) \cap I_<(a') \neq \emptyset$.**

Claim: $a' < -a'$.

Suppose not. Then both $I_<(a')$ and $I_<(-a')$ are isotropic and intersect nontrivially. Then since $a < -a$, it must be that $a \neq -a'$, meaning $I_<(a')$ intersects both $I_<(-a')$ and $I_<(a)$ nontrivially, contradicting that P had $S/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}$ a complete intersection.

As in type A (see Féray and Reiner [20, Theorem 10.6]), one decomposes P into four induced subposets:

$$P = \widehat{P} \sqcup P_{<a,a'} \sqcup (P_{<a} \smallsetminus P_{<a,a'}) \sqcup (P_{<a'} \smallsetminus P_{<a,a'}),$$

where $P_{<a,a'}$ is the induced subposet on i such that $\pm i < a$ and $\pm i < a'$ in $\widehat{G}(P)$, $(P_{<a} \smallsetminus P_{<a,a'})$ is the induced subposet on i such that $\pm i < a$ and $\pm i \not< a'$, similarly for $(P_{<a'} \smallsetminus P_{<a,a'})$ and $\widehat{P} = P \smallsetminus (P_{<a} \sqcup P_{<a'})$.

Observe that any element less than either a or a' is less than its negative. The signed posets $P_{<a} \smallsetminus P_{<a'}$ and $P_{<a'} \smallsetminus P_{<a}$ may be empty, but $P_{<a,a'}$ is not.

One constructs a signed poset Q using hanging, duplication and type B hanging and then shows that $Q = P$. There are two cases of the construction.

- Suppose there is $b \in \widehat{P}$ with $a < b$, $a' \not< b$ and $b \not< -b$. Construct Q as follows:
  (1) Start with $\widehat{P} \smallsetminus \{a', b\}$.
  (2) Hang $P_{<a,a'}$ below a in $\widehat{P} \smallsetminus \{a', b\}$.
  (3) Duplicate a.
  (4) Hang $P_{<a} \smallsetminus P_{<a'}$ and $P_{<a'} \smallsetminus P_{<a}$ below a and a', respectively.
  (5) Type B hang b from c, where $c \lessdot b$ in P.
- Suppose no such b exists. Construct Q as follows:
  (1) Start with $\widehat{P} \smallsetminus \{a', b\}$.
  (2) Hang $P_{<a,a'}$ below a in $\widehat{P} \smallsetminus \{a', b\}$.

(3) Duplicate a.

(4) Hang $P_{<a} \smallsetminus P_{<a'}$ and $P_{<a'} \smallsetminus P_{<a}$ below a and a', respectively.

In each case, each of the induced subposets used ($P_{<a,a'}, \widehat{P} \smallsetminus \{a'\}, \widehat{P} \smallsetminus \{a', b\}, P_{<a} \smallsetminus P_{<a'}, P_{<a'} \smallsetminus P_{<a}$,) is an initial complete intersection courtesy of Lemma 5.3.11, since they are induced subposets of an initial complete intersection. Consequently, Q is an initial complete intersection.

Next, one needs to show that P and Q are really the same poset. Certainly the restrictions of P and Q to $P_{<a,a'}$, $P_{<a} \smallsetminus P_{<a,a'}$ and $P_{<a'} \smallsetminus P_{<a,a'}$ are the same. It remains to check that when restricted to the vertices of $\widehat{P}$, P and Q are the same. Suppose $c > a$ and $c \not> a'$ in $\widehat{P}$. Consider two cases.

(b.i) **Suppose $c < -c$.** Recall that there must be at least one element, call it ℓ, such that $\ell < a$ and $\ell < a'$. Moreover, $\ell < -\ell$. This means that the induced subposet on a, b, ℓ is isomorphic to Figure 5.2(j).

(b.ii) **Suppose $c \not< -c$.** The existence of such a c means Q was constructed using the first construction. If c is not the b from the construction, one has that $a \in I_{\leq}(c) \cap I_{\leq}(b)$. However, $I_{\leq}(c)$ and $I_{\leq}(-c)$ also intersect nontrivially, a contradiction, unless $b = -c$. If $b = -c$, recall that there must be some $\ell \in P_{<a,a'}$. Taking the induced subposet of a, a', b, ℓ gives the signed poset shown in Figure 5.9. However, this poset is not an initial complete intersection, since $\{b, a, \ell\}$ intersects both $\{-b, a, \ell\}$ and $\{\ell, a'\}$ nontrivially, a contradiction, courtesy of Lemma 5.3.11.

It remains to check that given x, y in two different pieces of the decomposition, they share the same relation in P and in Q. The cases break down as they did in type A.

(b.I) **Suppose $x \in P_{<a} \smallsetminus P_{<a'}$ and $y \in P_{a'} \smallsetminus P_{<a}$.** Then x and y must be incomparable in both P and Q.

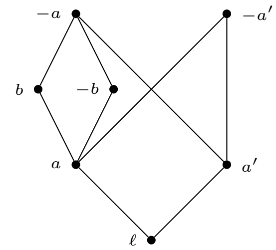


Figure 5.9: An induced subposet when $c > a, c \not> a', c < -c$ in case (b.ii) of the proof of Theorem 5.3.21

(b.II) **Suppose $x \in P_{<a} \setminus P_{<a'}$ (or $P_{<a'} \setminus P_{<a}$) and $y \in P_{<a,a'}$.** In this case, x and y are incomparable in Q by construction. If $x <_P y$, then $x \in P_{<a,a'}$, a contradiction. If $y <_P x$, then the induced subposet of a, x, y is isomorphic to the signed poset in Figure 5.2(j), contradiction. Thus, one has that x and y must also be incomparable in P.

(b.III) **Suppose $x \in P_{<a} \setminus P_{<a'}$ and $y \in \widehat{P}$.** Neither $y \leq_Q x$ nor $y \leq_P x$ is possible. Observe that $x \leq_Q y$ if and only if $a \leq_P y$ and $a \leq_P y$ implies $x \leq_P y$. Therefore, one must show that if $a \not\leq_P y$ then $x \not\leq_P y$. Suppose not, and $a \not\leq_P y$, but $x \leq_P y$. One knows that there exists $\ell \in P_{<a,a'}$. Suppose $\ell \not\leq_P y$. If $\ell < \ell'$, then the subposet of P induced by a, a', x, y, ℓ is isomorphic to the signed poset in Figure 5.2(n) a contradiction. If ℓ and $-\ell$ are not comparable, the subposet of P induced by a, y, x is isomorphic to the signed poset in Figure 5.2(j) or (k), a contradiction in both cases.

Thus, one must have that $\ell \leq_P y$. There are two cases:

- $y \not\geq_P a'$. Then the subposet of P induced by a, a', y, ℓ is isomorphic to the signed poset in Figure 5.2(m), a contradiction.
- $y \geq_P a'$. Then the subposet of P induced by a, y, ℓ is isomorphic to the signed poset in Figure 5.2(j), a contradiction.

(b.IV) **Suppose $x \in P_{<a'} \setminus P_{<a}$ and $y \in \widehat{P}$.** This case is the same as the previous case

with the roles of a and a' exchanged.

(b.V) **Suppose $x \in P_{<a,a'}$ and $y \in \widehat{P}$.** Begin by observing that $y \leq_Q x$ and $y \leq_P x$ are impossible. One has that $x \leq_Q y$ if and only if $a \leq_P y$ or $a' \leq_P y$. Both $a \leq_P y$ and $a' \leq_P y$ imply $x \leq_P y$, so one needs to show that if neither $a \not\leq_P y$ nor $a' \not\leq_P y$, then $x \not\leq_P y$. Suppose not. If $y < -y$, the subposet induced by x, y, a, a' is isomorphic to the forbidden signed poset in Figure 5.2(m).

If y and $-y$ are not comparable, then the subposet induced by x, y, a is isomorphic to the signed poset in either Figure 5.2(j) or (k), a contradiction.

(c) **Suppose that a *is* minimal and for all $a' \in \widehat{G}(P)$ not comparable to a with $I_<(a')$ isotropic, $I_<(a) \cap I_<(a') = \emptyset$.** One may assume that no a exists that falls into either of the earlier two cases. Then every $a \in \widehat{G}(P)$ with $a < -a$ is a minimal element of $\widehat{G}(P)$. One can then divide $[n]$ into $a_1, \ldots, a_k$ and $b_1, \ldots, b_j$, where $a_i < -a_i$ and b_ℓ and $-b_\ell$ are not comparable. Since P is an initial complete intersection, no a_i lies below more than one b_ℓ. There are two cases.

- Suppose a_1 and a_2 are covered by all the same elements (i.e. $-a_1, \ldots, -a_k$ and possibly b_ℓ for some ℓ). Let $P' = P \setminus \{a_1\}$. Then a_2 is a hanger in P' and P is obtained from P' by duplicating a_2.
- Suppose there is no such a_1, a_2. Then, there must be an a_i such that $b_\ell > a_i$ and b_ℓ covers no other a_s. Let $P' = P \setminus \{b_\ell\}$. Then P is obtained from P' by a type B hanging of b_ℓ from a_i.

□

5.3.4 Computing Rational Functions

Having characterized the initial complete intersections, attention now turns to calculating various rational function identities. One knows from Proposition 5.3.2 that the $U_J U_K$ as

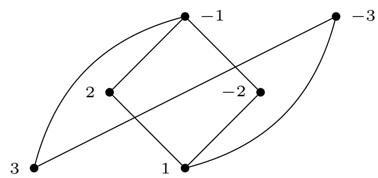


Figure 5.10: A signed poset

$\{J, K\}$ runs over $\Pi(P)$ form a minimal generating set for $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$. When P is an initial complete intersection, one then has

$$\mathrm{Hilb}(R_P^{\mathrm{wt}}, \boldsymbol{x}) = \mathrm{Hilb}(S_P^{\mathrm{wt}} / \mathrm{in}_{\preceq} I_P^{\mathrm{wt}}, \boldsymbol{x}) = \frac{\prod_{\{J,K\} \in \Pi(P)} (1 - \boldsymbol{x}^J \boldsymbol{x}^K)}{\prod_{J \in J_{\mathrm{conn}}(P)} (1 - \boldsymbol{x}^J)}. \tag{5.2}$$

One can then apply Proposition 2.2.14 to obtain Φ_P.

Corollary 5.3.22. Suppose P is an initial complete intersection. Then

$$\Phi_P(\boldsymbol{x}) = \frac{\prod_{\{J,K\} \in \Pi(P)} \langle \boldsymbol{x}, \chi_J + \chi_K \rangle}{\prod_{J \in J_{\mathrm{conn}}(P)} \langle \boldsymbol{x}, \chi_J \rangle}.$$

For example, consider the signed poset in Figure 5.10. Then $S_P^{\mathrm{wt}} = k[U_1, U_3, U_{12}, U_{1\bar{2}}]$ and $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}} = (U_{12} U_{1\bar{2}})$. Then

$$\mathrm{Hilb}(S_P^{\mathrm{wt}} / \mathrm{in}_{\preceq} I_P^{\mathrm{wt}}, \boldsymbol{x}) = \frac{1 - x_1^2}{(1 - x_1)(1 - x_3)(1 - x_1 x_2)(1 - x_1 x_2^{-1})},$$

and applying Proposition 2.2.14 gives

$$\Phi_P(\boldsymbol{x}) = \frac{2x_1}{x_1 x_3 (x_1 + x_2)(x_1 - x_2)}.$$

On the other hand, Table 5.1 gives the linear extensions of P, their descent sets and their major index. Then

w	$\mathrm{Des}(w)$	$\mathrm{maj}(w)$
$\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}$	$\emptyset$	0
$\begin{pmatrix}1 & 2 & 3\\ 1 & -2 & 3\end{pmatrix}$	$\{2\}$	2
$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$	$\{2\}$	2
$\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & -2\end{pmatrix}$	$\{3\}$	3
$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$	$\{1\}$	1
$\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & -2\end{pmatrix}$	$\{1,3\}$	4

Table 5.1: Linear extensions for the signed poset of Figure 5.10, their descents and major index.

$$\begin{aligned}
\Phi_P(\boldsymbol{x}) &= \sum_{w\in\mathcal{L}(P)} w\left(\frac{1}{x_1(x_1+x_2)\cdots(x_1+\cdots+x_n)}\right)\\
&= \frac{1}{x_1(x_1+x_2)(x_1+x_2+x_3)} + \frac{1}{x_1(x_1-x_2)(x_1-x_2+x_3)} + \frac{1}{x_1(x_1+x_3)(x_1+x_2+x_3)}\\
&+ \frac{1}{x_1(x_1+x_3)(x_1+x_3-x_2)} + \frac{1}{x_3(x_1+x_3)(x_1+x_2+x_3)} + \frac{1}{x_3(x_1+x_3)(x_1-x_2+x_3)}\\
&= \frac{2}{(x_1-x_2)(x_1+x_2)x_3} = \frac{2x_1}{x_1x_3(x_1-x_2)(x_1+x_2)}.
\end{aligned}$$

Altering the grading so $\deg U_J = |J|$ gives the following identity (c.f. Proposition 5.3.1).

Corollary 5.3.23. Suppose the signed poset P is an initial complete intersection. Then

$$\sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)} = [n]!_q \frac{\prod_{\{J,K\}\in\Pi(P)}[|J|+|K|]_q}{\prod_{J\in J_{\mathrm{conn}}(P)}[|J|]_q}.$$

Proof. Collapsing the $\mathbb{Z}^n$-grading to the $\mathbb{N}$-grading where $\deg U_J = |J|$, (5.2) transforms into

$$\mathrm{Hilb}(S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}, q) = \sum_{f\in\mathcal{A}(P)} q^{|f|} = \frac{\prod_{\{J,K\}\in\Pi(P)} 1-q^{|J|+|K|}}{\prod_{J\in J_{\mathrm{conn}}(P)} 1-q^{|J|}}.$$

Using Proposition 5.3.1 one then has

$$\sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)} = [n]!_q \frac{\prod_{\{J,K\}\in\Pi(P)} 1-q^{|J|+|K|}}{\prod_{J\in J_{\mathrm{conn}}(P)} 1-q^{|J|}} = [n]!_q \frac{\prod_{\{J,K\}\in\Pi(P)}[|J|+|K|]_q}{\prod_{J\in J_{\mathrm{conn}}(P)}[|J|]_q},$$

as claimed. □

Looking at the signed poset from Figure 5.10 once again, one has

$$\sum_{w\in\mathcal{L}(P)} q^{\mathrm{maj}(w)} = [3]!_q \frac{[2+2]_q}{[1]_q[1]_q[2]_q[2]_q} = \frac{[4]_q[3]_q}{[2]_q} = 1+q+2q^2+q^3+q^4,$$

matching the previous tabulation (see Table 5.1).

Lastly, one can alter the grading of $S_P^{\mathrm{wt}}/\operatorname{in}_{\preceq} I_P^{\mathrm{wt}}$ a third time, taking $\deg U_J = 1$ to obtain the following.

Corollary 5.3.24. Suppose P is a signed poset which is an initial complete intersection. Then

$$\sum_{f\in\mathcal{A}(P)} t^{\nu(f)} = \frac{(1-t^2)^{|\Pi(P)|}}{(1-t)^{|J_{\mathrm{conn}}(P)|}},$$

where $\nu(f)$ is the number of ideals in the unique expression of f as a sum of nontrivially intersecting connected ideals.

Chapter 6

Unfinished Business

6.1 Two triangulations of the weight cone

As was the case for Féray and Reiner in type A, the ideal $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$ suggests a triangulation of the weight cone K_P^{wt}. They explained (see [20, §11])

- that K_P^{wt} is a maximal cone in the normal fan of the graphic zonotope associated to the graph underlying the Hasse diagram,
- that the normal fan of the graphic zonotope is refined by the normal fan of the graph associahedron of the same graph and,
- lastly, that $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$ is the Stanley-Reisner ideal for the simplicial complex describing the triangulation of the weight cone by the the normal fan of the graph associahedron.

This section explains these two triangulations of the weight cone, the first indexed by linear extensions, the second by sets of pairwise trivially intersecting connected ideals, explains Zaslavsky's signed graph analogue of the graphic zonotope and how it gives a triangulation of K_P^{wt}, analogous to type A, but leaves open the problem of finding the correct definition of signed graph associahedron.

Definition 6.1.1. A *triangulation* of the cone K is a collection $T = \{\sigma_1, \ldots, \sigma_k\}$ of simplicial cones such that

- $\bigcup \sigma_i = K$;
- if $\sigma \in T$, then every face of σ is in T;
- for any $\sigma_i, \sigma_j \in T$,$\sigma_i \cap \sigma_j$ is a common face of σ_i and σ_j

The first triangulation of K_P^{wt} to be discussed is one suggested by Proposition 3.3.3, that where the maximal cones are unions of the $w\Phi^+$-partitions for $w \in \mathcal{L}(P)$, called the *P-partition triangulation.* We next describe a second triangulation called the *trivially intersecting ideals triangulation.*

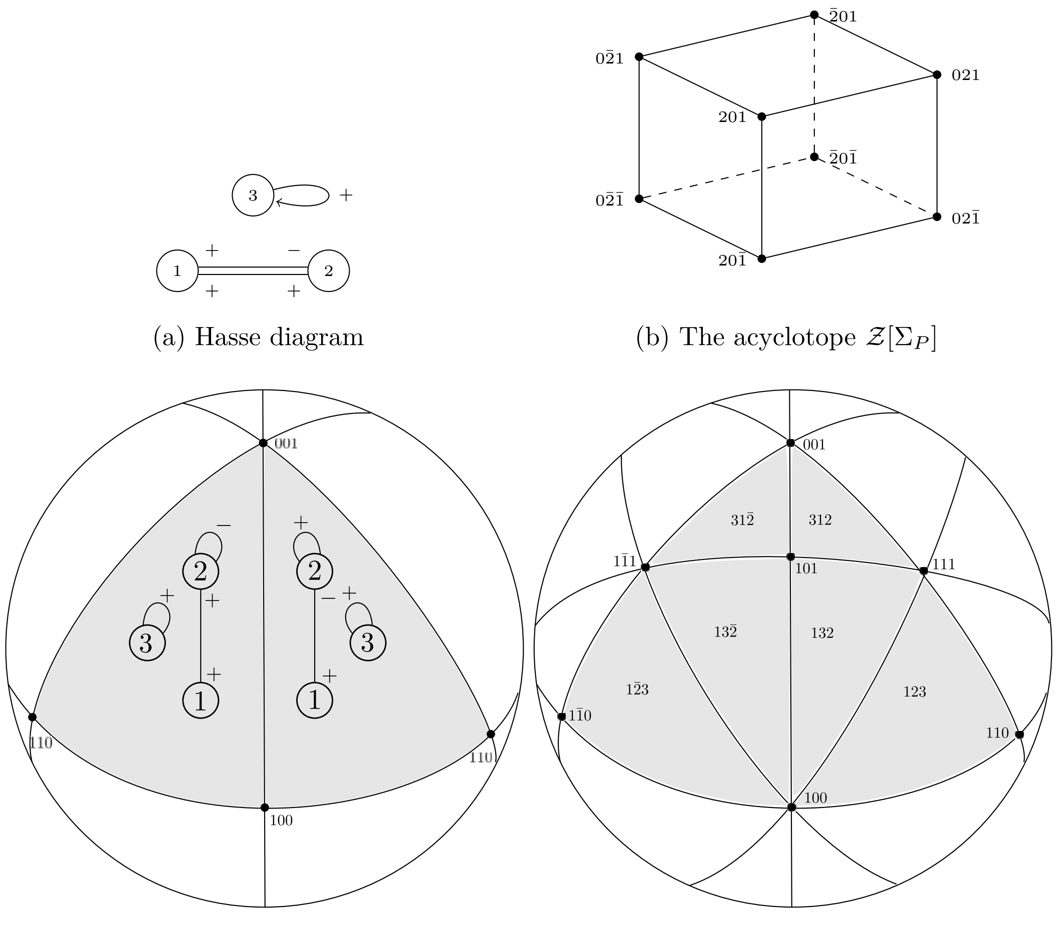
Proposition 6.1.2. *Suppose $P \subset \Phi_{B_n}$ is a signed poset. The weight cone K_P^{wt} is triangulated by cones $\{\sigma_A \colon \operatorname{span} A\}$, where A runs over all sets of connected ideals of P, the elements of which pairwise intersect trivially.*

Proposition 6.1.2 is an immediate consequence of Sturmfels [57, Theorem 8.3]. Moreover, as mentioned in Section 5.3.2, since $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$ is square-free, it is the Stanley-Reisner ideal of the complex on $J_{\mathrm{conn}}(P)$ whose faces are given by sets of ideals which pairwise intersect trivially.

Figure 6.1 shows the two different triangulations: the P-partition triangulation in Figure 6.1(d), where the maximal cones are indexed by the linear extensions of P, and the non-intersecting ideals triangulation of Proposition 6.1.2 in Figure 6.1(c), where the maximal cones are indexed by signed posets (more on these signed posets later).

In type A, the analogous triangulation is explained by Féray and Reiner in terms of the graphic zonotope.

Definition 6.1.3. A *zonotope* is a polytope that is the Minkowski sum of line segments. In



(a) Hasse diagram

(b) The acyclotope $\mathcal{Z}[\Sigma_P]$

(c) K_P^{wt} triangulated by sets of pairwise trivially intersecting ideals

(d) K_P^{wt} triangulated by $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$

Figure 6.1: The two different triangulations of K_P^{wt}

particular, if G is a graph, then

$$\mathcal{Z}[G] = \sum_{e=(i,j)\in G} [-(e_i - e_j), e_i - e_j],$$

is called the *graphic zonotope*.

Féray and Reiner note that the maximal cones of the normal fan $\mathcal{N}(\mathcal{Z}[G])$ are indexed by acyclic orientations of G. In particular, if P is a poset and G is the graph underlying its Hasse diagram, there is an orientation, call it ω, of G such that the corresponding maximal cone of the normal fan, $\mathcal{N}_\omega$, is, in fact, K_P^{wt}.

Definition 6.1.4. Let G be a graph. Its *graphical building set* $\mathcal{B}(G)$ is the collection of sets of vertices J where the vertex-induced subgraph $G|_J$ is connected.

The graphical building set is used to define the graph associahedron of Carr and Devadoss [12].

Definition 6.1.5. Suppose G is a graph. Its *graph associahedron* is

$$\mathcal{P}_G = \sum_{J\in\mathcal{B}(G)} \mathrm{conv}\{e_j : j \in J\}.$$

Féray and Reiner show the following.

Proposition 6.1.6 ([20, Proposition 11.7])**.** *Suppose P is a poset on $[n]$, G is the graph underlying its Hasse diagram and w is the orientation of G giving the Hasse diagram of P. Then the simplicial complex Δ_P having* $\mathrm{in}_{\preceq} I_P^{\mathrm{wt}}$ *as its Stanley-Reisner ideal describes the triangulation of the cone $\mathcal{N}_w$ in the fan $\mathcal{N}(\mathcal{Z}[G])$ by cones of the normal fan $\mathcal{N}(\mathcal{P}_G)$.*

The maximal cones of this triangulation are indexed by $\mathcal{B}(G)$-forests, forests F in which every principal ideal $F_{\leq i}$ is a connected ideal of P and, whenever i and j are incomparable in F, then the ideal $F_{\leq i}$ and $F_{\leq j}$ of P is disconnected.

One would hope for a similar understanding of the triangulation of Proposition 6.1.2. In the signed poset case, one can avail oneself of the analogue of the graphic zonotope—the acyclotope of Zaslavsky.

Definition 6.1.7. Suppose Σ is a signed graph and τ is a bidirection orienting Σ. Then *acyclotope* of Σ, written $\mathcal{Z}[\Sigma]$ is the polytope

$$\mathcal{Z}[\Sigma] = \sum_{e \in E} [-x_\tau(e), x_\tau(e)],$$

where $x_\tau(e)$ is the column vector associated to e in the incidence matrix of Σ.

Figure 6.1(b) gives an example of the acyclotope of the signed poset in Figure 6.1(a).

Proposition 6.1.8. *The hyperplane arrangement $H(\Sigma)$ whose fan corresponds to the normal fan of the graphic zonotope is:*

$$\begin{aligned} x_i &= \sigma(e) x_j && \textit{for an edge } e = (i, j) \\ x_i &= 0 && \textit{for a loop or half edge } e = (i, i),\ e = (i, -) \end{aligned}$$

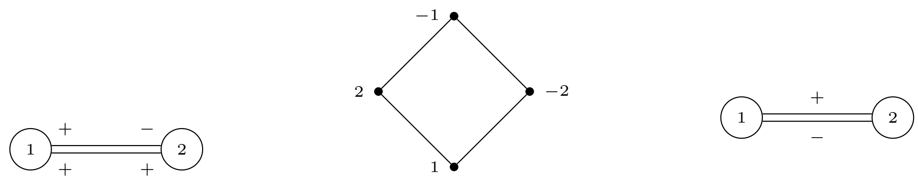
The regions of $H(\Sigma)$ correspond to various orientations of Σ.

Definition 6.1.9. A *cycle* of an oriented signed graph is a matroid circuit such that there is no vertex v such that all $\tau(v, e)$ coincide as e runs over the edges of the circuit incident to v. An orientation that contains no cycles will be said to be *acyclic*.

Zaslavsky generalized a result of Greene to the signed graph/acyclotope case in the following.

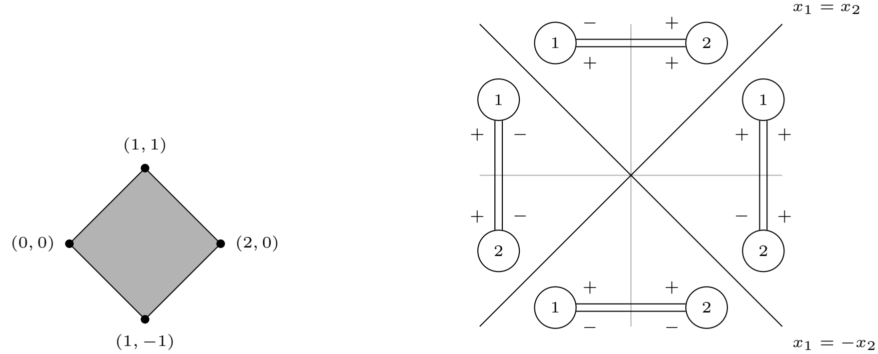
Theorem 6.1.10 (Zaslavsky [61, Theorem 4.4])**.** *Suppose Σ is a signed graph. There is a one-to-one correspondence between the regions of $H[\Sigma]$ and acyclic orientations of Σ.*

As an example, consider the signed poset, P, shown in Figure 6.2. Figure 6.2(c) shows Σ_P, the signed graph underlying the Hasse diagram of P (Figure 6.2(a)). As Σ_P has a single unbalanced cycle, so every orientation will be acyclic. Figure 6.3 shows the acyclotope



(a) Hasse diagram (b) $\widehat{G}(P)$ (c) Σ_P

Figure 6.2: $P = \{+e_1 - e_2, +e_1 + e_2, +e_1\}$



(a) the zonotope $\mathcal{Z}[\Sigma]$

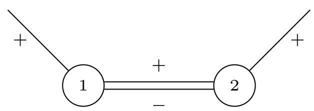
(b) $\mathcal{N}(\mathcal{Z}[\Sigma])$ with maximal cones labeled by acyclic orientations of Σ

Figure 6.3: The zonotop $\mathcal{Z}[\Sigma]$ and its Newton polytope for Σ from Figure 6.2

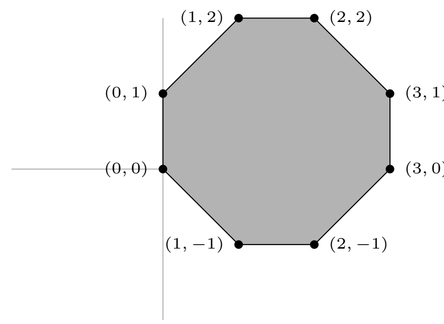
$\mathcal{Z}[\Sigma_P]$ and its normal fan $\mathcal{N}(\mathcal{Z}[\Sigma_P])$. Recall from Proposition 3.3.3 that the P-partitions of a signed poset are the disjoint union of the $w\Phi^+$-partitions for each $w \in \mathcal{L}(P)$. This corresponds to the triangulation of $\mathcal{N}(\mathcal{Z}[\Sigma_P])$ by the normal fan of the acyclotope of the complete graph, as in type A. There are a number of possible choices for a complete signed graph, but taking the lead from the type A braid arrangement, there is a clear choice.

Definition 6.1.11. Let $\pm K_n^B$ be the signed graph whose vertices are $[n]$ and whose edges are:

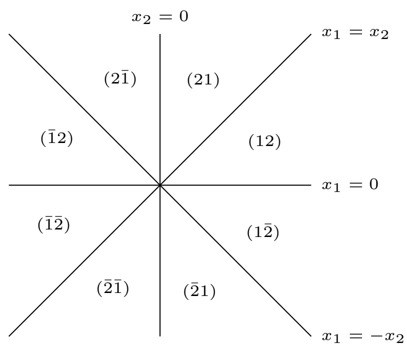
- an edge $\{i, j\}$ signed $+$ for all pairs $i, j \in [n]$,



(a) $\pm K_2^B$



(b) $\mathcal{Z}[\pm K_2^B]$



(c) $\mathcal{N}(\mathcal{Z}[\pm K_2^B])$

Figure 6.4: The complete signed graph $\pm K_2^B$, its acyclotope and the normal fan of the acyclotope

- an edge $\{i, j\}$ signed $-$ for all pairs $i, j \in [n]$
- a half edge at i signed $+$ for each $i \in [n]$.

Call $\pm K_n^B$ a *complete signed graph*.

One could opt to replace the half edges with loops, but in that case an orientation of a complete signed graph would, strictly speaking, correspond to Φ_{C_n} roots rather than Φ_{B_n}, though it would not alter the normal fan of the zonotope. Figure 6.4 shows $\pm K_2^B$, $\mathcal{Z}(\pm K_2^B)$ and $\mathcal{N}(\mathcal{Z}(\pm(K_2^B))$.

One sees that the example K_P^{wt} is triangulated by the cones spanned by $(12)\Phi_2^+$ and $(21)\Phi_2^+$, which are, in fact, the linear extensions of P.

Proposition 6.1.12. *Suppose $P \subset \Phi_{B_n}$ is a signed poset with Σ the signed graph underlying its Hasse diagram. Then $\mathcal{N}(\mathcal{Z}[\Sigma])$ is refined by $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$ and, if τ is the orientation of Σ corresponding to P, then $\mathcal{N}_\tau$ is triangulated by the cones of $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$ corresponding to the linear extensions of P.*

Proof. First, it is straightforward to see that $\mathcal{N}(\mathcal{Z}[\Sigma])$ is refined by $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$. The hyperplanes defining $\mathcal{N}(\mathcal{Z}[\Sigma])$ are a subset of the hyperplanes defining $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$.

Now, suppose τ is the orientation of Σ corresponding to P. By construction, the maximal cones of $\mathcal{N}(\mathcal{Z}[\pm K_n^B])$ are the $w\Phi^+$-partition cones for the elements of the Weyl group, i.e. the signed permutations. Thus, if the maximal cone corresponding to w lies in $\mathcal{N}_\tau$, by definition, w will be a linear extension of P. $\square$

As was noted above, one can index the maximal cones of the triangulation from Proposition 6.1.2 by certain signed posets, namely those whose weight cones are the maximal cones of the triangulation. By construction, these signed posets have simplicial and unimodular weight cones. They should be the signed analogue of type A's $\mathcal{B}(G)$-forests.

Question 6.1.13. What is the appropriated analogue of Carr and Devadoss's graph associahedron for signed graphs? Does its normal fan give the triangulation of K_P^{wt} given in Proposition 6.1.2?

At the moment, it appears an appropriate signed graph associahedron can be obtained by shaving the n-cube at faces corresponding to "signed tubes" of the signed graph, with a proof proceeding as that of Carr and Devadoss. However, all the details have yet to be written down.

6.2 The type C weight cone

Thus far, consideration of the weight cone, ideals and P-partitions has been restricted to those signed posets $P \subset \Phi_{B_n}$. However, though one can read the ideals from either $\widehat{G}_B(P)$

or $\widehat{G}_C(P^\vee)$, understanding one does not immediately lead to understanding the other, as the next two examples illustrate.

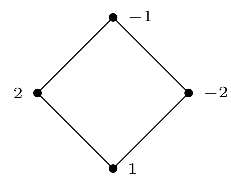


Figure 6.5: $P = \{e_1 - e_2, e_1 + e_2, e_1\}$

Consider the signed posets $P = \{e_1 - e_2, e_1 + e_2, e_1\}$ and $P^\vee = \{e_1 - e_2, e_1 + e_2, 2e_2\}$. Figure 6.5 illustrates $\widehat{G}(P)$. The connected ideals of P and $P^\vee$ are shown in Table 6.1. Recall that ideals live in the coweight lattice, so the maximal ideals of P and $P^\vee$ do not coincide. Inspecting the ideals reveals that all three connected ideals are required to generate the

ideal in $\widehat{G}(P)$	ideal in P	ideal in $P^\vee$
$\{1\}$	$(1,0)$	$(1,0)$
$\{1,2\}$	$(1,1)$	$(\frac{1}{2},\frac{1}{2})$
$\{1,-2\}$	$(1,-1)$	$(\frac{1}{2},-\frac{1}{2})$

Table 6.1: Type B and C ideals for $\widehat{G}(P)$ from Figure 6.5

semigroup $K_P^{\mathrm{wt}} \cap L_B^{\mathrm{cowt}}$. However, only $(\frac{1}{2},\frac{1}{2})$ and $(\frac{1}{2},-\frac{1}{2})$ are required to generate $K_P^{\mathrm{wt}} \cap L_C^{\mathrm{cowt}}$. Consequently, one sees K_P^{wt} is unimodular with respect to L_C^{cowt} but *not* with respect to L_B^{cowt}. In both cases, one has the triangulation into cones defined by collections of pairwise trivially intersecting ideals, even though K_P^{wt} is already simplicial and unimodular when viewed in type C.

Alone this is not enough to conclude that the arguments from Chapter 5 do not go through almost immediately in type C. Consider the signed poset in Figure 6.6. The connected ideals are $\{1\}, \{3\}, \{1,-2\}, \{1,2,3\}, \{1,-2,-3\}$. The toric ideal is the kernel of

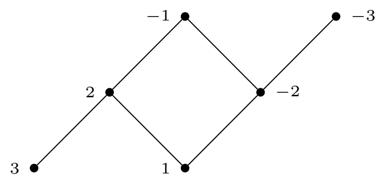


Figure 6.6: $\widehat{G}(P^\vee)$ for $P^\vee = \{e_1 - e_2, e_1 + e_2, e_1 + e_3, e_3 - e_2, 2e_1\}$

$\varphi\colon S^{\mathrm{wt}}_{P^\vee} \to k[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}]$ defined by

$$\begin{aligned}
\varphi(U_1) &= x_1^2\\
\varphi(U_3) &= x_3^2\\
\varphi(U_{1\bar{2}}) &= x_1^2 x_2^{-2}\\
\varphi(U_{123}) &= x_1 x_2 x_3\\
\varphi(U_{1\bar{2}\bar{3}}) &= x_1 x_2^{-1} x_3^{-1}.
\end{aligned}$$

Using Macaulay2, one sees that

$$\ker\varphi = (U_{1\bar{2}}U_{123}^2 - U_1^2 U_3, U_{123}U_{1\bar{2}\bar{3}} - U_1),$$

meaning that while the generators of the toric ideal are still indexed by pairs of nontrivially intersecting connected ideals, one needs to tweak the definition of $\mathrm{syz}(U_J, U_K)$ somewhat. Proposition 5.3.2 used that the leading term of $\mathrm{syz}(U_J, U_K)$ in type B is quadratic, which is clearly not the case here, though it is likely that a similar result holds and many of the results of Chapter 5 could be pushed through into type C.

6.3 On characterizing the R_P^{rt} complete intersections

In this section, attention returns to the root cone and its semigroup (refer to Chapter 4 for previous discussion). Boussicault and Féray [8, Theorem 7.7] had shown that Ψ_P factored for posets they called “gluings of diamonds along chains”, of which strongly planar posets were a subset. A poset was said to be a gluing of diamonds along chains if it could be decomposed into a collection of diamonds by means of disconnecting chains. A diamond was a cycle with a unique maximum and minimum. A disconnecting chain is a chain in the poset that partitions the vertices into three groups: the chain itself, and two other sets such that the paths between the two sets much pass through a vertex of the chain. For example, consider the poset in Figure 6.7. Certainly, it can be broken apart via disconnecting chains in the

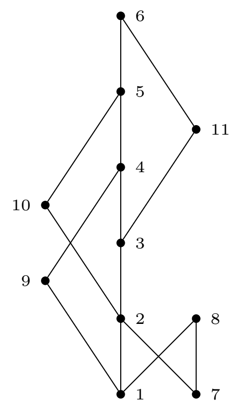


Figure 6.7: A poset which has R_P^{rt} a complete intersection but is not gluing of diamonds along chains

posets shown in Figure 6.8. The poset in Figure 6.8(d) is not a diamond, so is not covered by Boussicault and Féray’s result. However, the fact the poset of Figure 6.7 is a complete intersection is explained by Boussicault, Féray, Lascoux, and Reiner [9, Theorem 8.6], who give an algebraic explanation for the factorization via opening/closing notches—locating Boussicault and Féray’s disconnecting chains can serve as a guide for which notches to open.

An immediate consequence of [9, Theorem 8.6] is that a poset which can be broken apart into unicyclic components (those having one or no cycles) by opening notches must have R_P^{rt} a complete intersection. It turns out that this suffices to characterize the posets for which R_P^{rt} is a complete intersection, which was conjectured by the author of this thesis and V.

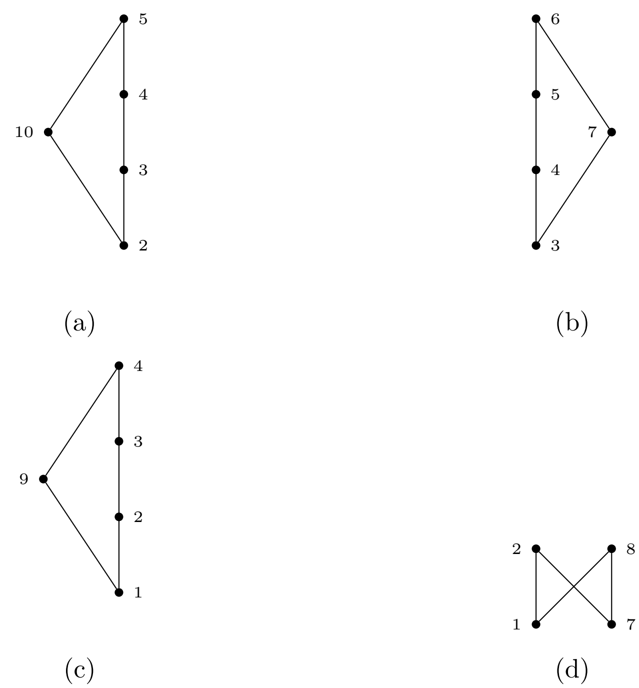


Figure 6.8: The poset of Figure 6.7 broken into unicycle components

Reiner and proved by Morris [42].

Theorem 6.3.1 ([42, Theorem 3]). *Suppose P is a poset. Its root cone semigroup ring R_P^{rt} is a complete intersection if and only if the Hasse diagram of P can be obtained from unicyclic posets by repeated gluings along chains.*

One would hope for a similar result in the signed poset case. Like in the poset case, it is clear that strong planarity is not a necessary condition for R_P^{rt} to be a complete intersection. After all, consider the signed poset in Figure 6.9. Its toric ideal is principal and there is no notch that can be opened, so R_P^{rt} must be a complete intersection, though the poset is clearly not strongly planar.

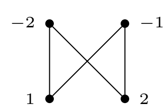


Figure 6.9: A signed poset which is not strongly planar, but has R_P^{rt} a complete intersection

Unicyclic posets, of course, have principal root cone toric ideals. Furthermore, cycles in the Hasse diagram of the poset correspond to circuits in its matroid. This turns out to be the key fact to identifying the correct signed analogue of "unicyclic".

Conjecture 6.3.2. *Suppose $P \subset \Phi_{B_n}$ (resp. $P^\vee \subset \Phi_{C_n}$) is a signed poset. R_P^{rt} (resp. $R_{P^\vee}^{\mathrm{rt}}$) is a complete intersection if and only if $\widehat{G}(P)$ can be broken into biconnected components $P_1, \ldots, P_k$, each of which has a Hasse diagram with at most one circuit, by opening a series of signed notches.*

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