Involutions on Baxter Objects, and q-Gamma Nonnegativity

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CHAPTER 1

Introduction

Baxter permutations originally arose in a 1964 paper of Glen Baxter [**6**], an analyst who was studying the conjecture that every pair of commuting continuous functions on the unit interval necessarily had a common fixed point. Say f and g are commuting continuous functions on the unit interval. Then it is equivalent to show that f and $f \circ g$ have a common fixed point. If $f \circ g$ has finitely many fixed points, $\{x_1, x_2, \ldots, x_N\}$, then $f(g(f(x_i))) = f(f(g(x_i))) = f(x_i)$ (first equality because f and g commute under composition, second equality because x_i is a fixed point of $f \circ g$). This means that $f(x_i)$ is itself a fixed point of $f \circ g$, and we get a permutation w of $\{1, \ldots, N\}$ by saying $w_i = j$ if $f(x_i) = x_j$.

Baxter showed that such permutations had some necessary conditions. In particular, $N = 2n - 1$ had to be odd, w sent odds to odds and evens to evens, the action of the permutation on the odds completely determined what happened on the evens, and the induced permutation on the odds could not contain certain subsequences. Over time, the permutations of length n on the odds avoiding certain subsequences (standardized to a permutation on $[n] := \{1, 2, \ldots, n\}$) are what have become known as Baxter permutations, and the larger permutations are called complete Baxter permutations, or w-admissible permutations. A few years later, Boyce and Huneke ([**13**] [**36**]) independently constructed counterexamples of commuting continuous functions with no common fixed points.

While the (now disproved) conjecture no longer provided motivation, there was still some interest in these permutations. Chung, Graham, Hoggatt, and Kleiman [**18**] were able to come up with an enumeration by creating a generating tree for these permutations, algebraically manipulating the resulting recursion relation, and then magically guessing the correct solution. They found that the number of Baxter permutations of length n was

$$B(n) := \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{1}$$

Shortly thereafter, Mallows was able to refine their recursion to show that the k^{th} summand gave the number of Baxter permutation with k rises [**40**], and then Viennot gave a combinatorial proof of this fact [**55**].

Since then, numerous other combinatorial objects have been found to be in bijection with Baxter permutations, some of which provide refined enumeration of Baxter permutations with respect to other statistics. Cori, Dulucq, and Viennot made bijections to shuffles of parenthesis systems, and pairs of binary trees satisfying a compatiblity relationship [**19**]. Dulucq and Guibert did work with stack words and standard Young tableaux of shape $3 \times n$ with no consecutive entries in any row [**21**], and later to triples of non-intersecting lattice paths [**22**]. Reading provided a Hopf algebra structure for "twisted" Baxter permutations as part of a larger work on lattice quotients of the weak order [**45**], and later gave an intrinsic description of this Hopf algebra in terms of diagonal rectangulations with Law [**38**]. There are also connections to plane bipolar orientations and 2-orientations of planar quadrangulations [**23**].

All of these combinatorial objects in bijection with Baxter permutations have a natural involution associated to them, so one natural question would be to count how many things in each combinatorial family are fixed under this involution.

- In Section 2, Theorem 2.2, we show that the bijections between these various combinatorial objects are equivariant with respect to the natural involutions. This means that the number of objects fixed by the natural involution in each combinatorial family is the same, and that it suffices to find the enumeration for only one combinatorial family.
- In Section 2, Theorem 2.3, we find the enumeration for one such combinatorial family, and show that it is an instance of the Stembridge "q=-1" phenomenon. Baxter permutations themselves have additional symmetries, such as being closed under taking inverses, that are not shared by the other families.
- In Section 2, Theorem 2.57, we give an overview of all symmetry classes of Baxter permutations, where the previous result covers Baxter permutations whose permutation matrix is fixed under 180° rotation, with Theorem 2.57 being a new result about those fixed under 90° rotation.

As part of this previous work, we came up with a natural q-analog for Baxter permutations with a fixed number of descents, and wanted to extend it to a natural q-analog for all Baxter permutations simultaneously. In particular, we wanted something something analogous to how the q-Narayana numbers can be combined to come up with a natural q-Catalan number [**28**]. This led to exploration as to various ways in which properties of Catalan objects could be extended to Baxter objects.

One such property is gamma nonnegativity, for a polynomial of degree n with symmetric coefficient sequence, which asks that it expands nonnegatively in the basis $\{t^i(1+t)^{n-2i}\}_{0\leq i\leq \lfloor n/2\rfloor}$. A motivation for gamma nonnegativity is geometric, coming from Gal's conjecture. Gal's conjecture says that the h-polynomial of any flag simplicial complex is gamma nonnegative, and it is further conjectured by Nevo and Petersen that the resulting gamma coefficients are not just positive, but also the f-vector of some associated simplicial complex [**42**].

Motivated by Gal's conjecture, there are a number of families of flag simplicial spheres and polytopes whose h-vectors are known to be gamma-nonnegative. Many of them arise from the classification of irreducible crystallographic root systems by Dynkin diagrams into three infinite families (types A,B, and D) and finitely many exceptional cases.

In one direction, one can consider the Coxeter complex, which is defined in terms of the parabolic subgroups of the corresponding Coxeter group. Stembridge has shown that graded Coxeter cones (and in particular, the Coxeter complex) are gamma nonnegative, and has given the expansions for the Coxeter complexes corresponding to the infinite irreducible families and the exceptional cases [**53**]. We can think of the Coxeter complex for different types as a generalization of the permutahedron, which is dual to the type A Coxeter complex.

In another direction, we have the associahedron. The associahedron is a polytope that can be described in terms of the triangulations of a regular polygon. Fomin and Zelevinsky [**27**] showed that finite-type cluster algebras have the same Dynkin classification, and associated to each diagram a cluster complex (dual to a generalized associahedra), for which the classical associahedra corresponds to type A, and the cyclohedron corresponds to type B. There are a number of results about gamma nonnegativity related to generalizations of the associahedron.

Postnikov, Reiner, and Williams [**44**] confirmed Gal's conjecture for chordal nestohedra, and computed the gamma vectors explicitly for the infinite families of the permutahedron, associahedron, and cyclohedron . More recently, work has been done been done by Nevo and Petersen to realize the gamma vectors of the Coxeter complexes, the associahedron, and the cyclohedron as the f-vectors of an associated complex [**42**]. Aisbett [**2**] was able to realize the gamma vectors of edge subdivisions of the cross polytope (which has the flag nestohedra mentioned in Postnikov, Reiner, and Williams as as special case) as the f-vector of an associated complex, which was independently proven by Volodin [**56**] in the dual case of 2-truncated cubes. Gorsky [**33**] then confirmed that while the type D associahedra are not nestohedra for $n \geq 4$, then can be realized as 2-truncated cubes, and are thus gamma nonnegative.

In Section 4, we will extend a number of these results and define a q-analog of gamma nonnegativity. In particular, the h-vector for many of the combinatorially defined complexes can be realized as the descent generating function over the underlying combinatorial objects. For many examples, we consider the joint distribution of these combinatorial objects with respects to descents and a second statistic (generally analogous to major index for permutations), and derive an expansion in terms of the polynomials $t^i \prod_{k=i}^{N-1-i}(1 + tq^{ka+b})$ (for some fixed a and b) whose coefficients are nonnegative polynomials in q.

- In Section 4.2.2, we prove such an expansion exists for the q-Narayana distribution, corresponding to Catalan objects.
- In Section 4.2.3, we prove that one such expansion exists for a natural polynomial associated to Baxter objects.
- In Conjecture 4.14, we conjecture that a second expansion exists for the natural polynomial associated to Baxter objects, which generalizes to an infinite family of polynomials, and includes the expansion proven for Catalan objects.
- In Sections 4.3 and 4.4, we recall previous results which show the existence of such an expansion for natural polynomials associated to permutations and signed permutations.
- In Section 4.5, we give such an expansion for a natural polynomial associated to the cyclohedron.
- In Section 4.6, we conjecture that such an expansion exists for a natural polynomial associated to involutive permutations.

CHAPTER 2

Involutions on Baxter Objects

2.1. Overview

The Baxter numbers are given by

$$B(n) := \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}}. \tag{2}$$

We will define

$$\Theta_{k,\ell} = \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}} \tag{3}$$

for $n = k + \ell + 1$ to refer to a single summand. The summand $\Theta_{k,\ell}$ counts many things, defined below, and illustrated in the Appendix:

(A): Baxter permutations in $\mathfrak{S}_n$ with k ascents and ℓ descents. [**18**]

(B): Baxter permutations in $\mathfrak{S}_n$ with k inverse ascents and ℓ inverse descents.

(C): Twisted Baxter permutations in $\mathfrak{S}_n$ with k inverse ascents and ℓ inverse descents. [**38**]

(D): Non-intersecting lattice paths from

$$\begin{aligned} &A_1 = (0, 2),\ A_2 = (1, 1), \text{ and } A_3 = (2, 0) \text{ to} \\ &B_1 = (k, \ell + 2), B_2 = (k + 1, \ell + 1),\ B_3 = (k + 2, \ell), \end{aligned} \tag{4}$$

which we will call *(k,ℓ)-Baxter paths.* [**22**]

(E): Standard Young tableaux of shape $3 \times n$ with no consecutive entries in any row, and k instances of $(i, i + 1)$ in the union of the first and third rows, which we will call *(k,ℓ)-Baxter tableaux.* [**21**]

(F): Diagonal rectangulations of size n, where k is the number of times the interior of the diagonal is intersected vertically, and ℓ is the number of times it is intersected horizontally. [**38**]

(G): Plane partitions in a $k \times \ell \times 3$ box, which we will call *Baxter plane partitions.*

Recall that a permutation $w = w_1 \dots w_n \in \mathfrak{S}_n$ has a descent at position i if $w_i > w_{i+1}$. A permutation w has an inverse descent at position i if w^{-1} has a descent as position i, which is equivalent to $i+1$ appearing to the left of i in w.

DEFINITION 2.1. Baxter permutations are those that avoid the patterns 3-14-2 and 2-41-3, where an occurrence of the pattern 3-14-2 in a permutation $w = w_1 \dots w_n$ means there exists a quadruple of indices $\{i, j, j+1, k\}$ with $i < j < j+1 < k$ and $w_j < w_k < w_i < w_{j+1}$ (and similarly for 2-14-3). Equivalently, it means there is an instance of the classical pattern 3142 where the elements representing 1 and 4 are adjacent in the original word.

When we only count subsequences with specified adjacencies in the original word, it is called a *vincular pattern*. For example, 25314 contains an instance of the classical pattern 2413, but not the vincular pattern 2-41-3.

For $n = 4$, there are $B(4) = 22$ Baxter permutations in $\mathfrak{S}_4$, with the only excluded ones being 2413 and 3142.

Twisted Baxter permutations have a syntactically similar definition, being those that avoid the vincular patterns 2-41-3 and 3-41-2.

Call these larger sets counted by $B(n)$ a set of *Baxter objects of order n*, and their subsets counted by $\Theta_{k,\ell}$ a set of *Baxter objects of order (k,ℓ)*. Each of these subsets has a natural involution that preserves k and ℓ:

- Conjugation by the longest permutation w_0 for (A), (B), and (C).
- Rotation by 180° about a central point for (D) and (F)
- Schützenberger evacuation for (E), which in the special case of a rectangular tableaux with N boxes corresponds to rotating the tableaux 180° and then replacing every label i with $N + 1 - i$.
- Taking the complement of a plane partition in the $k \times l \times 3$ box for (G).

Since Baxter permutations are closed under taking inverses [**38**], the map $w \mapsto w^{-1}$ provides an obvious bijection between Baxter objects (A) and (B). There are known bijections due to Dulucq and Guibert between the Baxter objects (A), (D) and (E) (see [**21**], [**22**]), and also between the objects (B), (C) and (F) due to Law and Reading (see [**38**]). We will also show the equivalence of objects (D) and (G). Chapter 2 is devoted to the proof of the following theorem:

THEOREM 2.2. *The given bijections between the above 7 classes of Baxter objects of order (k,ℓ), commute with their respective involutions.*

Since the bijections commute with the respective involutions, this means the number of Baxter objects of order (k,ℓ) fixed under involution is the same for all 7 classes of Baxter objects. Denote this common number $\Theta^{\circlearrowleft}_{k,\ell}$, and introduce a q-analogue of $\Theta_{k,\ell}$,

$$\Theta_{k,\ell}(q) := \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q} \tag{5}$$

where $n = k + \ell + 1$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q},$$

$$[m]!_q = [m]_q[m-1]_q \ldots [1]_q,$$

and

$$[j]_q = 1 + q + \ldots + q^{j-1}.$$

THEOREM 2.3. *$\Theta_{k,\ell}(q)$ lies in $\mathbb{N}[q]$, has symmetric coefficients, and satisfies $[\Theta_{k,\ell}(q)]_{q=-1} = \Theta^{\circlearrowleft}_{k,\ell}$.*

The proof of Theorem 2.3 is given in Section 2.3, using a result of Stembridge from the theory of plane partitions.

2.2. Proof of Theorem 2.2

In this section, we will show that the bijections between all our Baxter objects are equivariant. In each subsection, we will focus on showing the equivariance of bijections between two or three Baxter objects.

2.2.1. Objects (D) and (G). A plane partition is an array $(\pi_{i,j})_{i,j\geq 1}$ of non-negative integers with finitely many non-zero entries that weakly decrease along rows and columns. The plane partitions inside an $a \times b \times c$ box are those where $\pi_{i,j} \leq c$, and $\pi_{i,j} = 0$ if $i > a$ or $j > b$. Its complement in the $a \times b \times c$ box is the plane partition given by $\pi'_{i,j} = c - \pi_{a-i,b-j}$ for $1 \leq i \leq a$ and $1 \leq j \leq b$ and 0 elsewhere.

THEOREM 2.4. *There is a bijection between (k, ℓ)-Baxter paths and plane partitions in a $k \times \ell \times 3$ box, which equivariantly takes conjugation by w_0 to complementation of a plane partition.* [1]

PROOF. Each individual lattice path from A_i to B_i naturally corresponds to a partition λ_i inside of a $k \times \ell$ box, (our convention will be to take λ_i to be the part of the $k \times \ell$ box with A_i and B_i as corners that lies above the given lattice path).

[1]Thanks to Jang Soo Kim for noting this connection.

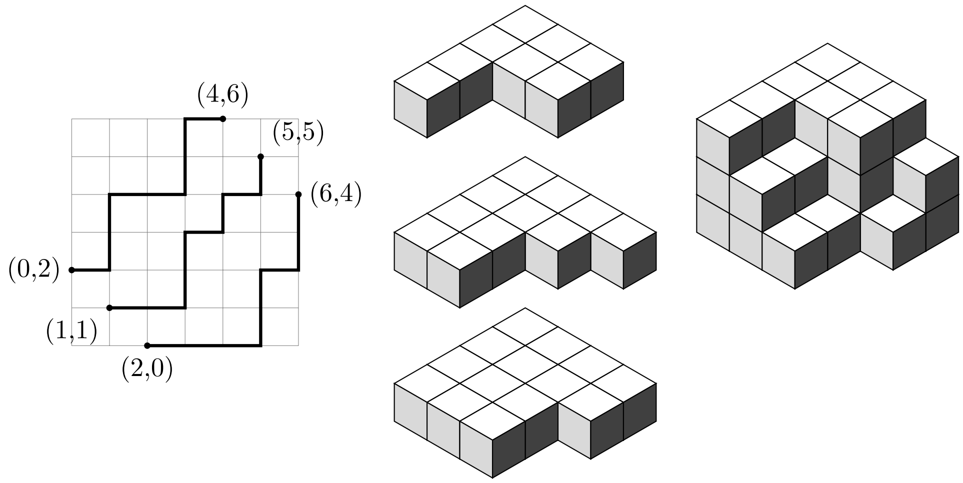

FIGURE 2.1. Example of the map from triples of non-intersecting lattice paths as in (4) to plane partitions in a $k \times \ell \times 3$ box, for $k = \ell = 4$. (Tikz code courtesy of Jang Soo Kim)

The non-intersecting condition is equivalent to requiring $\lambda_3 \subseteq \lambda_2 \subseteq \lambda_1$, which is precisely the condition necessary for a triple of partitions to form the layers of a plane partition when stacked. Additionally, one can see the involution on lattice paths (which is 180° rotation) corresponds to taking $\lambda_3 \subseteq \lambda_2 \subseteq \lambda_1$ to $\lambda_1^c \subseteq \lambda_2^c \subseteq \lambda_3^c$, where λ^c is the complement of λ in the $k \times \ell$ box, which is the same as taking the complement of the plane partition in the $k \times \ell \times 3$ box. $\square$

2.2.2. Objects (A), (D), and (E). One fundamental intermediate object in bijections between Baxter objects is a special sub-class of pairs of binary trees.

A *binary tree* is a rooted plane tree where every node has at most two children, denoted the left and right child. A *complete binary tree* is a binary tree where every node is either a leaf, or has has exactly two children. Let $\mathbb{BT}_n$ denote the set of binary trees with n nodes, and $\mathbb{CBT}_{2n+1}$ the set of complete binary trees on $2n+1$ nodes.

If we truncate all of the leaves from a complete binary tree on $2n+1$ nodes, we are left with a binary tree on n nodes. If we have a binary tree on n nodes, we can extend it to a binary tree on $2n+1$ nodes by adding leaves to every node with 0 or 1 children. The processes of truncation and extension are clearly inverse to each other.

DEFINITION 2.5. Let $\mathrm{Trunc} : \mathbb{CBT}_{2n+1} \mapsto \mathbb{BT}_n$ be the bijection from complete binary trees on $2n$+1 nodes to binary trees on n nodes obtained by truncating leaves, with inverse map called Extend. Let $\mathrm{Trunc}^{\times 2} : \mathbb{CBT}_{2n+1} \times \mathbb{CBT}_{2n+1} \mapsto \mathbb{BT}_n \times \mathbb{BT}_n$ (resp. $\mathrm{Extend}^{\times 2}$) be the corresponding maps on pairs of trees.

We call a leaf a *left* (resp. *right*) *leaf* if it is a left (resp. right) child of its parent.

DEFINITION 2.6. Let code : $\mathbb{CBT}_{2n+1} \mapsto \{0,1\}^{n-1}$ be the function that reads off the pattern of left and right leaves in a complete binary tree from left to right (excluding the left-most left leaf and right-most right leaf) by assigning a 1 to left leaves and a 0 to right leaves

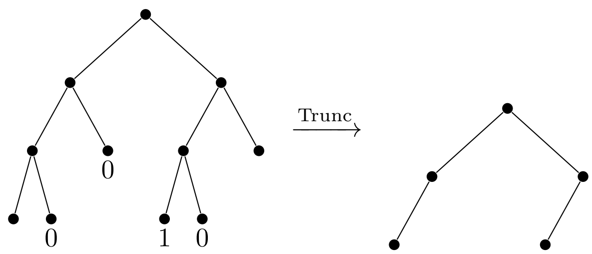

FIGURE 2.2. Example of Trunc taking an element of $\mathbb{CBT}_{11}$ with leaf code 0010 to an element of $\mathbb{BT}_5$.

We will mainly be interested in pairs of complete binary trees satisfying a compatibility relation.

DEFINITION 2.7. Let $\mathrm{Twin}_n \subset \mathbb{CBT}_{2n+1} \times \mathbb{CBT}_{2n+1}$ be the set of pairs of complete binary trees, (T_L, T_R), where $\mathrm{code}(T_L)$ is the same as $\mathrm{code}(T_R)$ if we interchange 0's and 1's. Let $\widetilde{\mathrm{Twin}_n}$ be the image of Twin_n under $\mathrm{Trunc}^{\times 2}$.

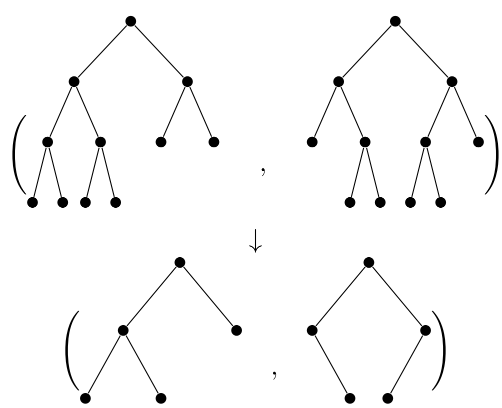

FIGURE 2.3. Example of map from Twin_n to $\widetilde{\mathrm{Twin}_n}$ for $n = 5$.

Clearly Twin_n and $\widetilde{\mathrm{Twin}_n}$ are in bijection.

Definition 2.8 ([**50**, §1.2]). Given a word $w = w_1w_2\dots w_n$ with distinct letters in $\mathbb{N}_{\geq 1}$, recursively define a binary tree $\mathrm{incr}(w)$ called the *increasing binary tree* for w by saying that if $w = uxv$ with $x = \min\{w_1, \dots w_n\}$, then $\mathrm{incr}(w)$ has x as its root, $\mathrm{incr}(u)$ as its left subtree, and $\mathrm{incr}(v)$ as its right subtree. Similarly, recursively define a binary tree $\mathrm{decr}(w)$ called the *decreasing binary tree* of w by saying that if $w = uxv$ with $x = \max\{w_1, \dots w_n\}$, then $\mathrm{decr}(w)$ has x as its root, $\mathrm{decr}(u)$ as its left subtree, and $\mathrm{incr}(v)$ as its right subtree.

While this process gives a labelled binary tree, we will only consider $\mathrm{incr}(w)$ and $\mathrm{decr}(w)$ to be the underlying unlabelled binary tree.

Definition 2.9. Let $\Psi : S_n \mapsto \mathbb{BT}_n \times \mathbb{BT}_n$ be the map that sends a permutation w to the pair of binary trees $(\mathrm{incr}(w), \mathrm{decr}(w))$.

Theorem 2.10 (Dulucq and Guibert, [**21**]). *$\Psi : \mathrm{Bax}_n \mapsto \widetilde{\mathrm{Twin}_n}$ is a bijection.*

It is known that if w has k ascents and ℓ descents, then $\mathrm{Extend}(\mathrm{incr}(w))$ will have $k + 1$ left leaves and $\ell + 1$ right leaves.

Definition 2.11. Say $w \in S_n$ is alternating if $w_1 < w_2 > w_3 < w_4 \dots$. Let $\mathrm{AltBax_{2n}}$ denote the set of alternating Baxter permutations of length $2n$.

Recall that alternating permutations have the property that $\mathrm{incr}(w)$ (resp. $\mathrm{decr}(w)$) is a complete binary tree if we add a left-most left leaf (resp. right-most right leaf) [**50**, Prop. 1.3.14].

Corollary 2.12. *The function Ψ is a bijection from alternating Baxter permutations of length $2n$ to all pairs of complete binary trees with $2n + 1$ nodes each.*

Definition 2.13. The natural involution on $\mathbb{CBT}_{2n+1} \times \mathbb{CBT}_{2n+1}$ (which has Twin_n as a subset) is taking the mirror reflection of each tree, and then swapping the two trees (see Figure 2.4).

Proposition 2.14. *Ψ equivariantly maps permutations with the action of conjugation by the longest element to pairs of twin trees with this involution action.*

This proposition is obvious from the definition of Ψ in terms of increasing and decreasing trees.

The equivalence of Baxter permutations fixed under conjugation by w_0 and triples of non-intersecting of lattice paths fixed under rotation, along with a number of other Baxter objects fixed under their respective involutions, is given by Felsner, Fusy, Noy, and Orden [**23**]. They follow from the fact that the bijections between the corresponding Baxter objects are all equivariant with respect to the natural involutions.

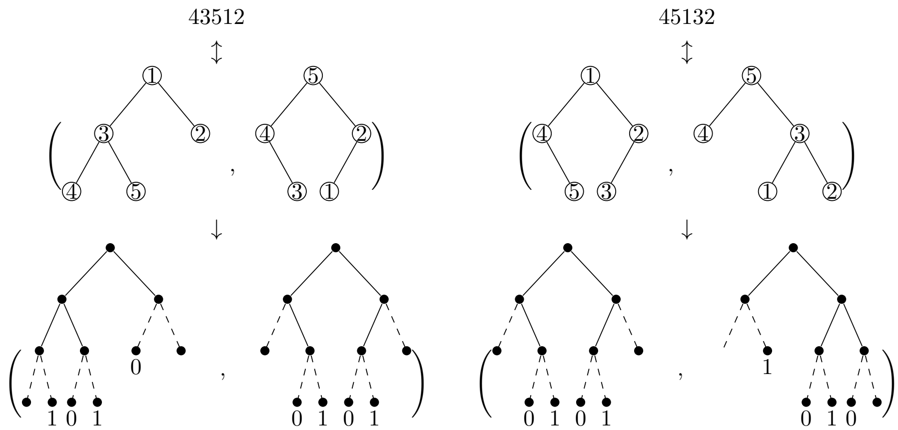

FIGURE 2.4. Map from Bax_n to Twin_n for $n = 5$, and the corresponding action under involution.

THEOREM 2.15 (Dulucq and Guibert, [**21**]). *There is a bijection between elements of* Twin_n *with* $k+1$ *left leaves and* $\ell+1$ *right leaves, and (k,ℓ)-Baxter paths.*

PROPOSITION 2.16. *The above bijection equivariantly takes the natural involution on* Twin_n *to rotation by* $180°$ *on triples of non-intersecting lattice paths.*

PROOF. Given a pair of twin trees (T_L, T_R), the first path (resp. third path) arises from reading the internal nodes of T_L (resp. T_R) in infix order, recording whether they are left or right children of their parents. The middle path is determined by $\mathrm{code}(T_L)$, which by the twin condition encodes the same information as $\mathrm{code}(T_R)$. It is not hard to see that the effect on this encoding of taking the mirror reflections of T_L and T_R and swapping them will be exactly the same as taking a rotation by $180°$ on triples of non-intersecting lattice paths. □

Felsner, Fusy, Noy, and Orden [**23**] have additionally shown the bijections to a number of other Baxter objects are also equivariant. One interesting Baxter family not included are Baxter tableaux, or $3 \times n$ standard Young tableaux with no consecutive entries in the same row, which we will now look at.

Cori, Dulucq, and Viennot [**19**] begin by working with a larger set of objects, counted not by $B(n)$, but by c_n^2, where $c_n = \frac{1}{n+1}\binom{2n}{n}$ is the *Catalan number.*

DEFINITION 2.17. Let $Y_n^{(k)}$ be the language of all words in $\{1, 2, \ldots, k\}$ such that each letter appears exactly n times, and for any prefix of the word, i appears at least as often as $i+1$. These are exactly the Yamanouchi words for standard Young tableaux of a $k \times n$ box[2].

A Yamanouchi word for a standard Young tableau is a word where the i^{th} letter indicates which row of the tableau i appears in.

Example 2.18. Let $P_{z,\overline{z}}$ be the language of well-formed parenthesis systems on the letters $\{z, \bar{z}\}$. If we think of 1's as being z's (corresponding to left parentheses) and 2's as being $\overline{z}$'s (corresponding to right parentheses), we can see that $P_{z,\overline{z}}$ corresponds to $\cup_{n\geq 0} Y_n^{(2)}$.

Evacuation can be defined for general standard Young tableaux, but in the special case of rectangular shape, it takes a particularly nice form.

Definition 2.19. Given a standard Young tableaux T of square shape with N boxes, let $\text{evac}(T)$ be the Young tableaux we get by rotating T by $180°$, and then replacing each label i with $N+1-i$.

This action also takes a nice form on the corresponding Yamanouchi words.

Definition 2.20. If $x = x_1 x_2 \dots x_{3n-1} x_{3n} \in Y_n^{(3)}$, then let

$$\text{evac}(x) = (3 - x_{3n})(3 - x_{3n-1}) \dots (3 - x_2)(3 - x_1).$$

Example 2.21. $\text{evac}(112323) = 121233$

We introduce an intermediate object, consisting of certain shuffles of two parenthesization systems.

Definition 2.22. Let Shuffle_{2n} be the set of all shuffles of $P_{a,\bar{a}}$ and $P_{b,\bar{b}}$ of length $2n$ such that for every prefix ending in b, the number of a's is strictly greater than the number of $\bar{a}$'s.

Example 2.23. $ab\bar{b}b\bar{a}a\bar{a}a\bar{b}\bar{a} \in \text{Shuffle}_{10}$

Theorem 2.24 (Cori, Dulucq, Viennot [**19**]). *There are bijections between* AltBax_{2n}*,* Shuffle_{2n}*, and pairs of complete binary trees with* $2n+1$ *nodes each.*

In particular, each set has c_n^2 *objects.*

The bijection $\beta : \text{AltBax}_{2n} \to \text{Shuffle}_{2n}$ will later be recalled in Definition 2.30.

Later, Dulucq and Guibert showed that there was an additional bijection to a special class of Yamanouchi words.

[2]They are also referred to as *stack words*, as they encode the permutations that can be sorted with $k-1$ stacks [**32**].

Definition 2.25. Let $Y_n^{(3)}(22)$ be the subset of $Y_n^{(3)}$ consisting of Yamanouchi words avoiding the consecutive pattern 22 (corresponding to Young tableaux with no consecutive entries in the middle row). Let $Y_n^{(3)}(11,22,33)$ be the subset of $Y_n^{(2)}$ consisting of Yamanouchi words avoiding the consecutive patterns 11, 22, or 33 (corresponding to Baxter tableaux, which have no consecutive entries in any row).

Theorem 2.26 (Dulucq and Guibert [**21**]). *There is a bijection between* $\mathrm{Shuffle}_{2n}$ *and* $Y_n^{(3)}(22)$. *This bijection is given by the map that sends* $\alpha = \alpha_1\alpha_2\ldots\alpha_{2n} \in \mathrm{Shuffle}_{2n}$ *to* $f(\alpha) = \Phi(\alpha_1)\Phi(\alpha_2)\ldots\Phi(\alpha_{2n})$, *where*

$$\begin{cases} \Phi(a) = 1 \\ \Phi(b) = 21 \\ \Phi(\bar{a}) = 23 \\ \Phi(\bar{b}) = 3 \end{cases}$$

Example 2.27.

$$\begin{aligned} &\Phi(\ a\ \ b\ \ \bar{b}\ \ b\ \ \bar{a}\ \ a\ \ \bar{a}\ \ a\ \bar{b}\ \ \bar{a}\ \ ) \\ =\ &\ 1\ 21\ 3\ 21\ 23\ 1\ 23\ 1\ 3\ 23 \end{aligned}$$

It is not immediately clear that all of the maps are necessarily equivariant with respect to their natural involutions. We will show that the original bijections of Cori, Dulucq, and Viennot on the objects counted by c_n^2 (AltBax_{2n}, $\mathrm{Shuffle}_{2n}$, $\mathbb{CBT}_{2n+1} \times \mathbb{CBT}_{2n+1}$, $Y_n^{(3)}(22)$) are equivariant with respect to their involutions. Then an equivariant bijection from the Baxter tableaux (or equivalently, Yamanouchi words in $Y_n^{(3)}(11,22,33)$) to Twin_n is obtained by restricting the equivariant bijection from Yamanouchi words in $Y_n^{(3)}(22)$ to all pairs of complete binary trees.

$$\begin{tikzcd}
 & \mathbb{CBT}_n \times \mathbb{CBT}_n \arrow[r, leftrightarrow, "\Psi"] & \mathrm{AltBax}_{2n} \arrow[r, leftrightarrow, "\beta"] & \mathrm{Shuffle}_{2n} \arrow[r, leftrightarrow, "f"] & Y_n^{(3)}(22) \\
\mathrm{Bax}_n \arrow[r, leftrightarrow, "\mathrm{Extend}^{\times 2} \circ \Psi"] & \mathrm{Twin}_n \arrow[u, hook] \arrow[rrr, leftrightarrow] & & & Y_n^{(3)}(11,22,33) \arrow[u, hook]
\end{tikzcd}$$

Figure 2.5. Diagram indicating the maps between objects counted by c_n^2, and their restriction to Baxter objects.

Lastly, we show that the bijection between the Twin_n and Bax_n is equivariant, making the composite map from Baxter tableaux to Baxter permutations equivariant.

First, we note that it is trivial to check that the map from Baxter permutations to alternating Baxter permutations of length $2n$ equivariantly takes conjugation by

w_0 on S_n to conjugation by w_0 on S_{2n}. The map Ψ sends a Baxter permutation w of length n to a pair of twin trees in Twin_n, $\big(\mathrm{incr}(w), \mathrm{decr}(w)\big)$, equivariantly. But by Corollary 2.12, Ψ^{-1} will equivariantly map this to AltBax_{2n}. So it suffices to check that the map from AltBax_{2n} to $Y_n^{(3)}(22)$ is equivariant.

PROPOSITION 2.28. *An equivalent formulation for the Baxter condition on permutations of length n says that for every $p \in [n-1]$, we can either write the permutation as*

$$\pi = \pi' p \pi_- \pi_+ (p+1) \pi'' \quad \text{or} \quad \pi = \pi'(p+1)\pi_+\pi_- p \pi'',$$

where the (possibly empty) subsequence π_- (resp. π_+) consists of values less than p (resp. greater than $p+1$).

EXAMPLE 2.29. For $\pi = 5671342 \in \mathrm{Bax}_7$ and $p = 4$, we have $\pi' = \emptyset$, $\pi_+ = 67$, $\pi_- = 13$, and $\pi'' = 2$.

The proof of this is straighforward, and left to the reader.

This allows us to construct a map from AltBax_{2n} to $\mathrm{Shuffle}_{2n}$, which is in fact the bijection referred to in Theorem 2.24.

DEFINITION 2.30. Let $\beta : \mathrm{AltBax}_{2n} \mapsto \mathrm{Shuffle}_{2n}$ be the map defined as follows: Given a $\pi \in \mathrm{AltBax}_{2n}$, for each $p \in [2n-1]$, look at the relative order of p and $p+1$, whether π_- is empty, and whether π_+ is empty. We call this triple of information the *type* of p (with respect to π). If $\beta(\pi) = \alpha$, for each of these 8 types, there are two possible strings of length two that $\alpha_p\alpha_{p+1}$ could be, listed in the figure below. Starting with $\alpha_1 = a$, we can recursively construct α by noting that only one of the two choices for $\alpha_p\alpha_{p+1}$ will be consistent with what we already know α_p must be.

Type	π						$\alpha_p\alpha_{p+1}$
Type 1	π'	p	$\emptyset$	$\emptyset$	$p+1$	π''	$a\bar{a}$ or $b\bar{a}$
Type 2	π'	p	$\emptyset$	π_+	$p+1$	π''	ab or bb
Type 3	π'	p	π_-	$\emptyset$	$p+1$	π''	$\bar{a}\bar{a}$ or $\bar{b}\bar{a}$
Type 4	π'	p	π_-	π_+	$p+1$	π''	$\bar{a}b$ or $\bar{b}b$
Type 5	π'	$p+1$	$\emptyset$	$\emptyset$	p	π''	$a\bar{b}$ or $b\bar{b}$
Type 6	π'	$p+1$	$\emptyset$	π_-	p	π''	$\bar{a}\bar{b}$ or $\bar{b}\bar{b}$
Type 7	π'	$p+1$	π_+	$\emptyset$	p	π''	aa or ba
Type 8	π'	$p+1$	π_+	π_-	p	π''	aa or ba

FIGURE 2.6. Relations between $\pi \in \mathrm{AltBax}_{2n}$, $\alpha \in \mathrm{Shuffle}_{2n}$, $\mathrm{incr}(\pi)$, and $\mathrm{decr}(\pi)$.

Example 2.31. As a working example, we will start with $\pi = 2314 \in \mathrm{AltBax}_4$.

For $p = 1$, we see that $p+1$ occurs before p, π_+ is non-empty, and π_- is empty. This means $p = 1$ is type 7, and so $\alpha_1\alpha_2$ is either aa or ba. But since a shuffle word has to start with a, we know $\alpha_1\alpha_2 = aa$.

For $p = 2$, we see that p occurs before $p+1$, and that π_+ and π_- are both empty. This means $p = 2$ is type 1, and so $\alpha_2\alpha_3$ is either $a\bar{a}$ or $b\bar{a}$. Only the first case is consistent with us previously finding $\alpha_2 = a$, so $\alpha_3 = \bar{a}$.

For $p = 3$, we see that p occurs before $p+1$, π_- is non-empty, and π_+ is empty. This mean $p = 3$ is type 3, and so $\alpha_3\alpha_4$ is either $\bar{a}\bar{a}$ or $\bar{b}\bar{a}$. Only the first choice is consistent with $\alpha_3 = \bar{a}$, so $\alpha_4 = \bar{a}$.

Thus, we see that $\beta(2314) = aa\bar{a}\bar{a}$.

Since this map is a bijection, as knowing the type for each $\alpha_p\alpha_{p+1}$ uniquely determines α, knowing the type for each p is enough to recover what the original alternating Baxter permutation is.

Thus, we will find it convenient to encode elements of AltBax_{2n} and $\mathrm{Shuffle}_{2n}$ as words of length n on the letter set $\{1, 2 \ldots, 8\}$, where the p^{th} letter indicates the type of p in w (resp. the type of $\alpha_p\alpha_{p+1}$).

Example 2.32. For $\pi = 2314$, as $p = 1$ was the type 7, $p = 2$ was the type 1, and $p = 3$ was the type 3, we would encode this element as 713.

Theorem 2.33. *The bijection between* AltBax_{2n} *and* $Y_n^{(3)}(22)$ *is equivariant with respect to conjugation by* w_0 *and evacuation.*

Proof. The bijection of Dulucq and Guibert from AltBax_{2n} to $Y_n^{(3)}(22)$ is a composition of a map β from AltBax_{2n} to $\mathrm{Shuffle}_{2n}$ and the previously defined f from $\mathrm{Shuffle}_{2n}$ to $Y_n^{(3)}(22)$. So we need to show that if $f(\beta(w)) = x$, then $f(\beta(w_0ww_0)) = \mathrm{evac}(x)$.

We note that the elements in the intermediate set, $\mathrm{Shuffle}_{2n}$, have no natural involution associated to them. However, we can define an involution on $\mathrm{Shuffle}_{2n}$ by mapping it bijectively to a set with a natural involution, doing an involution there, and then mapping it back.

This gives us two possible ways of defining an involution on $\mathrm{Shuffle}_{2n}$ that are not obviously the same. One option is $\alpha \mapsto \beta(w_0\beta^{-1}(\alpha)w_0)$, induced from AltBax_{2n}. The other is $\alpha \mapsto f^{-1}(\mathrm{evac}(f(\alpha)))$, induced from $Y_n^{(3)}(22)$. Proving equivariance is equivalent to showing that these two induced involutions are the same.

First, we describe the involution on $\mathrm{Shuffle}_{2n}$ induced from conjugation by w_0 on the alternating Baxter permutations.

If we know what type p is in the original word π, we can readily figure out the type of $n - p$ in the involuted word, $\hat{\pi} = w_0\pi w_0$, as:

- p appears before $p+1$ in π iff $n-p$ appears before $n+1-p$ in $\hat{\pi}$.
- π_- is empty iff $\hat{\pi}_+$ is empty.
- π_+ is empty iff $\hat{\pi}_-$ is empty.

Thus, if p is of type 1,2,3,4,5,6,7,8 in the original word, then $n-p$ will be of type 1,3,2,4,5,7,6,8 (respectively)

This means that the involution on Shuffle_{2n} induced from AltBax_{2n} corresponds to reversing the encoded word, swapping 2's and 3's, and swapping 7's and 6's.

Example 2.34. The encoded word for $\pi = 2314$ is 713, so the encoded word for $w_0\pi w_0 = 1423$ should be 216. Sure enough, in 1423, $p=1$ is of type 2, $p=2$ is of type 1, and $p=3$ is of type 6.

Now, we consider the relationship between Shuffle_{2n} and $Y_n^{(3)}(22)$. Say we have $f(\alpha) = a$. Each of the $2n$ letters of α corresponds to one of the $2n$ instances of 1 and 3 in a, and additionally keeps track of whether or not that instance of 1 or 3 is preceded by a 2.

Let $\hat{\alpha} = f^{-1}(\text{evac}(f(\alpha))$, representing the involution on Shuffle_{2n} induced by $Y^{(3)}(22)$.

Proposition 2.35. *$\hat{\alpha}_{2n+1-p}$ corresponds to a 1 (resp. 3) if and only if α_p corresponds to a 3 (resp. 1).*

Proof. Doing evacuation on a Yamanouchi word corresponds to reversing the word, and swapping 1's and 3's. So if the p^{th} occurrence of either a 1 or a 3 in $x \in Y^{(3)}(22)$ is a 1 (resp. 3) the $(2n+1-p)^{th}$ occurence of either a 1 or a 3 in $\text{evac}(x)$ will be a 3 (resp. 1). So if $\hat{\alpha} = f^{-1}(\text{evac}(f(\alpha))$, then $\hat{\alpha}_{2n+1-p}$ will be either $\bar{a}$ or $\bar{b}$ (resp. a or b). $\square$

Proposition 2.36. *The 1 or 3 in $\hat{\alpha}_{2n+1-p}$ will be preceded by a 2 if and only if α_{p+1} corresponds to something that is preceded by a 2.*

This easily follows from the fact that evacuation reverses the word.

Say we know whether α_p corresponds to a 1 or a 3, and what α_{p+1} is. There are 8 different cases, and for each case there are two possibilities for what $\alpha_p\alpha_{p+1}$ could be (depending on whether or not the 1 or 3 in α_p is preceded by a 2 or not). One can see that these are exactly the 8 different types from Figure 2.2.2.

Example 2.37. Say we know that α_p corresponds to a 1, that α_{p+1} corresponds to a 1, and α_{p+1} is preceded by a 2. Then $\alpha_p\alpha_{p+1}$ could be ab or bb, corresponding to type 2.

By Proposition 2.35, we can determine whether $\hat{\alpha}_{2n-p}$ and $\hat{\alpha}_{2n+1-p}$ correspond to 1's or 3's. By Proposition 2.36, we can also determine whether $\hat{\alpha}_{2n+1-p}$ is preceded by a 2 or not. Thus, we can determine which of the 8 different cases from Figure 2.2.2 $\hat{\alpha}_{2n-p}\hat{\alpha}_{2n+1-p}$ corresponds to. One can check case-by-case that we get the same correspondence as before.

Example 2.38. Say we know that $\alpha_p\alpha_{p+1}$ is of type 2 as in the previous example. Then α_p corresponding to a 1 means $\hat{\alpha}_{2n+1-p}$ corresponds to a 3. And α_{p+1} corresponding to a 1 means $\hat{\alpha}_{2n-p}$ corresponds to a 3. Finally, α_{p+1} being preceded by a 2 means $\hat{\alpha}_{2n+1-p}$ is preceded by a 2. Thus, $\hat{\alpha}_{2n-p}\hat{\alpha}_{2n+1-p}$ could be $\bar{a}\bar{a}$ or $\bar{b}\bar{a}$, corresponding to type 3.

□

Corollary 2.39. *The bijection between* Twin_n *and* Bax_n *is equivariant.*

Corollary 2.40. *Let $n = k + \ell + 1$. The bijection from (k,ℓ)-Baxter permutations to (k,ℓ)-Baxter tableaux is equivariant with respect to conjugation by w_0 and evacuation.*

Remark 2.41. Although it is not our primary interest, the following corollary also allows us to count how many alternating Baxter permutations of length $2n$ and standard $3 \times n$ Young tableaux avoiding 22 are fixed under evacuation.

Corollary 2.42. *The number of alternating Baxter permutations of length $2n$ fixed under conjugation by w_0 and the number of $3 \times n$ standard Young tableaux with no consecutive entries in the middle row fixed under evacuation are both equal to c_n, the Catalan number.*

Proof. We have an equivariant bijection between pairs of complete binary trees and these two objects, so it suffices to count how many pairs of complete binary trees are fixed under their involution. The involution on pairs of trees is given by reflecting each tree horizontally and then swapping the order of the pair. So a pair fixed under involution is completely determined by the first tree, and it is clear that each complete binary tree yields a pair fixed under involution (by pairing a tree T with a reflected copy of itself). So there are as many pairs of complete binary trees fixed under involution as there are pairs of complete binary trees, of which there are c_n.

□

In Figure 2.7, one can see that the original alternating Baxter permutation is fixed under conjugation by w_0, and that the right tree is the mirror image of the left tree.

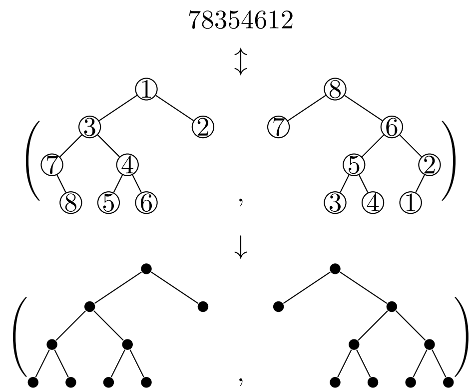

FIGURE 2.7. Map from alternating Baxter permutations of length $2n$ to pairs of complete binary trees with $2n+1$ nodes.

2.2.3. Objects (A) and (B). Next, we show the equivalence of objects (A) and (B), using the following fact.

PROPOSITION 2.43 (Law and Reading [**38**], Corollary 4.2). *A permutation w lies in Bax_n if and only if w^{-1} lies in Bax_n.*

If one were dealing with regular pattern avoidance, this would be trivial, because a permutation w contains an instance of 2413 (resp. 3142) if and only if w^{-1} contains an instance of 3142 (resp. 2413). However, one has to do some extra work to check that the analogous statement holds when one has the extra adjacency conditions of vincular patterns.

PROPOSITION 2.44. *The map $w \mapsto w^{-1}$ gives a bijection between Baxter permutations with k descents and Baxter permutations with k inverse descents that commutes with conjugation by w_0.*

PROOF. Conjugation by w_0 commutes with $w \mapsto w^{-1}$, since $w_0^{-1} = w_0$. □

While this result on its own is elementary, it is important because the previous Baxter families all had statistics that naturally corresponded to ascents/descents, while the remaining Baxter families all have statistics that will correspond to inverse ascents/inverse descents.

2.2.4. Objects (B) and (C). There is another class of Baxter objects known as *twisted Baxter permutations*. While Baxter permutations avoid the patterns 3-14-2 and 2-41-3, twisted Baxter permutations avoid the patterns 3-41-2 and 2-41-3. Even though the two pairs of patterns look similar, it is not immediately obvious that they should be so closely related. Section 8 of Law and Reading's paper [**38**] provides a bijection between the two that relies on looking at fibers of the lattice congruence Θ_{3412} on the weak order for S_n, as we next explain.

DEFINITION 2.45. For $w \in S_n$, let $\mathrm{Inv}(w)$ be the set of inversions, or pairs (w_i, w_j) with $1 \leq i < j \leq n$ such that $w_j < w_i$. We say that $u \leq v$ in the weak order if $\mathrm{Inv}(u) \subseteq \mathrm{Inv}(v)$.

THEOREM 2.46 (Corollary 3.1.4, [**10**]). *The covering relations for the weak order on S_n come precisely from the pairs of permutations that differ only in two adjacent entries.*

We need the following proposition, which follows immediately from Proposition 8.1 in their paper.

DEFINITION 2.47. A 3-14-2 $\rightarrow$ 3-41-2 move on a permutation is an action that takes an instance of the pattern 3-14-2, and switches the adjacent entries in the middle so the subsequence corresponds to an instance of the pattern 3-41-2. That is to say, if $w = w_1 \dots w_n$ has an instance of 3-14-2 corresponding to the subsequence $w_i w_j w_{j+1} w_k$, then a 3-14-2 $\rightarrow$ 3-41-2 move would send w to $w_1 \dots w_{j-1} w_{j+1} w_j w_{j+2} \dots w_n$.

PROPOSITION 2.48. *Given a twisted Baxter permutation, it will be the maximal element in its fiber over Θ_{3412}, the corresponding Baxter permutation will be the unique minimal element, and the fiber will consist of all permutations attainable from the twisted Baxter permutation by making any sequence of (3-14-2 $\rightarrow$ 3-41-2) moves.*

COROLLARY 2.49. *The number of twisted Baxter permutations of length n with k inverse descents is equal to the number of Baxter permutations of length n with k inverse descents.*

PROOF. The moves that get us from a twisted Baxter permutation to a Baxter permutation will never change the number of inverse descents. Swapping the elements playing the role of 1 and 4 in adjacent positions will never change the relative order of i and $i+1$ for any i. $\square$

COROLLARY 2.50. *A twisted Baxter permutation and its corresponding Baxter permutation are each fixed under conjugation by w_0 if and only if their common fiber is fixed under conjugation by w_0.*

PROOF. Since the fibers of this congruence can be described as the orbit of all possible (3-14-2 $\leftrightarrow$ 3-41-2) moves, conjugation by w_0 will map fibers to fibers. $\square$

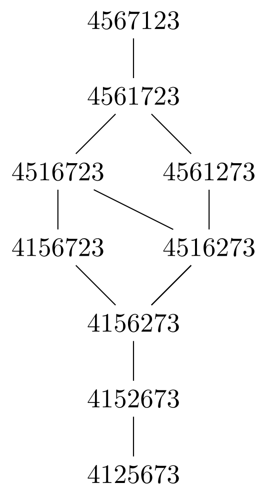

Figure 2.8. Fiber of the congruence θ_{3412} with the Baxter permutation 4567123 as its maximal element, and the twisted Baxter permutation 4125673 as its minimal element.

2.2.5. Objects (C) and (F).

Definition 2.51. A *diagonal rectangulation* of size n is a subdivision of an $n \times n$ square into n rectangles (with lattice points for corners) such that the interior of every rectangle intersects a fixed diagonal of the square.

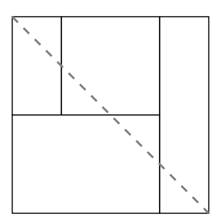

Figure 2.9. A diagonal rectangulation of size $n = 4$

We next check to see that the bijection between twisted Baxter permutations and diagonal rectangulations given in Section 6 of Law and Reading [**38**] preserves the indicated statistic, and will equivariantly take conjugation by w_0 to 180° rotation. We again have the intermediate object of pairs of twin trees.

The map from twisted Baxter permutations to pairs of twin trees used by Law and Reading is equivalent to $w \mapsto \text{Extend}^{\times 2}(w^{-1})$ (they respectively call these the *upper* and *lower planar binary trees*). Conjugation by w_0 on twisted Baxter permutations will correspond to the same involution on pairs of trees defined in Definition 2.13. Also, if a twisted Baxter permutation w has k inverse ascents, w^{-1}

will have k ascents, and $\mathrm{incr}(w^{-1})$ will have k left leaves (excluding the left-most one), preserving the statistic.

A diagonal rectangulation is made by gluing the two trees together. In particular, one draws the trees so that all the leaves are evenly spaced on the lowest level, and all intersections make right angles. Then the twin tree condition guarantees that if we turn the left tree upside-down, it will match up with the right tree to form a diagonal rectangulation (see Figure 2.10).

It is then obvious that the involution on pairs of trees corresponds to 180° rotation on diagonal rectangulations, and that k left leaves in the left tree (excluding the left-most one) will correspond to the k vertical intersections with the interior of the diagonal.

$$w = 3124, w^{-1} = 2314$$

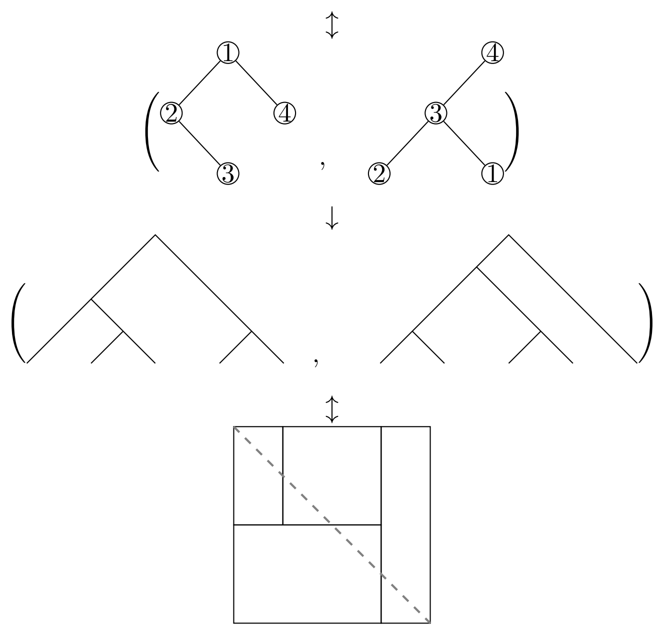

FIGURE 2.10. Map from twisted Baxter permutations to diagonal rectangulations

2.3. Proof of Theorem 2.3

By Theorem 2.2, if we want to count the number of objects fixed under involution for any Baxter object, we only have to find the number of objects fixed under involution for one family of Baxter objects. This is easiest for Baxter plane partitions. MacMahon gave a closed formula for the generating function of plane partitions inside a box, weighted by number of boxes.

THEOREM 2.52 ([**51**],Theorem 7.21.7). *Fix a, b, and c, and let $|\pi|$ be the total number of boxes in a plane partition. Then*

$$\sum_{\pi} q^{|\pi|} = \prod_{\substack{i=1,\ldots,a\\ j=1,\ldots,b\\ k=1,\ldots,c}} \frac{[i+j+k-1]_q}{[i+j+k-2]_q} \tag{6}$$

where π runs over all plane partitions that fit in an $a \times b \times c$ box.

We will write the above sum as $M(a,b,c;q)$. One can check that for Baxter plane partitions, this gives the previously defined q-analogue of $\Theta_{k,\ell}$.

COROLLARY 2.53.

$$M(k,\ell,3;q) = \sum_{\pi} q^{|\pi|} = \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q} = \Theta_{k,\ell}(q) \tag{7}$$

where π runs over all plane partitions that fit in a $k \times \ell \times 3$ box.

In particular, this tells us that $\Theta_{k,\ell}(q)$ is indeed a polynomial with symmetric, non-negative integer coefficients, which is not immediately obvious from the definition.

Additionally, we have the following theorem of Stembridge.

THEOREM 2.54 (Stembridge, Example 2.1, [**52**]). *The number of self-complementary plane partitions that fit inside an $a \times b \times c$ box is $M(a,b,c;-1)$.*

By setting $a = k$, $b = \ell$, and $c = 3$, we get the following result.

THEOREM 2.55. $\Theta^{\circlearrowleft}_{k,\ell} = [\Theta_{k,\ell}(q)]_{q=-1}$

Although Theorem 2.55 follows from Stembridge's result without any further computation, it turns out that it agrees with formulas for $\Theta^{\circlearrowleft}_{k,\ell}$ given previous by Felsner, Fusy, Orden, and Noy [**23**], after correcting one of the cases of their formula, and applying a hypergeometric summation, as we explain next.

THEOREM 2.56.

(1) *If k and ℓ are odd, then $\Theta^{\circlearrowleft}_{k,\ell} = 0$*

(2) *If k and ℓ are even, with $k = 2\kappa$ and $\ell = 2\lambda$, then for $N = \kappa + \lambda$,*

$$\begin{aligned}\Theta^{\circlearrowleft}_{2\kappa,2\lambda} &= \sum_{r\geq 0} \frac{2r^3}{(N+1)(N+2)^2}\binom{N+2}{\kappa+1}\binom{N+2}{\kappa-r+1}\binom{N+2}{\kappa+r+1} \\ &= \frac{\binom{N+1}{\kappa}\binom{N+1}{\kappa+1}\binom{N}{\kappa}}{(N+1)} \end{aligned}\tag{8}$$

(3) *If k is odd and ℓ is even*[3]*, with $k=2\kappa+1$ and $\ell=2\lambda$, then for $N=\kappa+\lambda$,*

$$\begin{aligned}
\Theta^{\circlearrowleft}_{2\kappa+1,2\lambda} &= \Theta^{\circlearrowleft}_{2\kappa,2\lambda} \\
&\quad + \sum_{r\geq 1} \frac{(\lambda-r+1)r(r+1)(2r+1)}{(\kappa+2+r)(N+1)(N+2)^2}\binom{N+2}{\kappa+1}\binom{N+2}{\kappa-r+1}\binom{N+2}{\kappa+r+1} \\
&= \frac{\binom{N+1}{\kappa}\binom{N+1}{\kappa+1}\binom{N+1}{\kappa+1}}{(N+1)}
\end{aligned} \tag{9}$$

(4) *If k is even and ℓ is odd, with $k=2\kappa$ and $\ell=2\lambda+1$, then*

$$\Theta^{\circlearrowleft}_{2\kappa,2\lambda+1} = \Theta^{\circlearrowleft}_{2\lambda+1,2\kappa},$$

which is the same as (9) with κ and λ switched.

Before embarking on the proof, we review the approach used by Felsner, Fusy, Orden, and Noy. They counted non-intersecting triples of lattice paths as in (4) fixed under rotation. The rotation will be about the point $(\frac{k}{2},\frac{\ell}{2})$, and a rotationally invariant Baxter path will be uniquely determined by what it does below the line $x+y=\frac{k+\ell}{2}$. In [**23**], the authors show that all rotationally invariant Baxter paths arise from triples of lattice paths from A_1, A_2, and A_3 to specific points below the line $x+y=\frac{k+\ell}{2}$, which depend on the parity of k and ℓ, and also a parameter r. These triples of lattice paths can be counted by the Gessel-Viennot-Lindstrom lemma [**31**], and they obtain their formula by summing the resulting expressions over all possible parameters r for k and ℓ of fixed parity, resulting in the first parts of (8) and (9).

Proof of Theorem 2.56.

Proof of Assertion 1. When k and ℓ are both odd, one can easily see $\Theta^{\circlearrowleft}_{k,\ell}=0$. In terms of rotationally invariant lattice paths, the central point of rotation will have two non-integral coordinates. In order for the path from A_2 to meet up with itself upon rotation, it would have to go through this point, but lattice paths always have at least one integral coordinate. One can also look at the plane partition model, where the $k\times\ell\times 3$ box has an odd number of boxes, so the size of any plane partition in that box must have opposite parity of its complement. Correspondingly, we check that $[\Theta_{k,\ell}(q)]_{q=-1}=0$. In this case, the denominator of (5) only has one factor of $1+q$, coming from $\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q$, whereas the numerator will have two factors of $(1+q)$, coming from each of $\begin{bmatrix} n+1 \\ k \end{bmatrix}_q$ and $\begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q$.

[3]This case corrects Proposition 7.4, part iii in Felsner, Fusy, Orden, and Noy

Proof of Assertion 2. When $k = 2\kappa$ and $\ell = 2\lambda$ are both even, the resulting summation in (8) is (after factoring out a constant) the hypergeometric series

$$ {}_5F_4\left[\begin{matrix} 2 & 2 & 2 & -\kappa, & -\lambda \\ & 1 & 1 & \kappa+3 & \lambda+3 \end{matrix}\middle| 1\right], $$

where we recall that

$$ {}_rF_s\left[\begin{matrix} a_1 & a_2 & \dots & a_r \\ b_1 & b_2 & \dots & b_s \end{matrix}\middle| z\right] = \sum_{n=1}^{\infty} \frac{(a_1)_n \dots (a_r)_n}{(1)_n (b_1)_n \dots (b_s)_n} z^n, $$

for $(x)_n = x(x+1)\dots(x+n-1)$.
This can be evaluated using the formula for a well-poised ${}_5F_4$ [**5**, (4.4.1) p.27],

$$ (10) \qquad {}_5F_4\left[\begin{matrix} a & \frac{1}{2}a+1 & c & d & e \\ & \frac{1}{2}a & 1+a-c & 1+a-d & 1+a-e \end{matrix}\middle| 1\right] $$
$$ = \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{\Gamma(1+a)\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}. $$

By choosing $a = c = 2$, $d = -\kappa$, and $e = -\lambda$, one can check this gives (8).

Proof of Assertion 3. When $k = 2\kappa+1$ is odd and $\ell = 2\lambda$ is even, the summation is (again, after factoring out a constant) the hypergeometric series

$$ {}_4F_3\left[\begin{matrix} 3, & 5/2, & 1-\lambda, & -\kappa; \\ & 3/2, & \lambda+3, & \kappa+4 \end{matrix}\middle| 1\right]. $$

This can also be evaluated using (10), but with choice of parameters $a = 3$, $c = 1-\lambda$, $d = -\kappa$, and $e = 2$ (note that as $e = 1+a-e$, the ${}_5F_4$ reduces to a ${}_4F_3$), and one can check that this gives (9).

Proof of Assertion 4. We exploit natural symmetry that forces $\Theta^{\circlearrowleft}_{k,\ell} = \Theta^{\circlearrowleft}_{\ell,k}$.

$\square$

2.4. Other Symmetry Classes

It is clear that if $w = w_1w_2\dots w_n$ is a Baxter permutation, then $w_nw_{n-1}\dots w_1$ and $(n+1-w_1)(n+1-w_2)\dots(n+1-w_n)$ will also be Baxter permutations, and we previously mentioned that w^{-1} will also be a Baxter permutation. These actions correspond to reflecting the permutation matrix for w horizontally, vertically and across a diagonal (respectively), so Baxter permutations have the full action of the dihedral group of a square acting on them. In the previous section, we were considering how many Baxter permutations were fixed under 180° rotation of the permutation matrix. While the other dihedral actions on Baxter permutations do not seem to correspond to any natural actions on other Baxter objects, we can still ask how many Baxter permutations are fixed under each action.

The horizontal and vertical reflections are not interesting, because other than when $n = 1$, no permutation will ever be fixed by those actions.

Baxter permutations fixed under reflection across the diagonal correspond to self-involutive Baxter permutations, and these have previously been considered. Bousqet-Mélou came up with enumerative formulas for the number of fixed-point free self-involutive Baxter permutations of length $2n$, which has the surprisingly simple closed formula $b_n = \frac{3\cdot 2^{n-1}}{(n+1)(n+2)}\binom{2n}{n}$, as well as refined enumeration with respect to various statistics [**11**]. Later, Fusy used planar maps to give a combinatorial proof of the enumeration for b_n, as well as a closed-form multivariate enumeration for all self-involutive Baxter permutations [**12**].

The last remaining conjugacy class of dihedral actions on Baxter permutations is the one corresponding to 90° rotations, which we now consider.

Theorem 2.57. *The number of Baxter permutations of length n fixed under 90° rotation of its permutation matrix is $2^m C_m$ (where C_m is the Catalan number) if $n = 4m+1$, and zero otherwise.*

To prove this, we will recall and extend the method of generating tress originally used by Chung, Graham, Hoggat, and Kleiman to enumerate all Baxter permutations.

2.4.1. Generating Tree for Classes of Permutations. Say we have a family of permutations that is closed under removing the largest entry. Then every permutation in the family of length n arises uniquely from taking a permutation of length $n-1$ in the family, and inserting the letter n into an admissible position.

Definition 2.58. We say that the *generating tree* of a family of permutations closed under removing the largest entry is the tree whose nodes are the permutations in the family, and the parent of each node is the permutation obtained by removing the largest entry.

In many cases, one can obtain enumeration results by analyzing this tree.

The family of permutations avoiding some set of classical patterns in always closed under taking any subword (in particular, removing the largest element), so we can construct a generating tree.

Example 2.59. Consider the set of permutations that avoid the classical pattern 231. It is not hard to see given a 231 avoiding permutation, the only place one can insert a new largest label into a permutation and still avoid the pattern 231 is immediately to the left of a left-to-right maxima or at the end of the permutation.

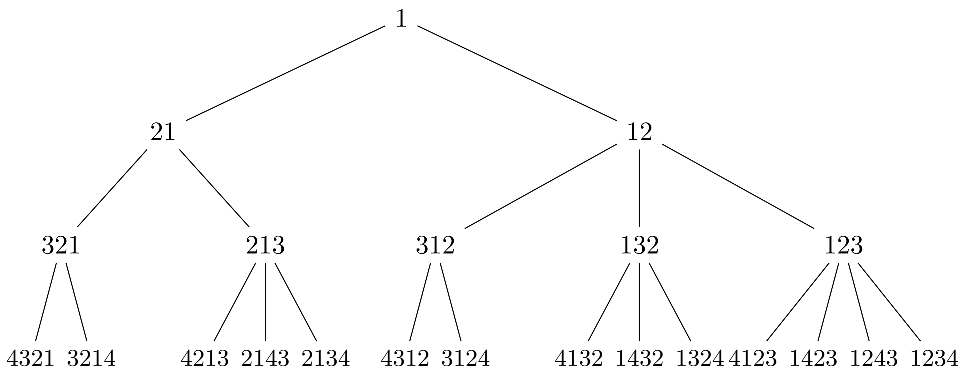

FIGURE 2.11. The beginning of the generating tree for 231-avoiding permutations.

We say that w_i is a left-to-right maxima of w if $w_i > w_k$ for all $k < i$. So the number of children a permutation has in the generating tree depends only on this statistic.

Furthermore, inserting a new largest label has a predictable effect on the number of left-to-right maxima of the resulting permutation. If a permutation has $k+1$ left-to-right maxima, then it will have $k+1$ children with $2, 3, \ldots, k+2$ left-to-right maxima.

Thus, an abstract tree with nodes labelled by integers that has root 1 and the property that every node $k+1$ has children labelled $2, 3, \ldots k+2$ will be isomorphic to the generating tree for 231 avoiding permutations. This tree is known as the *Catalan tree* [**57**], and is known to have rank sizes corresponding to the Catalan numbers. See Figures 2.4.1 and 2.4.1 for the generating tree of 231 avoiding permutations, and the associated Catalan tree.

Similarly, we could consider permutations that avoided the classical pattern 132. In this case, the places where we could insert a new largest label are immediately to the right of a right-to-left maxima (where we say that w_i is a right-to-left maxima of w if $w_i > w_k$ for all $k > i$) or at the beginning of the permutation. We again have the same predictable effect on number of right-to-left maxima by inserting a new largest label into a fixed permutation in all possible ways, and we again get a generating tree isomorphic to the Catalan tree.

2.4.2. Generating Tree for Baxter permutations. Baxter permutations are given by a vincular pattern, where we have adjacency issues to consider, so it is not immediately obvious that they are closed under removing the largest label.

LEMMA 2.60. *If w is a Baxter permutation, and we remove its largest label, then the result is still a Baxter permutation*

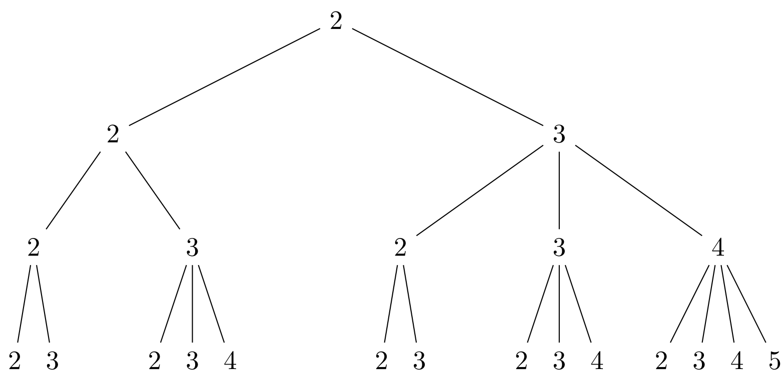

FIGURE 2.12. The beginning of the Catalan tree

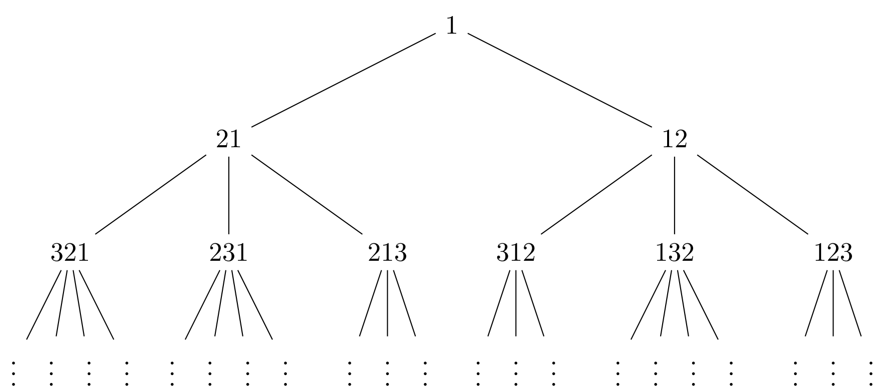

FIGURE 2.13. The beginning of the generating tree for Baxter permutations

PROOF. Say $w = w_1 \dots w_n$ is a Baxter permutation, and we remove $w_i = n$ to get $\bar{w} = w_1 \dots \hat{w}_i \dots w_n$. If $\bar{w}$ is not a Baxter permutation, then WLOG say there is an instance of $2-41-3$. That means there is a subsequence $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ with $i_1, i_2, i_3, i_4 \neq i$, $w_{i_3} < w_{i_1} < w_{i_4} < w_{i_2}$, and i_2 is adjacent to i_3 in $\hat{w}$. The only way i_2 can be adjacent to i_3 in $\hat{w}$ is if $i_2 + 1 = i_3$, or if $i_3 + 2 = i + 1 = i_2$. In the first case, the subsequence i_1, i_2, i_3, i_4 would be an instance of $2-41-3$ in w, a contradiction of our assumption. In the second case, the subsequence i_1, i, i_3, i_4 would be an instance of $2-41-3$ in w, again a contradiction. $\square$

Therefore, every Baxter permutation of length n uniquely arises from taking a Baxter permutation of length $n-1$ and inserting n into an admissible position.

The admissible places where we can insert a new largest label into a Baxter permutation are immediately to the left of a left-to-right maxima, and immediately to the right of a left-to-right maxima.

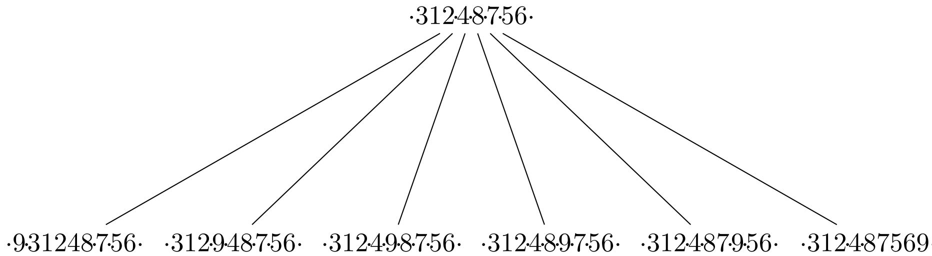

FIGURE 2.14. Branching of generating tree at $w = 31248756$, with insertion points marked.

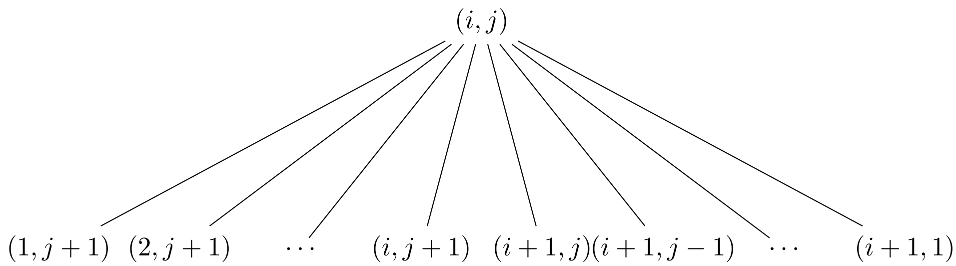

FIGURE 2.15. Rule for generating tree isomorphic to Baxter permutations

The resulting Baxter permutation will also have a predictable number of left-to-right and right-to-left maxima. Say w has left-to-right maxima $x_1 < x_2, \ldots < x_i = n$ and right-to-left maxima $n = y_j > y_{j-1} > \ldots > y_1$. If we insert $n+1$ to the left of x_k, the resulting permutation will have left-to-right maxima $x_1 < \ldots < x_{k-1} < n+1$, and right-to-left maxima $n+1 > n = y_j > \ldots > y_1$. If we insert $n+1$ to the right of y_k, the resulting permutation will have left-to-right maxima $x_1 < x_2 < \ldots x_i = n < n+1$, and right-to-left maxima $n+1 > y_{k-1} > \ldots > y_1$.

This means that the number of children a given Baxter permutation has (and how many children those children will have, and so on) is entirely encoded by the number i of left-to-right maxima, and the number j of right-to-left maxima. Thus, the tree with root $(1,1)$, and the property that every node (i,j) has children $(1, j+1), (2, j+1), \ldots (i, j+1), (i+1, j), (i+1, j-1), \ldots (i+1, 1)$ will be isomorphic to the generating tree for Baxter permutations.

COROLLARY 2.61. *Baxter permutations have the same number of descents as inverse descents.*

PROOF. If n is inserted to the left of a left-to-right maxima, then either $n+1$ is being added to the front of the word, or it is being inserted into an ascent. In either case, the resulting permutation will have one more descent than the original one. But since n is always the rightmost left-to-right maxima, $n+1$ is being

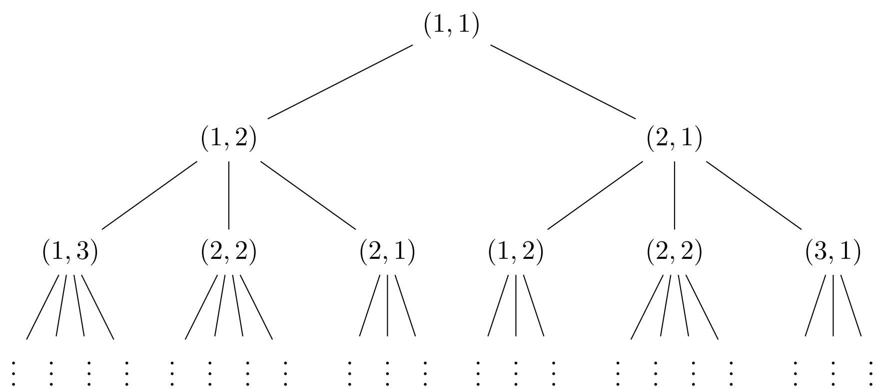

FIGURE 2.16. The beginning of the generating tree isomorphic to the one on Baxter permutations.

inserted to the left of n, so we are also creating one new inverse descent. Similarly, $n+1$ being inserted to the right of a left-to-right maxima creates no new descents nor inverse descents. Since the act of inserting a new largest label preserves the difference between number of descents and number of inverse descents, and the base of our generating tree has the same number of descents as inverse descents, all permutations in the generating tree have the same number of descents as inverse descents. □

2.4.3. Generating Tree for Baxter permutations fixed under $180°$ **rotation.** First, let us consider the generating tree of all Baxter permutations fixed under $180°$ rotation.

A permutation w of length n being fixed under $180°$ rotation means that if $w_i = j$, then $w_{n+1-i} = n+1-j$. Also, by the same logic we used to see that we could remove n from a Baxter permutation and still be a Baxter permutation, we can see that we can remove 1 from a Baxter permutation (and decrease all remaining labels by one) and still be a Baxter permutation.

Combining these two things, we can see that if we remove n and 1 (and then decrease all the labels by 1) from a Baxter permutation fixed under $180°$ rotation, then we will still have a Baxter permutation fixed under $180°$ rotation. So again, we can construct a generating tree.

Note that in this case, we are removing two entries at a time, so we will have separate generating trees for when n is even and when n is odd. We will use the convention that the generating tree for n even has the empty permutation $\emptyset$ of length 0 as its root, with children 12 and 21.

We already have a combinatorial rule for when we can insert a new largest entry into a Baxter permutation and still be a Baxter permutation, so now we come up

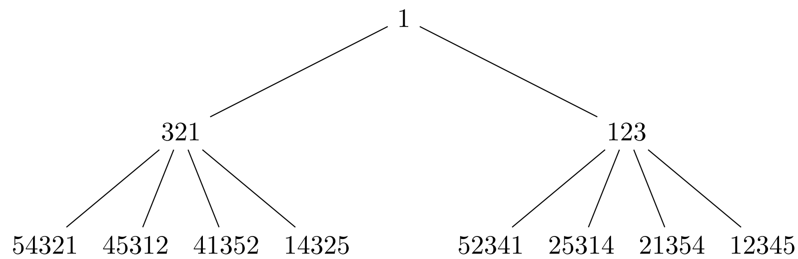

Figure 2.17. Generating Tree for Baxter permutations of odd length fixed under conjugation by the longest element

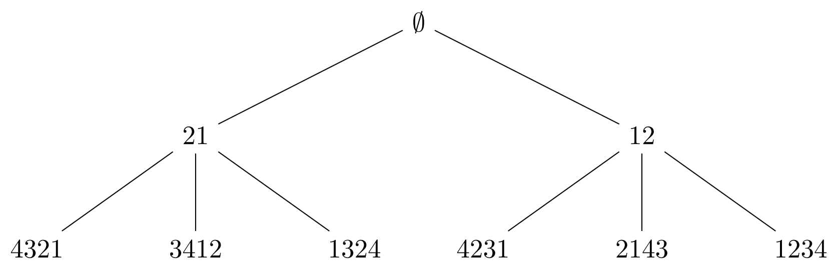

Figure 2.18. Generating Tree for Baxter permutations of even length fixed under conjugation by the longest element

with a combinatorial rule for when we can insert a new smallest entry into a Baxter permutation and still be a Baxter permutation. By inserting a new smallest entry, we mean that we increase all the labels in the existing permutation by 1, and then insert a new entry with label 1, so that if the original permutation was a standard permutation of n letters on $[n]$, then the result will be a standard permutation on $[n+1]$.

Lemma 2.62. *Inserting a new smallest label into position j into a permutation is equivalent to rotating the permutation matrix $180°$, inserting n into position $n+1-j$, and then rotating the permutation matrix $180°$ again.*

Consequently, given a Baxter permutation w, the admissible places we can insert a new smallest label are immediately to the left of a left-to-right minima, or immediately to the right of a right-to-left minima.

This proof makes it clear that if we can insert n into position i of w, then we can insert 1 into position $n+1-i$ of w_0ww_0. So if w is fixed under conjugation by the longest element, then we can insert $n+1$ into a position if and only if we can insert 1 into the complementary position. However, we need to check to make sure that we can still insert 1 into a complementary position after we have inserted $n+1$.

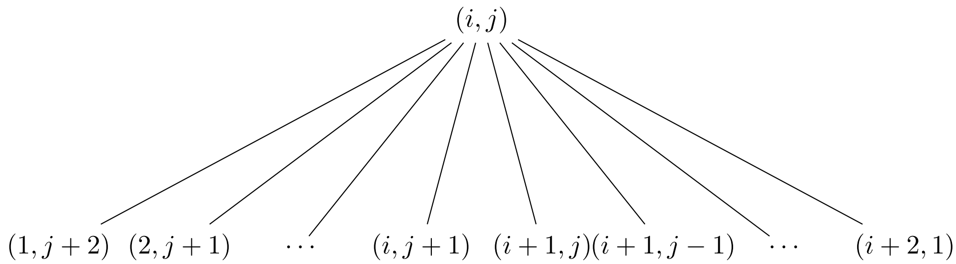

FIGURE 2.19. Rule for generating tree isomorphic to Baxter permutations fixed under conjugation by longest element.

If i is less than $n/2$, then the complementary place we want to insert 1 will be shifted right by 1. If i is greater than $n/2$, then the complementary place we want to insert 1 will still be $n+1-i$. In either of these cases, the act of inserting n will not affect being able to insert 1 into the complementary position, because the combinatorial rule for inserting 1 depends on things being left-to-right and right-to-left minima, and inserting a new largest label will not affect that.

As a kind of boundary case, we have the situation where n is even and $i = n/2$, so we are inserting $n+1$ into the middle of the word. Then we could insert 1 either immediately to the left or right of $n+1$ and still have a permutation fixed under conjugation by the longest element. However, exactly one of these choices will correspond to a Baxter permutation.

WLOG, say we inserted $n+1$ to the right of a left-to-right maxima, $w_{n/2}$. Then $w_{n/2+1}$ will be a right-to-left minima, even after we insert $n+1$. So we can insert 1 to the left of $w_{n/2+1}$, which will be immediately to the right of $n+1$. This means that $w_{n/2} > w_{n/2+1}$, or else the subword $w_{n/2}(n+1)1w_{n/2+1}$ would be a copy of the vincular pattern $2-41-3$. Thus, if we inserted 1 on the other side of $n+1$, we would be making a subword that was an instance of the forbidden vincular pattern $3-14-2$.

Now, we want to make an isomorphic generating tree that doesn't require us to keep track of the permutation in full, analogous to what we did with all Baxter permutations.

Again, we only need to keep track of the number of left-to-right and right-to-left maxima. Each of these corresponds to a place where we can insert $n+1$, and then we know there will be a complementary place we can insert 1 to stay fixed under conjugation by the longest element. We know how inserting $n+1$ will affect the number of left-to-right and right-to-left maxima. Inserting 1 will in general not create any new left-to-right or right-to-left maxima, except in the case where we are adding 1 to the beginning or end of the word. Thus, we get the branching rule as seen in Figure 2.4.3.

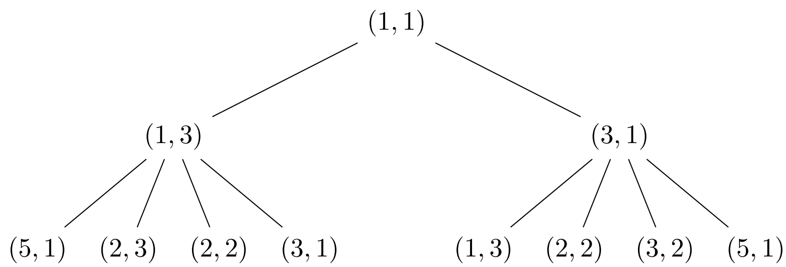

FIGURE 2.20. Isomorphic generating Tree for Baxter permutations of odd length fixed under conjugation by the longest element

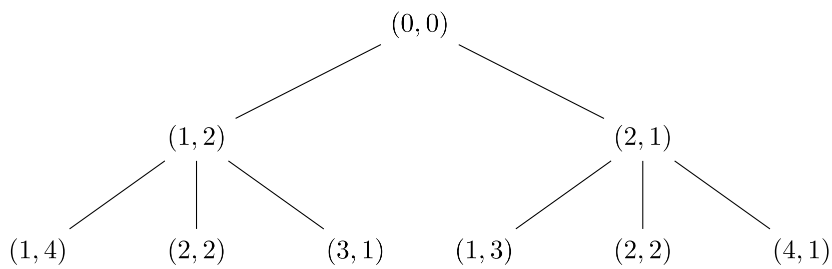

FIGURE 2.21. Isomorphic generating Tree for Baxter permutations of even length fixed under conjugation by the longest element

In principle, one could try and analyze this branching rule to come up with an algebraic formula for the number of Baxter permutations fixed under conjugation by the longest element. While it may give a refined enumeration for the number of Baxter permutations fixed under conjugation by the longest element with a given number of left-to-right and right-to-left maxima, it is unlikely that the resulting expression for the entire set would be as elegant as the "$q = -1$" formula in Theorem 2.3. In practice, this is more of a stepping stone to the case of Baxter permutations fixed under $90°$ rotation, where the rules for insertion are more technical, but the resulting branching structure has a transparent formula.

2.4.4. Generating Tree for Baxter permutations fixed under $90°$ rotation. For a permutation to be fixed under $90°$ rotation, it is equivalent to say that if $w_i = j$, then $w_j = n+1-i$, $w_{n+1-i} = n+1-j$, and $w_{n+1-j} = i$. If we consider the cycle structure of this permutation, in general it makes a 4-cycle $(i, j, n+1-i, n+1-j)$. If this were to degenerate into a smaller cycle, we would have that $i = n+1-i$. This forces to $n = 2i+1$ to be odd, and it also forces $i = j$, which means it actually degenerates to a single central fixed point. Thus, a permutation fixed under this action must have length $4m$ or $4m+1$.

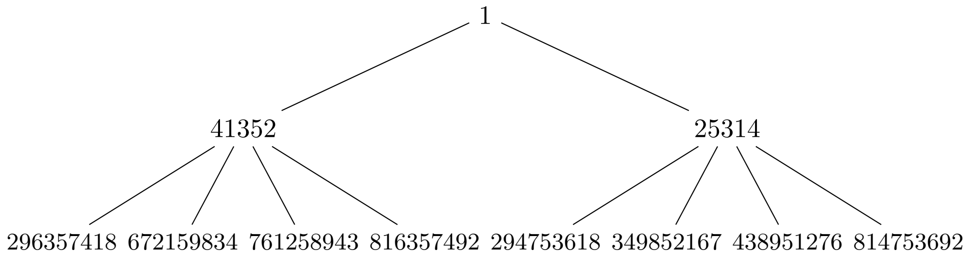

FIGURE 2.22. Start of generating tree for Baxter permutations fixed under $90°$ rotation.

If w is a Baxter permutation of length n with k descents, then by Corollary 2.61, w^{-1} will have k descents, and w_0w^{-1} will have $n-1-k$ descents. So for a Baxter permutation to be fixed by this action, we must have $k = n-1-k$, which implies that n must be odd, and along with our previous observation must be $n = 4m+1$. Thus, a Baxter permutation fixed under $90°$ rotation will consist of a single central fixed point, and four cycles of the form $(i, j, n+1-i, n+1-j)$.

In particular, for $n > 1$, such a permutation will have a four cycle of the form $(1, j, n, n+1-j)$, which means the permutation starts with j, has n in the j^{th} position, 1 in the $(n+1-j)^{th}$ position, and $n+1-j$ at the end. We already know that we can remove n and 1 from a Baxter permutation and still be a Baxter permutation. It's not hard to see that we can also remove the first element or the last element from a Baxter permutation and still be a Baxter permutation. So if we take a Baxter permutation fixed under $90°$ rotation and the remove the largest label, the smallest label, the first label, and the last label, then we will still have a Baxter permutation, and it will still be fixed under $90°$ rotation. Thus, we can create a generating tree, with the identity permutation on 1 element as the root

In order to create a four cycle, we have to come up with a combinatorial rule for when we can insert a letter at the beginning (resp. end) of a Baxter permutation, and still have it be a Baxter permutation. To insert a letter j at the beginning of a permutation w of length n, we mean that we increase all the labels greater than or equal to j in w by 1, and then prepend j, so the result is a standard permutation on $[n+1]$.

LEMMA 2.63. *Inserting j at the end of a word is equivalent to rotating the permutation matrix $90°$ clockwise, inserting n into position $n+1-j$, and the rotating the permutation matrix $90°$ counter-clockwise.*

Similarly, inserting j at the beginning of a word is equivalent to rotating a permutation matrix $90°$ counter-clockwise, inserting n into position j, and then rotating back $90°$ clockwise.

Consequently, we can insert j at the end (resp. beginning) of a Baxter permutation and still have it be a Baxter permutation if and only if all entries smaller than j appear to the left (resp. right) of j, or if all entries bigger than $j-1$ appear to the right (resp. left) of $j-1$.

Note that inserting j at the end (resp. beginning) of Baxter permutation can possibly decrease the number of right-to-left (resp. left-to-right) maxima, as any previous left-to-right (resp. right-to-left) maxima that was less than j will no longer be one after j is inserted at the end (resp. beginning).

THEOREM 2.64. *For a Baxter permutation fixed under $90°$ rotation, for every admissible position we can insert a new largest label and still have a Baxter permutation, it is also possible to insert a new smallest label, a new beginning label, and a new final label so that the result is a Baxter permutation fixed under $90°$ rotation.*

PROOF. Without loss of generality, assume we are inserting n to the right of a right-to-left maxima. The procedure for when we can insert $n+1$ to the left of a left-to-right maxima is the same, except we reverse the order of the word, follow the procedure for inserting $n+1$ to the right of a right-to-left maxima, and then reverse the order of the resulting word.

Say w is a Baxter permutation of length n fixed under $90°$ rotation, with a right-to-left maxima at w_j. This means that we could insert $n+1$ into position $j+1$, and by Lemma 2.62 we could also insert 1 into position $n-j$, and by Lemma 2.63 we could insert $n-j$ at the end or $j+1$ at the beginning. Specifically, since we know that we'd be inserting $n+1$ to the right of a left-to-right maxima, we know that w_{n+j-1} must be a right-to-left minima, and that all entries larger than j appear to the left of j, and that all entries smaller than $n-j$ appear to the left of $n-j$. We also know that as a left-to-right maxima, w_j must be at least $n-j$, since there are $n-j$ things to its right that must be smaller.

But we need to check that we can perform all four insertions sequentially in a way so that the result is a Baxter permutation fixed under $90°$ rotation with a new four cycle added.

We will seperately consider the cases with $j+1 < n/2$, and $j+1 > n/2$. Note that since n has to be odd, we don't have to deal with the special case of $j+1 = n/2$.

First, suppose $j+1 < n/2$. We insert $j+1$ at the beginning first, which increases any labels that were $j+1$ or higher by 1. So now, we want to insert $n+1-j$ at the end. We have to check that all labels less than $n+1-j$ are to the left of $n+1-j$. Since in the original permutation we had that all labels less than $n-j$ were to the left of $n-j$, when we add 1 to all labels $j+1$ or higher, we will have that all labels less than $n+1-j$ (except possibly $j+1$) are to the left of $n+1-j$. But since we add $j+1$ to the beginning of the word, it will also certainly be to the left of $n+1-j$. So we may insert $(n+1-j)$ at the end.

We now have a permutation of length $n+2$, which will still be fixed under 180° rotation. So by the previous section, if we can insert a new largest label into some position, we know we can insert a new smallest label into the complementary position. The $(j+1)^{st}$ entry in this permutation will be w_j+2, as inserting two smaller labels increased its label by 2, and inserting a label at the beginning shifted it right by one. We need to check that this is still a right-to-left maxima. The only thing we did that could have changed this is inserting $n+1-j$ at the end. However, since $w_j \geq n-j$, we have $w_j+2 \geq n+2-j$, so adding $n+1-j$ will not keep it from being a right-to-left-maxima. Thus, we can insert a new largest label into position $j+2$, and also a new smallest label into the complementary position $(n+1-j)$.

After all of these steps, we will now have $j+2$ in the first position, $n+4$ in position $j+2$, 1 in position $(n+4)-(j+2)$, and $(n+2-j)$ at the end, which creates the desired four cycle.

Now, suppose $j+1 > n/2$. Again, we insert $j+1$ at the beginning. Now, we want to insert $n-j$ at the end. Inserting $j+1$ will not affect any labels $n-j$ or smaller, so we will still have that all labels less than $n-j$ are to the left of $n-j$.

Again, we have a permutation of length $n+2$ fixed under 180° rotation, so it suffices to show we can place a new largest label, and it will automatically follow that a new smallest label can go in the complementary position. Consider the $(j+1)^{st}$ entry of this permutation, which was originally w_j. We claim this is still a right-to-left maxima. The only thing that could have changed this fact is inserting $n-j$ at the end. Since $w_j \geq n-j$, this label would at least be increased by 1 when we inserted $n-j$. It could possibly also be increased by 1 when we inserted $j+1$, but what's important is that the $(j+1)^{st}$ entry is at least $n+1-j$, and thus having $n-j$ at the end will not prevent it from being a left-to-right maxima.

After all of these steps, we will now have $j+3$ in the first position, $n+4$ in position $j+3$, 1 in position $(n+4)-(j+3)$, and $n+1-j$ at the end, which creates the desired four cycle.

$\square$

Now, we want analyze how doing these four insertions changes the number of left-to-right and left-to-right maxima.

LEMMA 2.65. *If w is a Baxter permutation fixed under 90° rotation, then w has the same number of left-to-right and right-to-left maxima. In particular, if w has left-to-right maxima in positions $x_1 < x_2 < \ldots < x_j$ and right-to-left maxima at positions $y_j < y_{j-1} < \ldots < y_1$, and we do a four cycle insertion corresponding to* *being able to insert a new largest label to the right of w_{y_i} (or to the left of w_{x_i}), then the resulting Baxter permutation fixed under 90° rotation will have $i+1$ left-to-right maxima and $i+1$ right-to-left maxima.*

Proof. We note that $w_j < w_i$ if and only if j appears to the left of i in w^{-1} if and only if i appears to the left of j in $w_0 w^{-1}$. Thus, if $w = w_0 w^{-1}$, we have a right-to-left maxima in position j if and only if j is a left-to-right maxima (and similarly, a left-to-right maxima in position j if and only j is a right-to-left maxima). So if w is fixed by this action, it must have the same number of right-to-left maxima as left-to-right maxima.

Additionally, this gives a bijection between right-to-left maxima that were originally in w that are later killed by $n+3$, and left-to-right maxima that are killed by the $j+2$ or $j+3$ at the beginning of the word. Similarly, there is a bijection between right-to-left maxima originally in w that are later killed by the final entry, and left-to-right maxima originally in w later killed by 1.

Since 1 always ends up on the interior of the word, it will never be a left-to-right maxima, and so the final entry will also never kill anything that was originally a right-to-left maxima. Since we (WLOG) did an insertion corresponding to putting a new largest label to the right of w_{y_i}, $n+3$ will kill the right-to-left maxima $w_{y_i}, \dots w_{y_j}$. Thus, we will have $i+1$ right-to-left maxima; the new right-most entry, the $i-1$ original right-to-left maxima not killed by $n+3$, and $n+3$.

□

We now have enough information to analyze the generating tree for Baxter permutations fixed under rotation by 90° degrees. If a Baxter permutation fixed under rotation by 90° degrees has $i+1$ left-to-right maxima and $i+1$ right-to-left maxima, then it will have $2i+2$ children. There will be $i+1$ children with number of left-to-right (and right-to-left) maxima being $2, 3, \dots i+2$ corresponding to inserting a new largest label to the left of a left-to-right maxima, and $i+1$ children with number of left-to-right (and right-to-left) maxima being $2, 3, \dots i+2$ corresponding to inserting a new largest label to the right of a right-to-left maxima.

Thus, this generating tree is almost like the Catalan tree, except each parent with label $i+1$ has two (not one) children with a label between 2 and $i+2$, and our root will have label 1. This implies that the number of elements of a given rank m must be $2^m C_m$. See Figure 2.4.4.

2.4.5. Remarks. The fact that this enumeration has such an elegant closed formula means that it is likely that there is an underlying combinatorial bijection. However, as with Chung, Graham, Hoggat, and Kleiman, the method of generating trees does not make such an interpretation transparent.

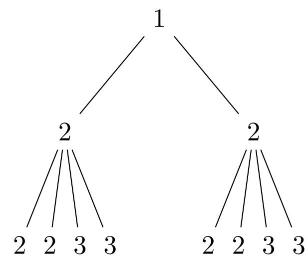

Figure 2.23. The beginning of the doubled Catalan tree

Additionally, one might hope that it is possible to extend our previous "q=-1" result for Baxter permutations fixed under 180° rotation to an instance of the cyclic sieving phenomenon. That is to say, finding a polynomial $f(q)$ where gives an enumeration of Baxter permutations (perhaps with respect to some statistics), $f(-1)$ counts how many of these Baxter permutations are fixed under 180° rotation, and $f(i) = f(-i)$ counts how many of them are fixed under 90° rotation. However, the natural candidate of $\Theta_{k,\ell}(q)$ does not work, and it does not appear that it can be easily modified to give such a result.

2.4.6. Other Conjectures. Asinowski, Barequet, Bousquet-Mélou, Mansour, and Pinter studied equivalence classes of floorplans (ie, tiling a rectangle by smaller rectangles) that never have four corners meet [**3**]. They considered an equivalence relations on the "rooms" of the floorplan, and showed the equivalence classes corresponded to Baxter permutations. They also considered an equivalence corresponding to the segments on the interior of a floorplan, and showed that equivalence refined the equivalence on rooms, and that the equivalence classes were in bijection with $2-14-3$ and $3-41-2$ avoiding permutations. They also showed that for a given floorplan, the Baxter permutation corresponding to the room equivalence class and the permutation avoiding $2-14-3$ and $3-41-2$ coming from the segment equivalence class could naturally be combined to form a complete Baxter permutation.

So in some sense, the corresponding permutation avoiding $2-14-3$ and $3-41-2$ is the 'missing' part of the Baxter permutation, and the family of permutations avoiding $2-14-3$ and $3-41-2$ may be of some interest. Asinowski, Barequet, Bousquet-Mélou, Mansour, and Pinter gave an enumeration formula for this family of permutations.

Theorem 2.66 (Asinowski, Barequet, Bousquet-Mélou, Mansour, and Pinter). *The number a_n of permutations of length n avoiding the patterns $2-14-3$ and $3-41-2$ is given by*

$$a_n = \sum_{i=0}^{\lfloor (n+1)/2 \rfloor} (-1)^i \binom{n+1-i}{i} B(n+1-i)$$

where $B(n)$ is the number of Baxter permutations of length n.

Unfortunately, the signs mean that we don't get a sum that is graded with respect to a statistic like descents, as we had for Baxter permutations. It would be nice to have some of nonnegative enumerative formula for these permutations.

One can check that this family of permutations is closed under all reflections and rotations of the corresponding permutation matrix. So again, we can ask how many of these permutations are fixed under certain actions.

For 180° rotation, there does not appear to be any nice formula.

For reflection across a diagonal (corresponding to taking the inverse of a permutation), we have some interesting conjectures.

CONJECTURE 2.67. *Let GD_n be the set of Grand Dyck paths, or words of length $2n$ with exactly n 0's and n 1's. Then the number of self-involutive permutations avoiding the patterns $2-14-3$ and $3-41-2$ is the number of Grand Dyck paths that avoid the consecutive sequences 101 and 010 (zig-zag avoiding).*

This comes from an apparent match with OEIS entry A078678.

CONJECTURE 2.68. *The number of fixed-point free self-involutive permutations of length $2n$ avoiding the patterns $2-14-3$ and $3-41-2$ is equal to the number of permutations of length n avoiding the patterns $2-14-3$ and $3-41-2$.*

We can also consider 90° rotation.

CONJECTURE 2.69. *Consider all rooted trees with positive integer weights on the nodes, so that the weight of a parent is equal to the sum of the weights of its children, and the weight of a tree is the weight of its root.*

Then the number of permutations of length $4k$ or $4k+1$ avoiding the patterns $2-14-3$ and $3-41-2$ fixed under 90° rotation is equal to the number of trees of weight k.

This comes from an apparent match with OEIS entry A118376.

Unfortunately, there is no clear intuition for why any of these conjectures should be true, other than numerical evidence.

CHAPTER 3

Hoggatt Numbers

3.1. Definitions

The formula (5) gives a meaningful q-analog for $\Theta_{k,\ell}$, and we would like to extend it to a q-analog for Baxter numbers. A natural way in which one can generalize is inspired by placing Baxter numbers within the family of Hoggatt sums [**24**].

Let $M(k,\ell,m)$ be the number of plane partitions that fit in a $k\times\ell\times m$ box, which we will call the *MacMahon numbers*. Via MacMahon's plane partition formula given in (6), these can be simply expressed as

$$M(k,\ell,m) = \frac{\prod_{i=0}^{m-1}\binom{k+\ell+m-1}{k+i}}{\prod_{j=1}^{m-1}\binom{k+\ell+m-1}{j}}. \tag{11}$$

We also consider a natural q-shift of MacMahon's formula,

$$M(k,\ell,m;q) = q^{m\binom{k+1}{2}}\sum_{\pi} q^{|\pi|}, \tag{12}$$

where we sum over all plane partitions π in a $k\times\ell\times m$ box. Note that $M(k,\ell,m;1) = M(k,\ell,m)$.

Definition 3.1. We will call

$$H_n^{(m)} = \sum_{k+\ell=n-1} M(k,\ell,m)$$

the *Hoggatt sum (of level m)*,

$$H_n^{(m)}(q) = \sum_{k+\ell=n-1} M(k,\ell,m;q)$$

the *q-Hoggatt sum (of level m)*, and

$$H_n^{(m)}(q,t) = \sum_{k+\ell=n-1} M(k,\ell,m;q)t^k$$

the (q,t)-*Hoggatt sum (of level m)*.

3.1.1. Level 1: Subsets.

For $m=1$, the MacMahon numbers are

$$M(k,\ell,1) = \binom{k+\ell}{k},$$

the *binomial coefficients*. These have combinatorial interpretation

$$M(k,\ell,1) = \left|\binom{[k+\ell]}{k}\right|,$$

the size of the set of k element subsets of $[k+\ell] := \{1,2,\ldots,k+\ell\}$.
It has a natural q-analogue

$$M(k,\ell,1;q) = q^{\binom{k+1}{2}} \begin{bmatrix} k+\ell \\ k \end{bmatrix}_q$$

with combinatorial interpretation

$$M(k,\ell,1;q) = \sum_{\binom{[k+\ell]}{k}} q^{\Sigma S},$$

where $\Sigma S = \sum_{i\in S} i$.
The Hoggatt sum is

$$H_n^{(1)} = 2^{n-1},$$

with combinatorial interpretation being the total number of subsets of $[n-1]$.
The q-Hoggatt sum is

$$H_n^{(1)}(q) = \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = (-q;q)_{n-1} = (1+q)(1+q^2)\ldots(1+q^{n-1}),$$

with combinatorial interpretation

$$H_n^{(1)}(q) = \sum_{S\subseteq[n-1]} q^{\Sigma S}.$$

The (q,t)-Hoggatt sum is

$$H_n^{(1)}(q,t) = \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q t^k = (-tq;q)_{n-1} = (1+tq)(1+tq^2)\ldots(1+tq^{n-1}),$$

with combinatorial interpretation

$$H_n^{(1)}(q,t) = \sum_{S\subseteq[n-1]} q^{\Sigma S} t^{|S|}.$$

Geometrically, we have that $H_n^{(1)}(1,t) = (1+t)^{n-1}$ is the h-vector for the octahedron, the flag simplicial polytope dual to the cube.

In Coxeter-theoretic terms, we have $H_n^{(1)}(q,t) = \sum_{J\subseteq[n-1]} q^{\mathrm{maj}(w^J)} t^{\mathrm{des}(w^J)}$, where w^J is the maximal length coset representative for the parablic coset W_J inside $W = A_{n-1}$.

And in terms of pattern avoiding permutations, we can realize $\{w^J | J \subseteq [n-1]\}$ as the set of permutations in S_n that avoid the classical patterns 132 and 231, and $H_n^{(1)}(q,t)$.

3.1.2. Level 2: Catalan Objects. For $m = 2$, the MacMahon numbers are

$$M(k,\ell,2) = \frac{1}{k+\ell}\binom{k+\ell}{k}\binom{k+\ell}{k+1},$$

the *Narayana numbers*.

They have q-analogue

$$M(k,\ell,2;q) = \frac{q^{k^2+k}}{[k+\ell+1]_q}\begin{bmatrix} k+\ell+1 \\ k \end{bmatrix}_q \begin{bmatrix} k+\ell+1 \\ k+1 \end{bmatrix}_q,$$

the q-Narayana numbers [**28**].

The Hoggatt sum is

$$H_n^{(2)} = \frac{1}{n+1}\binom{2n}{n},$$

the Catalan number.

The q-Hoggatt sum is

$$H_n^{(2)}(q) = \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q,$$

the *q-Catalan number* [**28**].

And the (q,t)-Hoggatt sum is

$$H_n^{(2)}(q,t) = \sum_{k=0}^{n-1} \frac{q^{k^2+k}}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q t^k,$$

the q-Narayana distribution.

$H_n^{(2)}(1,t)$ is the h-polynomial for the original type A associahedron. It can also be thought of as a generating function over Catalan objects with respect to a natural statistic (Dyck paths of length $2n$ with respect to peaks, etc.).

A standard q-analog of this polynomial is what we have defined as $H_n^{(2)}(q,t)$, which is a generating function for the q-Narayana numbers [**28**]. This polynomial has a number of combinatorial interpretations. For example, it gives the joint distribution of Dyck paths with respect to descents and major index. Additionally, the work of Stump [**54**] shows that we have

$$\mathrm{Cat}(n,q,t) = \sum_{w \in S_n(231)} q^{\mathrm{maj}(w)+\mathrm{maj}(w^{-1})} t^{\mathrm{des}(w)}, \tag{13}$$

where $S_n(231)$ is the set of all permutations of length n that avoid the pattern 231 (w avoids the pattern 231 if there does not exist a triple of indices $1 \leq i < j < k \leq n$ such that $w_k < w_i < w_j$).

3.1.3. Level 3: Baxter Objects. For $m = 3$, the MacMahon numbers are

$$M(k, \ell, 3) = \Theta_{k,\ell},$$

which have numerous combinatorial interpretations as given in the introduction. They have q-analogue

$$M(k, \ell, 3; q) = \Theta_{k,\ell}(q)$$

.
The Hoggatt sum is

$$H_n^{(3)} = B(n),$$

the Baxter number.
The q-Hoggatt sum is

$$H_n^{(3)}(q) = \sum_{k=0}^{n} q^{3\binom{k+1}{2}} \Theta_{k,\ell}(q),$$

as a q-Baxter number.
And the (q,t)-Hoggatt sum is

$$H_n^{(3)}(q,t) = \sum_{k=0}^{n-1} \frac{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ k+2 \end{bmatrix}_q}{\begin{bmatrix} n+1 \\ 1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q} q^{3\binom{k+1}{2}} t^k.$$

$H_n^{(3)}(q,t)$ can conjecturally can be expressed as a generating function of Baxter permutations similar to the one for 231-avoiding permutations in Equation (13), as we next explain.

Definition 3.2. Given a permutation w, let $\mathrm{IDB}(w) = \{w_{i+1}^{-1} : w_{i+1}^{-1} < w_i^{-1}\}$ be the set of *inverse descent bottoms*, and let $\mathrm{IDT}(w) = \{w_i^{-1} - 1 : w_{i+1}^{-1} < w_i^{-1}\}$ be the set of *inverse descent tops*.

In simpler terms, an inverse descent in a permutation is an instance of $i + 1$ appearing to the left of i, and for each such inverse descent pair, we call the position of $i + 1$ an inverse descent bottom, and the position of i (minus one) the inverse descent top.

Definition 3.3. Let

$$\mathrm{imaj}_B(w) = \sum_{i \in IDB} i$$

and

$$\operatorname{imaj}_T(w) = \sum_{i \in IDT} i.$$

We note that this is simply the notion of descent bottoms (resp. tops) and sum of descent bottoms (resp. tops) applied to the inverse of a permutation as mentioned in [**26**] and [**4**].

Conjecture 3.4.

$$\operatorname{Bax}_n(q,t) = \sum_{w \in \operatorname{Bax}_n} q^{\operatorname{imaj}_B(w) + \operatorname{maj}(w) + \operatorname{imaj}_T(w)} t^{\operatorname{des}(w)}$$

This conjecture would be a corollary to a related conjecture, explained next, that gives a bijection between Baxter permutations and non-intersecting triples of lattice paths in terms of IDB, Des, and IDT.

Given a permutation $w \in S_n$, define a triple of lattice paths, each consisting of $n-1$ steps, as follows. One path starts at $(2,0)$, whose i^{th} step is to to the right if $i \in IDB(w)$, and up otherwise. The second path starts at $(1,1)$, and whose i^{th} step is to the right if $i \in \operatorname{Des}(w)$, and up otherwise. The third path starts at $(0,2)$, and whose i^{th} step is to the right if $i \in \operatorname{IDT}(w)$, and up otherwise.

Conjecture 3.5. *The above correspondence gives a bijection between Baxter permutations of length n with k descents, and non-intersecting triples of lattice paths from $(0,2)$, $(1,1)$, $(2,0)$ to $(k, n-k+1)$, $(k+1, n-k)$, $(k+2, n-k-1)$.*

This conjectured bijection appears to be fundamentally different from the classical bijection given by Dulucq and Guibert [**22**] involving twin binary trees.

Example 3.6. Let $w = 2147563$. Then $\operatorname{IDB}(w) = \{1,3,4\}$, $\operatorname{Des}(w) = \{1,4,6\}$, and $\operatorname{IDT}(w) = \{1,5,6\}$.

For higher Hoggatt levels ($m \geq 4$), it is not known if there are any combinatorial interpretations other than plane partitions.

Remark 3.7. Even though Hoggatt levels $m = 1, 2$ have representations in terms of permutations avoiding 231 and 312 and permutations avoiding 231 (respectively), they do not have the same interesting symmetries as Baxter permutations.

- In general, 231 permutations are not closed under taking inverses. The ones that are fixed under this action must also necessarily avoid the pattern 312. All permutations avoiding 231 and 312 are equal to their own inverse.
- Similarly, in general, 231 permutations are not closed under 180° rotation. The ones that are fixed under this action must also necessarily avoid 312. If we identify permutations avoiding 231 and 312 of length n with subsets

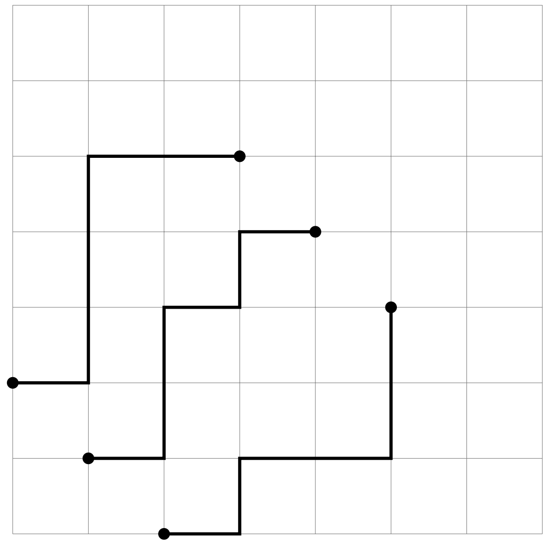

FIGURE 3.1. Image of $w = 2147563$ under conjectured bijection.

of $[n-1]$, then rotating the permutation matrix $180°$ degrees corresponds to sending $I \subseteq [n-1]$ to $\{n-i\}_{i\in I}$, so the number of fixed points will be $2^{\lfloor n/2 \rfloor}$.

- Neither 231 avoiding permutations nor 231 and 312 avoiding permutations are generally closed under $90°$ rotation. Additionally, there are no permutations in either class fixed under this action for $n > 1$. Any permutation of length $n > 1$ fixed under this action will have a non-degenerate 4-cycle of the form $(1, j, n, n+1-j)$ with $j \neq n/2$. If $j < n/2$, then the subsequence $j \dots n \dots 1$ will form an instance of 231, and if $j > n/2$, then the subsequence $j \dots n \dots n+1-j$ will form an instance of 231.

3.2. Real Rootedness

Additionally, one can look at the specialization of the (q,t)-Hoggatt numbers at $q = 1$. At level 1, we get

$$H_n^{(1)}(1,t) = \sum_{k=0}^{n-1} \binom{n-1}{k} t^k = (1+t)^{n-1}.$$

At level 2, we get

$$H_n^{(2)}(1,t) = \sum_{k=0}^{n-1} \frac{\binom{n}{k}\binom{n}{k+1}}{\binom{n}{1}} t^k,$$

the Narayana distribution. And for level 3, we get

$$H_n^{(3)}(1,t) = \sum_{k=0}^{n-1} \frac{\binom{n+1}{k}\binom{n+1}{k+1}\binom{n+1}{k+2}}{\binom{n+1}{1}\binom{n+1}{2}} t^k,$$

the generating function for Baxter permutations with respect to descents.

THEOREM 3.8. *$H_n^{(m)}(1,t)$ has only real roots for all m.*

This follows from the theory of multiplier sequences.

DEFINITION 3.9. Say that $\{a_k\}_{k\geq 0}$ is a multiplier sequence if for every polynomial $\sum_{k=0}^{n} b_k t^k$ with all real roots, $\sum_{k=0}^{n} a_k b_k t^k$ also has all real roots.

Multiplier sequences satisfy some basic properties, which are outlined in Craven and Csordas [**20**].

PROPOSITION 3.10. *Let $\{a_k\}_{k\geq 0}$ be a multiplier sequence.*

(a) *$\sum_{k=0}^{n} a_k t^k$ is real rooted.*
(b) *$\{a_{k+r}\}_{k\geq 0}$ for $r \geq 0$ is also a multiplier sequence.*
(c) *If $\{b_k\}_{k\geq 0}$ is another multiplier sequence, then $\{a_k b_k\}_{k\geq 0}$ is also a multiplier sequence.*
(d) *If $\{a_r, a_{r+1}, \ldots, a_{r+s}\}$ is a segment of a multiplier sequence, and $\sum_{k=0}^{n} b_k t^k$ is real rooted with $n \leq s$, then $\sum_{k=0}^{n} a_{r+s-k} b_k t^k$ is real rooted.*

PROOF OF THEOREM 3.8. When $m = 1$, $H_n^{(1)}(1,t) = \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!} = (1+t)^{n-1}$, which is obviously real rooted.

For $m > 1$, we can write $H_n^{(m)}(1,t)$ as

$$\sum_{k=0}^{n-1} \frac{f_m(n)! t^k}{\prod_{i=0}^{m-1} (k+i)!(n-1-k-i)!},$$

where

$$f_m(n) = \frac{(n+m-2)!^m}{\prod_{j=1}^{m-1} \binom{n+m-2}{j}}.$$

It suffices to show that we can obtain $H_n^{(m)}(1,t)$ by applying multiplier sequences to a known real-rooted polynomial.

Consider the polynomial

$$\frac{f_m(n)(1+t)^{n-1}}{(n-1)!} = \sum_{k=0}^{n+1-m} \frac{f_m(n) t^k}{k!(n-1-k)!},$$

which is obviously real-rooted, and of degree $n-1$.

It is well known that $\{\frac{1}{k!}\}_{k\geq 0}$ is a multiplier sequence [**15**, Theorem 2.4.1]. Proposition 3.10 (b) tells us that for $1 \leq i \leq m-1$, $\{\frac{1}{(k+i)!}\}_{k\geq 0}$ will be a multiplier sequence, and so

$$\sum_{k=0}^{n-1} \frac{f_m(n)t^k}{(n-1-k)!\prod_{i=0}^{m-1}(k+i)!}$$

will be real rooted.

Now apply Proposition 3.10 (d) to the segments $\{\frac{1}{(i)!}, \ldots, \frac{1}{(n-1+i)!}\}$ for $1 \leq i \leq m-1$ to this polynomial. This shows that

$$\sum_{k=0}^{n-1} \frac{(n+m-2)!t^k}{\prod_{i=0}^{m-1}(k+i)!(n-1-k-i)!} = H_n^{(m)}(1,t)$$

is real-rooted.

□

3.3. Gamma Nonnegativity

One can show that any polynomial of degree n with nonnegative symmetric coefficients and all real roots must expand nonnegatively in the basis

$$\{t^i(1+t)^{n-2i}\}_{0\leq i\leq \lfloor n/2\rfloor}$$

(Branden [**14**], Gal [**29**, Remark 3.1.1], Stembridge [**53**, Section 1.4]). While Gal proved that the h-polynomial is not real-rooted for flag spheres of dimension 5 or higher, he did make a conjecture about this weaker property.

Gal's Conjecture (Gal, [**29**]). *The h-polynomial for any flag (generalized) homology sphere is gamma nonnegative.*

Following Gal [**29**], one can consider the expansion of polynomials of degree n with symmetric coefficients in terms of the basis $\{t^i(1+t)^{n-2i}\}_{0\leq i\leq \lfloor \frac{n}{2}\rfloor}$.

Definition 3.11. Let $h(t) = a_0 + a_1t + \ldots a_nt^n$ be a polynomial of degree $\leq n$ with symmetric coefficients ($a_i = a_{n-i}$). Then if we write

$$h(t) = \sum_{i=0}^{\lfloor n/2\rfloor} \gamma_i t^i(1+t)^{n-2i},$$

we call $\gamma(h(t)) := \left(\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor (n+1)/2\rfloor}\right)$ the *gamma vector*, and we say that $h(t)$ is *gamma nonnegative* if all of the γ_i are nonnegative.

Since each polynomial $t^i(1+t)^{n-2i}$ has coefficient sequence that is symmetric about $t^{n/2}$ and unimodal, a polynomial being gamma nonnegative implies that its coefficient sequence is symmetric and unimodal. It is also worth pointing out that if a polynomial has symmetric integer coefficients, then the gamma vector will necessarily consist of integers.

Corollary 3.12. *$H_n^{(m)}(1,t)$ is γ-nonnegative.*

Proof. It is not hard to see that $H_n^{(m)}(1,t)=\sum_{k=0}^{n-1} a_k t^k$ will have symmetric coefficient sequence, as a_k counts plane partitions in a $k\times(n-1-k)\times m$ box, and a_{n-1-k} counts plane partitions in a $(n-1-k)\times k\times m$ box. Gamma nonnegativity now follows from Theorem 3.8, via [**29**, Remark 3.1.1]. $\square$

In general, we can represent $H_n^{(m)}(1,t)$ as a hypergeometric series, particularly,

$$H_n^{(m)}(1,t) = {}_mF_{m-1}\left[\left.\begin{matrix} -n+1 & -n & \dots & -n-m+2 \\ & m & \dots & 2 \end{matrix}\right| (-1)^m t\right].$$

For small values of m, we can use known hypergeometric transformations to come up with an explicit expression for the gamma-nonnegativity expansion.

For $m=1$, we have already seen that $H_n^{(1)}(1,t)=\sum_{k=0}^{n-1}\binom{n-1}{k}t^k=(1+t)^{n-1}$, by the standard binomial theorem.

For $m=2$, by a standard quadratic transformation (as seen in [**44**, Prop 11.14]), one can see that this polynomial has a gamma nonnegative expansion of

$$H_n^{(2)}(1,t) = \sum_{k=0}^{\lfloor \frac{n-1}{2}\rfloor} C_k \binom{n-1}{2k} t^k (1+t)^{n-1-2k},$$

where $C_k=\frac{1}{k+1}\binom{2k}{k}$ is the Catalan number.

For $m=3$, we can apply the well-poised ${}_3F_2$ quadratic tranformation [**5**, p. 97]

$$\begin{aligned}(1-z)^a {}_3F_2&\left[\left.\begin{matrix} a & b & c \\ & 1+a-b & 1+a-c \end{matrix}\right| z\right] \\ &= {}_3F_2\left[\left.\begin{matrix} \frac{1}{2}a & \frac{1}{2}(a+1) & 1+a-b-c \\ & 1+a-ba & 1+a-c \end{matrix}\right| -\frac{4z}{(1-z)^2}\right].\end{aligned}$$

Setting $z=-t$, $a=-n+1$, $b=-n$, $c=-n-1$ gives a gamma nonnegative expansion of

$$H_n^{(3)}(1,t) = \sum_{k=0}^{\lfloor n/2\rfloor} \gamma_i t^i (1+t)^{n-2i}$$

for

$$\gamma_i = \frac{(n+3)_i(1-n)_{2i}}{(1)_i(2)_i(3)_i}.^{1} \tag{14}$$

[1]Thanks to Dennis Stanton for suggesting this transformation

We note that each γ_i is an integer as its part of the gamma expansion for a symmetric polynomial with integer coefficients, though integrality is not *a priori* obvious by looking at the closed formula.

CHAPTER 4

q-Gamma Nonnegativity

4.1. Introduction

Now, we will establish a bivariate notion of gamma nonnegativity.

Definition 4.1. Call a bivariate polynomial

$$P(q,t)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}a_{r,s}q^r t^s$$

with integer coefficients *symmetric* (with center of symmetry $(M/2,N/2)$) if there exist $M,N\in\mathbb{Z}_{\geq 0}$ such that $P(q,t)=q^M t^N P(\frac{1}{q},\frac{1}{t})$ (or equivalently, $a_{r,s}=a_{M-r,N-s}$).

Note that this is not the same as $P(q,t)=P(t,q)$.

These symmetric polynomials are multiplicative, in the sense that the product of a symmetric polynomial (with center of symmetry $(M_1/2,N_1/2)$) and a symmetric polynomial (with center of symmetry $(M_2/2,N_2/2)$ will again be a symmetric polynomial (with center of symmetry $((M_1+M_2)/2,(N_1+N_2)/2)$.

Definition 4.2. Let $P(q,t)$ be a symmetric bivariate polynomial in $\mathbb{Z}[q,t]$ with center of symmetry $(M/2,N/2)$. Then if there exist nonnegative integers a and b and $\gamma_i^{(a,b)}(q)\in\mathbb{Z}[q]$ such that

$$P(q,t)=\sum_{i=0}^{\lfloor N/2\rfloor}\gamma_i^{(a,b)}(q)t^i\prod_{k=i}^{N-1-i}(1+tq^{ka+b}),$$

we say that $P(q,t)$ has a *q-gamma expansion (of type (a,b))*, and call

$$\gamma^{(a,b)}(P):=\left(\gamma_0^{(a,b)}(q),\gamma_1^{(a,b)}(q),\ldots,\gamma_{\lfloor N/2\rfloor}^{(a,b)}(q)\right)$$

its *q-gamma vector* (of type (a,b)).

We say that $P(q,t)$ is *q-gamma nonnegative (of type (a,b))* if furthermore each $\gamma_i^{(a,b)}(q)$ lies in $\mathbb{Z}_{\geq 0}[q]$.

This is clearly a q-analog of regular gamma expansions/gamma nonnegativity, as specializing a q-gamma expansion of $P(q,t)$ to $q=1$ will give a regular gamma expansion of $P(1,t)$, and if $P(q,t)$ is q-gamma nonnegative, then $P(1,t)$ will be gamma nonnegative (as in Definition 3.11).

Definition 3.11 is a nonnegative expansion of a symmetric polynomial with center of symmetry $(0, N/2)$ and nonnegative coefficients into a distinguished basis of nonnegative symmetric polynomials that also have center of symmetry $(0, N/2)$.

PROPOSITION 4.3. *If $P(q,t)$ has center of symmetry $(M/2, N/2)$ and has a q-gamma expansion of type (a,b),*

$$P(q,t) = \sum_{i=0}^{\lfloor N/2 \rfloor} \gamma_i^{(a,b)}(q) t^i \prod_{k=i}^{N-1-i} (1 + tq^{ka+b}).$$

Then we necessarily have that $\gamma_i^{(a,b)}(q)$ is symmetric with center of symmetry $(\frac{M-A}{2}, 0)$, for

$$A = \sum_{k=i}^{N-1-i} ka + b = (ai + b)(N - 2i) + \frac{a(N-2i)(N-1-2i)}{2}.$$

PROOF. Consider the smallest i for which $\gamma_i^{(a,b)}(q) = \sum b_j q^j \neq 0$, call it i_0. Then if we think of $P(q,t) = \sum_{s=0}^{N} \sum_{r=0}^{M} a_{r,s} q^r t^s$ as a polynomial in t with coefficients in $\mathbb{Z}[q]$, the only term from the q-gamma expansion that contributes to t^{i_0} and t^{N-i_0} will be $\gamma_{i_0}^{(a,b)}(q) t^i \prod_{k=i_0}^{N-1-i_0}(1 + tq^{ka+b})$. Thus, the coefficient of t^{i_0} must be $\gamma_{i_0}^{(a,b)}(q)$, so we have $b_j = a_{j,i_0}$. Similarly, the coefficient of t^{N-i_0} must be $\gamma_{i_0}^{(a,b)}(q) q^A$, where

$$A = \sum_{k=i_0}^{N-1-i_0} ka + b = (ai_0 + b)(N - 2i_0) + \frac{a(N-2i_0)(N-1-2i_0)}{2}.$$

This tells us that $a_{j+A,N-i_0} = b_j$. Combining these two statements with the symmetry of $P(q,t)$ gives $b_j = a_{j,i_0} = a_{M-j,N-i_0} = b_{M-A-j}$, which is exactly the statement that $\gamma_{i_0}^{(a,b)}(q)$ is symmetric with center of symmetry $(\frac{M-A}{2}, 0)$.

Now, repeat on

$$P(q,t) - \gamma_{i_0}^{(a,b)}(q) t^{i_0} \prod_{k=i_0}^{N-1-i_0} (1 + tq^{ka+b}),$$

which will have q-gamma expansion

$$\sum_{i=i_0+1}^{\lfloor N/2 \rfloor} \gamma_i^{(a,b)}(q) t^i \prod_{k=i}^{N-1-i} (1 + tq^{ka+b}).$$

$\square$

We note that this proposition implies that each summand of a q-gamma expansion will have the same center of symmetry as the original polynomial.

It will sometimes be convenient to use the notation of the Pochhammer symbol. We define

$$\begin{aligned}(x)_i =& x(x+1)\cdots(x+i-1)\\(x;q)_i =& (1-x)(1-xq)\dots(1-xq^{i-1}).\end{aligned}$$

Using the Pochhammer symbol, we can equivalently say that $P(q,t)$ has a q-gamma expansion if there exist nonnegative integers a and b and $\gamma_i^{(a,b)}(q) \in \mathbb{Z}[q]$ such that

$$P(q,t) = \sum_{i=0}^{\lfloor N/2 \rfloor} \gamma_i^{(a,b)}(q)\ t^i\ (-tq^{ai+b};q^a)_{N-2i}$$

If $P(q,t)$ is q-gamma nonnegative, then setting $q = 1$ makes $P(1,t)$ a univariate polynomial with symmetric coefficients and nonnegative gamma vector $\big(\gamma_0(1), \gamma_1(1), \dots, \gamma_{\lfloor N/2 \rfloor}(1)\big)$, so this is a q-analog of gamma nonegativity.

Remark 4.4. Note that given a $P(q,t)$ that is q-gamma nonnegative, the corresponding a and b are not uniquely determined. For example,

$$\begin{aligned}P(q,t) &= 1 + q^3t + q^4t + q^5t + q^6t + q^9t^2\\&= (1+tq^4)(1+tq^5) + (q^3+q^6)t\\&= (1+tq^3)(1+tq^6) + (q^4+q^5)t\end{aligned}$$

has nonnegative $\gamma^{(1,4)}$ and $\gamma^{(3,3)}$ vectors.

Remark 4.5. There is a slightly different notion of q-gamma nonnegativity that has been used by Krattenthaler and Wachs. They say that a polynomial $P(q,t) \in \mathbb{N}[q][t]$ of degree d in t is *q-gamma nonnegative of index r* if there exist $\gamma_k(q) \in \mathbb{N}[q]$ such that

$$P(q,t) = \sum_{k=0}^{\lfloor d/2 \rfloor} q^{r\binom{k+1}{2}} \gamma_k(q) t^k \prod_{i=k+1}^{d-k} (1+q^{ri}t)$$

The notion of q-gamma nonnegativity that we use is more general, as anything that is q-gamma nonnegative of index r in the sense that they have defined will be q-gamma nonnegative of type (r,r) in the sense that we have defined (with the $\gamma_j(q)$ differing only by a factor of q), but the converse is not necessarily true. However, many results we have here could be expressed in either framework. They are also able to get some additional results from their framework.

Theorem 4.6 (Krattenthaler and Wachs). [1] *Let $P(q,t) \in \mathbb{N}[q][t]$ be q-gamma nonnegative of index r of degree n. Then if $\widetilde{P}(q,t)$ is the polynomial obtained from $P(q,t)$ by multiplying the coefficient of t^j by $q^{-r\binom{j+1}{2}}$, then $\widetilde{P}(q,t) = \sum_{i=0}^{n} a_i(q)t^i$ is palindromic (i.e., $a_i(q) = a_{n-i}(q)$) and q-unimodal (i.e., $a_{i+1}(q) - a_i(q) \in \mathbb{N}[q]$ for $0 \leq i < n/2$).*

4.2. Hoggatt Families

Our original motivation for q-gamma nonnegativity arose from examples contained in Hoggatt families.

Proposition 4.7. *$H_n^{(m)}(q,t)$ is symmetric, with center of symmetry $(M,N) = (m\binom{n}{2}, n-1)$.*

Proof. Let
$$H_n^{(m)}(q,t) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} a_{r,s}q^r t^s.$$

Take $M = m\binom{n}{2}$ and $N = n-1$. Define an involution on the set of plane partitions that contribute to $H_n^{(m)}(q,t)$ by pairing π in the $k \times (n-1-k) \times m$ box with $\bar{\pi}^c$ in the $(n-1-k) \times k \times m$ box, where $\bar{\pi}$ is the plane partition in $(n-1-k) \times k \times m$ box naturally identified with π.

The first plane partition π will contribute a term of $q^{|\pi|+m\binom{k+1}{2}}t^k$, while the second plane partition $\bar{\pi}^c$ will contribute a term of $q^{km(n-1-k)-|\pi|+m\binom{n-k}{2}}t^{n-1-k}$.

One can check that

$$km(n-1-k) - |\pi| + m\binom{n-k}{2} = m\binom{n}{2} - \left(|\pi| + m\binom{k+1}{2}\right),$$

which shows that $a_{r,s} = a_{M-r,N-s}$, and $H_n^{(m)}(q,t)$ is symmetric.

□

In general, we have that

$$H_n^{(m)}(q,t) = {}_m\phi_{m-1}\left[\begin{matrix} q^{-n+1} & q^{-n} & \dots & q^{-n-m+2} \\ & q^m & \dots & q^2 \end{matrix}\middle| q, (-1)^m(tq^{mn+\binom{m}{2}})\right].$$

Again, for small values of m, we are able to do some explicit computations.

4.2.1. Subsets. When $m=1$, by the q-binomial theorem, we get

$$H_n^{(1)}(q,t) = \sum_{k=0}^{n-1} q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q t^k = (-tq;q)_{n-1} = (1+tq)(1+tq^2)\dots(1+tq^{n-1}),$$

[1] Personal communication with Michelle Wachs

which shows that $H_n^{(1)}(q,t)$ is q-gamma nonnegative of type (1,1).

4.2.2. Catalan Objects. For $m=2$, we have the following.

THEOREM 4.8.

$$H_n^{(2)}(q,t) = \sum_{k=0}^{n-1} \frac{q^{k^2+k}}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q t^k \tag{15}$$

$$= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} C_k(q^2) \begin{bmatrix} n-1 \\ 2k \end{bmatrix}_q q^{k(k+2)} t^k \prod_{i=k}^{n-2-k} (1+tq^{2i+2}),$$

where $C_k(q) = \frac{1}{[k+1]_q} \begin{bmatrix} 2k \\ k \end{bmatrix}_q$ *is a q-Catalan number [**28**], and thus* $\mathrm{Cat}_n(q,t)$ *is q-gamma nonnegative of type (2,2).*

REMARK 4.9. This result was independently obtained by Christian Krattenthaler and Michelle Wachs.

PROOF. From the definition, we can see that

$$\mathrm{Cat}_n(q,t) = {}_2\phi_1 \left[\begin{matrix} q^{-n} & q^{-n+1} \\ & q^2 \end{matrix} \middle| q, tq^{2n+1} \right].$$

Now we apply Lemma 4.10 below, setting $x = tq^{2n}$, $A = q^{-n+1}$, and $B = q^{-n}$. $\square$

LEMMA 4.10.

$${}_2\phi_1 \left[\begin{matrix} A & B \\ & \frac{Aq}{B} \end{matrix} \middle| q, \frac{Ax}{B} \right] = \sum_{k=0}^{\infty} \frac{(A^2;q^2)_{2k}(-A/B;q)_{2k} q^{-2\binom{k}{2}} x^k}{(q^2;q^2)_k (A^2q^2/B^2;q^2)_k (-A;q)_{2k}} \frac{(-A^2q^{2k}x;q^2)_\infty}{(-q^{-2k}x;q^2)_\infty},$$

as a formal power series in x.

COROLLARY 4.11. *If we set $A = q^{-N}$ for N a positive integer, then we have*

$${}_2\phi_1 \left[\begin{matrix} q^{-N} & B \\ & \frac{q^{-N+1}}{B} \end{matrix} \middle| q, \frac{xq^{-N}}{B} \right] = \sum_{k=0}^{\lfloor N/2 \rfloor} \frac{(q^{-2N};q^2)_{2k}(-q^{-N}/B;q)_{2k} q^{-2\binom{k}{2}} x^k}{(q^2;q^2)_k (-q^{2-2N}/B;q)_k (-q^{-N};q)_{2k}} (-q^{-2N+2k}x;q^2)_{N-2k}$$

as a polynomial identity.

PROOF. Many thanks to Dennis Stanton for helping to derive this identity.

Our goal is to show that the coefficient of x^N on each side of the equation is the same. First, we look at the right hand side. The q-binomial theorem [**30**][II.3] says

$$ {}_1\Phi_0[a;-;q,z] = \frac{(az;q)_\infty}{(z;q)_\infty}, $$

so by setting $z = -xq^{-2k}$ and $a = A^2q^{4k}$, we can expand the last part of the right hand side as

$$ \frac{(-A^2q^{2k}x;q^2)_\infty}{(-q^{-2k}x;q^2)_\infty} = \sum_j \frac{(A^2q^{4k};q^2)_j}{(q^2;q^2)_j}(-xq^{-2k})^j. $$

Then the coefficient of x^N on the right hand side is

$$ \sum_{k=0}^{N} \frac{(A^2;q^2)_{2k}(-A/B;q)_{2k}q^{-2\binom{k}{2}}x^k}{(q^2;q^2)_k(A^2q^2/B^2;q^2)_k(-A;q)_{2k}} \frac{(A^2q^{4k};q^2)_{N-k}}{(q^2;q^2)_{N-k}}(-xq^{-2k})^{N-k}. $$

Simplifying this expression gives

$$ \frac{(A^2;q^2)_N}{(q^2;q^2)_N} \sum_{k=0}^{N} \frac{(A^2q^{2N};q^2)_k(-A/B;q^2)_k(-Aq/B;q^2)_k(q^{-2N};q^2)_kq^{2k}}{(q^2;q^2)_k(A^2q^2/B^2;q^2)_k(-A;q^2)_k(-Aq;q^2)_k} $$

which can also be written as

$$ \frac{(A^2;q^2)_N}{(q^2;q^2)_N} {}_4\phi_3 \left[\left. \begin{matrix} A^2q^{2n} & -A/B & -Aq/B & q^{-2N} \\ & A^2q^2/B^2 & -A & -Aq \end{matrix} \right| q^2, q^2 \right]. $$

Using Singh's quadratic transformation [**30**][III.22],

$$ {}_4\phi_3 \left[\left. \begin{matrix} a^2 & b^2 & c^2 & d^2 \\ & a^2b^2q & -cd & -cdq \end{matrix} \right| q^2, q^2 \right] = {}_4\phi_3 \left[\left. \begin{matrix} a^2 & b^2 & c & d \\ & abq^{1/2} & -abq^{1/2} & -cd \end{matrix} \right| q, q \right] $$

with choice of parameters $a^2 = Aq/B$, $b^2 = -A/B$, $c^2 = A^2q^{2N}$, and $d^2 = q^{-2N}$, this expression becomes

$$ \frac{(A^2;q^2)_N}{(q^2;q^2)_N} {}_4\phi_3 \left[\left. \begin{matrix} -A/B & -Aq/B & Aq^N & q^{-N} \\ & Aq/B & -Aq/B & -A \end{matrix} \right| q, q \right]. $$

Cancelling out the like terms, the ${}_4\phi_3$ becomes a ${}_3\phi_2$. Using the q-Saalschütz sum formula ([**30**][II.12]),

$$ {}_3\phi_2 \left[\left. \begin{matrix} a & b & q^{-n} \\ & c & abc^{-1}q^{1-n} \end{matrix} \right| q, q \right] = \frac{(c/a;q)_n(c/b;q)_n}{(c;q)_n(c/ab;q)_n}, $$

for choice of parameters $a = -A/B$, $b = Aq^N$, $c = -A$, and $n = N$, our expression reduces to

$$\frac{(A^2;q^2)_N}{(q^2;q^2)_N}\frac{(B;q)_N(q^{-N};q)_N}{(-A;q)_N(Bq^{-N}/A;q)_N}$$

After making some standard reductions, this readily becomes

$$\frac{(A;q)_N(B;q)_N}{(q;q)_N(Aq/B;q)_N}(A/B)^N,$$

which is the desired coefficient of x^N on the left hand side of the original equation.

□

4.2.3. Baxter Objects. For $m=3$, in terms of q-gamma nonnegativity, we have the following theorem.

THEOREM 4.12.

$$H_n^{(3)}(q,t)=\sum_{i=0}^{\lfloor (n-1)/2\rfloor}\gamma_i(q)t^i\prod_{k=i}^{n-2-i}(1+tq^{n+1+k})$$

for

$$\gamma_i(q)=q^{3\binom{i}{2}+3i}\frac{(q^{n-2i};q)_{2i}(q^{n+3};q)_i}{(q^3;q)_i(q^2;q)_i(q;q)_i} \tag{16}$$

and thus $H_n^{(3)}(q,t)$ *has a q-gamma expansion of type* $(1,n+1)$.

PROOF. Note that $H_n^{(3)}(q,t)={}_3\Phi_2\left[\begin{matrix}q^{-n+1} & q^{-n} & q^{-n-1}\\ & q^2 & q^3\end{matrix}\middle| q,-tq^{3n+3}\right]$. Using the Sears-Carlitz transformation of a terminating well-poised ${}_3\phi_2$ [**30**, (III.14)],

$$\begin{aligned}&{}_3\Phi_2\left[\begin{matrix}a & b & c\\ & aq/b & aq/c\end{matrix}\middle| q,\frac{aqz}{bc}\right]=\\ &\frac{(az;q)_\infty}{(z;q)_\infty}{}_5\Phi_4\left[\begin{matrix}a^{1/2} & -a^{1/2} & (aq)^{1/2} & -(aq)^{1/2} & aq/bc\\ & aq/b & aq/c & az & q/z\end{matrix}\middle| q,q\right]\end{aligned}$$

for choice of parameters $a=q^{-n+1}$, $b=q^{-n}$, $c=q^{-n-1}$, and $z=-tq^{2n}$, the left hand side is readily seen to be the formula for $H_n^{(3)}(q,t)$, and the right hand side reduces to our desired expression by standard reductions.

□

CONJECTURE 4.13. $\gamma_i(q)\in\mathbb{Z}_{\geq 0}[q]$ *(as in Equation (16)), and thus the above q-gamma expansion is q-gamma nonnegative.*

Interestingly enough, there appears to be a second q-gamma expansion for Baxter permutations, which appears to generalize to all Hoggatt levels.

Conjecture 4.14. *For every $m \geq 3$, $H_n^{(m)}(q,t)$ is q-gamma nonnegative of type (m,m).*

The examples of q-gamma nonnegativity given for $m = 1, 2$ are both of this form. This would be an extension of Corollary 3.12, which asserts the necessary condition that this polynomial be gamma nonnegative when we set $q = 1$.

4.3. Permutations

Another example of q-gamma nonnegativity comes from looking at permutations.

Definition 4.15. Say that a permutation $w = w_1 \dots w_n$ has a *descent at position* i if $w_{i+1} < w_i$. Let $\operatorname{Des}(w)$ be the set of all descents of w, let $\operatorname{des}(w) = |\operatorname{Des}(w)|$ and define the *major index* by $\operatorname{maj}(w) = \sum_{i \in \operatorname{Des}(w)} i$.

For example, if $w = 631542$, then $\operatorname{Des}(w) = \{1, 2, 4, 5\}$, $\operatorname{des}(w) = 4$, and $\operatorname{maj}(w) = 12$.

The classical Eulerian polynomial is given by

$$A_n(t) = \sum_{w \in S_n} t^{\operatorname{des}(w)}.$$

This relates to Gal's conjecture, as $A_n(t)$ is known to be the h-vector for the permutahedron. It has been proven combinatorially by Shapiro, Woan, and Getu [**47**] that it has gamma nonnegativity expansion

$$A_n(t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i t^i (1+t)^{n-1-2i},$$

where γ_i is the number of permutations in S_n with no consecutive descents, $n-1$ is not a descent, and i peaks, where the number of peaks is defined by

$$\operatorname{pk}(u) = |\{2 \leq i \leq n-1 | u_{i-1} < u_i > u_{i+1}\}|.$$

There is another version of this formula due to Foata and Schützenberger [**25**] which shows that

$$A_n(t) = 2^{-n-1} \sum_{u \in S_n} (4t)^{\operatorname{pk}(u)} (1+t)^{n-1-2\operatorname{pk}(u)}.$$

Our polynomial of interest will be

$$A_n(q,t) = \sum_{w \in S_n} q^{\operatorname{maj}(w)} t^{\operatorname{des}(w)}.$$

Theorem 4.16 (Han, Jouhet, Zeng [**35**]). *There exists an expansion*

$$A_n(q,t) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \gamma_i^{(0,1)}(q) t^i \prod_{k=i}^{n-1-i} (1 + tq^k),$$

where $\gamma_i^{(0,1)}(q) \in \mathbb{Z}_{\geq 0}[q]$*. In particular,* $A_n(q,t)$ *is q-gamma nonnegative of type (0,1).*

Their proof relies on the q-Carlitz identity, which states that

$$\sum_{k \geq 0} [k+1]_q^n t^k = \frac{\sum_{w \in S_n} q^{\mathrm{maj}(w)} t^{\mathrm{des}(w)}}{\prod_{i=0}^n (1 - tq^i)},$$

and they do not give a combinatorial interpretation.

Example 4.17.

$$\begin{aligned}
A_2(q,t) =& 1 + tq \\
A_3(q,t) =& (1+tq)(1+tq^2) + (q+q^2)t \\
A_4(q,t) =& (1+tq)(1+tq^2)(1+tq^3) + (2q+4q^2+2q^3)t(1+tq^2) \\
A_5(q,t) =& (1+tq)(1+tq^2)(1+tq^3)(1+tq^4) + (3q+8q^2+8q^3+3q^4)t(1+tq^2)(1+tq^3) \\
&+ (2q^3+4q^4+4q^5+4q^6+2q^7)t^2
\end{aligned}$$

Remark 4.18. Shareshian and Wachs have come up with similar gamma nonnegativity results [**48**, Remark 5.5], but with excedance playing the primary role, where $\mathrm{exc}(w)$ is the number of i for which $w_i > i$. In particular, if we let

$$A_{n,k}^{\mathrm{maj,des,exc}} = \sum_{\substack{w \in S_n \\ \mathrm{fix}(w) = k}} q^{\mathrm{maj}(w)} p^{\mathrm{des}(w)} t^{\mathrm{exc}(w)}$$

with $A_n^{\mathrm{maj,des,exc}}$ representing the same sum with no restriction on number of fixed points, then

- $A_{n,0}^{\mathrm{maj,des,exc}}(q,p,q^{-1}t)$ has coefficients in $\mathbb{Z}_{\geq 0}[q,p]$ when expanded in the basis $\{t^d(1+t)^{n-2d}\}_{d=0}^{\lfloor n/2 \rfloor}$.
- $A_{n,k}^{\mathrm{maj,des,exc}}(q,1,q^{-1}t)$ has coefficients in $\mathbb{Z}_{\geq 0}[q]$ when expanded in the basis $\{t^d(1+t)^{n-k-2d}\}_{d=0}^{\lfloor (n-k)/2 \rfloor}$.
- $A_n^{\mathrm{maj,des,exc}}(q,1,q^{-1}t)$ has coefficients in $\mathbb{Z}_{\geq 0}[q]$ when expanded in the basis $\{t^d(1+t)^{n-1-2d}\}_{d=0}^{\lfloor (n-1)/2 \rfloor}$.

We note that none of these results can be specialized to give Conjecture 4.16, and while these results could be phrased as being q-gamma nonnegative of type (0,0), typically our results are q-gamma nonnegative of type (a,b) with a strictly positive.

Remark 4.19. It is also worth pointing out that Theorem 4.16 does not immediately follow from a specialization of the cd-index. For a bounded and ranked poset, one can define a polynomial in non-commuting variables **a** and **b** called the ab-index. In the case where the poset is the face lattice of a polytope, the ab-index is known to be a polynomial in the expressions $\mathbf{c} = \mathbf{a} + \mathbf{d}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ called the cd-index [**50**].

In the case of the permutahedron, one can think of the ab-index as being the generating function over all permutations, where to each permutation $w = w_1 \dots w_n$, we assign a monomial $u_w = u_1 \dots u_{n-1}$, where u_i is a if $i \notin \mathrm{Des}(w)$ and b if $i \in \mathrm{Des}(w)$. One can see that specializing **a** to 1 and **b** to t will give $A_n(t)$, and if we make the corresponding changes to **c** and **d**, then the cd-index will specialize to a gamma nonnegativity expansion.

Similarly, if we specialize **a** to 1 and **b** to tq^i when b is the i^{th} term in its monomial, we will get $A_n(q,t)$. Then one can replace c with $(1 + tq^i)$ and d with $(tq^j + tq^{j+1})$ (where i and j depend on the position of c and d in a given monomial) to come up with a q-analog of the gamma positivity expansion for $A_n(q,t)$. However, the pieces of this expansion do not have central symmetry with respect to both q and t, and there does not appear to be a systematic way to combine terms to give our expansion.

For example, consider S_4. Using the cd-index $\phi_4 = c^3 + 2cd + 2dc$ and making the appropriate specializations, we would get the expansion

$$A_4(q,t) = (1+tq)(1+tq^2)(1+tq^3) + 2(1+q)q^2(1+tq) + 2q(1+q)(1+tq^3).$$

The first summand has our desired central symmetry with respect to both q and t, but the latter two summands do not.

4.4. Signed Permutations

We define a signed permutation of length n to be a bijection $f : \pm[n] \mapsto \pm[n]$, for $\pm[n] = \{-n, \dots, -1, 1, \dots n\}$, satisfying $f(-i) = -f(i)$. Typically, we will refer to a signed permutation in one line notation as $w = w_1 \dots w_n$, where $w_i = f(i)$, and we use bars to denote negated elements. For example, the bijection with $f(1) = -3$, $f(2) = 1$, and $f(3) = -2$ would be denoted at $\bar{3}1\bar{2}$. Let B_n be the set of all signed permutations of length n.

Definition 4.20. We say a signed permutation $w = w_1 \dots w_n$ has a *descent* at position i if $w_i > w_{i+1}$, and a descent at position 0 if $w_1 < 0$. Let $\mathrm{Des}_B(w) \subseteq \{0, 1, \dots, n-1\}$ be the set of descents of w, and $\mathrm{des}_B(w) = |\mathrm{Des}_B(w)|$. For example, $\mathrm{Des}_B(\bar{3}1\bar{2}) = \{0, 2\}$.

THEOREM 4.21. *[**43**, Petersen, Prop 4.16]*

$$\sum_{w\in B_n} t^{des_B(w)} = \sum_{u\in S_n}(4t)^{lpk(u)}(1+t)^{n-2lpk(u)},$$

where $lpk(u) = \#\{i \in [1, n-1] : u_{i-1} < u_i > u_{i+1}\}$*, with the convention that* $u_0 = 0$.

One natural statistic that arises when looking at signed permutations is called $fmaj$, originally defined by Adin and Roichman [**1**].

DEFINITION 4.22. For a signed permutation $w \in B_n$, let $\operatorname{maj}_B(w) = \sum_{i\in \operatorname{Des}_B(w)} i$, and let $\operatorname{neg}(w) = \#\{i|w_i < 0\}$. Then

$$\operatorname{fmaj}(w) = 2\operatorname{maj}_B(w) + \operatorname{neg}(w).$$

Part of the motivation for defining this statistic is that it is equidistributed with Coxeter length on signed permutations, similar to how the major index on permutations is equidistributed with its Coxeter length (inversion number), as shown by MacMahon [**39**].

THEOREM 4.23 (Han, Jouhet, Zeng [**35**]).

$$B_n(q,t) = \sum_{w\in B_n} q^{\operatorname{fmaj}(w)} t^{\operatorname{des}_B(w)} = \sum_{i=0}^{\lfloor n/2\rfloor} \gamma_i^{(2,1)}(q) t^i \prod_{k=i}^{n-1-i}(1+tq^{2k+1}),$$

where $\gamma_i^{(2,1)}(q) \in \mathbb{Z}_{\geq 0}[q]$*. In particular,* $B_n(q,t)$ *is q-gamma nonnegative of type (2,1).*

Again, this follows from a type-B version of the q-Carlitz identity [**17**], which states that

$$\sum_{k\geq 0}[2k+1]_q^n t^k = \frac{\sum_{w\in B_n} q^{\operatorname{fmaj}(w)} t^{des_B(w)}}{\prod_{i=0}^n (1-tq^{2i})}.$$

EXAMPLE 4.24.

$$\begin{aligned}
B_2(q,t) =&(1+tq)(1+tq^3) + (q^2+2q^3+q^4)t\\
B_3(q,t) =&(1+tq)(1+tq^3)(1+tq^5) + (2q+5q^2+6q^3+5q^4+2q^5)t(1+tq^3)\\
B_4(q,t) =&(1+tq)(1+tq^3)(1+tq^5)(1+tq^7)\\
&+ (3q+9q^2+15q^3+18q^4+15q^5+9q^6+3q^7)t(1+tq^3)(1+tq^5)\\
&+ (2q^4+7q^5+11q^6+13q^7+14q^8+13q^9+11q^{10}+7q^{11}+2q^{12})
\end{aligned}$$

Remark 4.25. Our previous two examples correspond to type A (permutations) and type B (signed permutations). It would be interesting to know if there is a similar result for the third infinite family of irreducible Coxeter groups (type D, corresponding to signed permutations with an even number of signs). There are a number of potential statistics for type D one could consider (see [**7**], [**8**], [**9**], [**41**]) that are analagous to major index and fmaj. However, none of them seem to give the desired type of result generalizing the type D Eulerian polynomial.

4.5. Cyclohedron

Another example comes from the cyclohedron. It can be thought of as the dual polytope to a type B version of the associahedron, which corresponds to centrally symmetric triangulations of a $2n$-gon. It can also be realized as the graph-associahedron for the n-cycle graph, as defined by Carr and Devadoss [**16**]. Its h-polynomial was computed by Simion [**49**] to be:

$$\mathrm{Cyc}_n(t) = \sum_{k=0}^{n} \binom{n}{k}^2 t^k.$$

Let $\mathfrak{D}_n$ be the set of binary sequences in $\{0,1\}^{2n}$ with the same number of 0's as 1's. These can be combinatorially interpreted as lattice paths from $(0,0)$ to (n,n) using only steps $E = (1,0)$ and $N = (1,0)$. As with permutations, for $a = a_1 \ldots a_{2n} \in \mathfrak{D}_n$, we let $\mathrm{Des}(w) = \{i | a_{i+1} < a_i\}$, $\mathrm{des}(a) = |\mathrm{Des}(a)|$, and $\mathrm{maj}(a) = \sum_{i \in \mathrm{Des}(w)} i$. This h-vector can be interpreted as the descent generating function for $\mathfrak{D}_n$, or equivalently, the generating function for the lattice paths with respect to number of EN corners.

This polynomial has a gamma nonnegativity expansion, which can be computed by a hypergeometric transformation (as in [**44**], via Lemma 4.1 in [**46**] with $r = n, a_1 = a_2 = 1$) to be

$$\mathrm{Cyc}_n(t) = \sum_{r=0}^{\lfloor n/2 \rfloor} \binom{n}{r, r, n-2r} t^r (1+t)^{n-2r}.$$

If we consider the joint distribution with respect to des and maj, we get

$$\mathrm{Cyc}_n(q,t) = \sum_{a \in \mathfrak{D}_n} q^{\mathrm{maj}(a)} t^{\mathrm{des}(a)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q^2 q^{k^2} t^k.$$

This can be viewed as a special case of Theorem 1 in [**37**] for $\mu_1 = \mu_2 = 0$, $\lambda_1 = \lambda_2 = n$, $c = -n$, $d = n$.

THEOREM 4.26.

$$\mathrm{Cyc}_n(q,t) = \sum_{r=0}^{\lfloor n/2 \rfloor} \gamma_r(q) t^r \prod_{k=r}^{n-1-r} (1+tq^{2k+1}),$$

for

$$\gamma_r(q) = q^{(-r^2+2r+2nr)} \frac{(q^{-n};q)_{2r}(-1;q)_{2r}}{(q^2;q^2)_r(q^2;q^2)_r} \tag{17}$$

and thus has a q-gamma expansion of type (2,1).

REMARK 4.27. This result was independently obtained by Michelle Wachs and Christian Krattenthaler.

PROOF. We again apply Lemma 4.10, this time with choice of parameters $A = B = q^{-n}$, and $x = tq^{2n+1}$

□

THEOREM 4.28 (Krattenthaler and Wachs). [2] $\gamma_r(q) \in \mathbb{Z}_{\geq 0}[q]$ *(for* $\gamma_r(q)$ *as in Equation 17), and thus the above q-gamma expansion is q-gamma nonnegative.*

4.6. Involutions

One interesting example that is not obviously the h-vector for a flag simplicial polytope has to do with involutive permutations. We say a permutation $w \in S_n$ is an involution if $w^2 = 1$, and let I_n denote the set of all involutions in S_n. As with all permutations, we can consider the descent generating function,

$$I_n(t) = \sum_{w \in I_n} t^{\mathrm{des}(w)}.$$

Guo and Zeng have proved that this polynomial is unimodal, but they additionally conjectured that it is gamma nonnegative.

CONJECTURE 4.29 (Guo,Zeng [**34**]).

$$I_n(t) = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \gamma_r t^r (1+t)^{n-1-2r}$$

for $\gamma_r \geq 0$.

However, it appears that even more is true.

CONJECTURE 4.30.

$$I_n(q,t) = \sum_{w \in I_n} q^{maj(w)} t^{des(w)} = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} \gamma_r^{(1,1)}(q) t^r \prod_{k=r}^{n-2-r} (1+tq^{k+1}),$$

[2]Private communication with Michelle Wachs

where $\gamma_r^{(1,1)}(q) \in \mathbb{Z}_{\geq 0}[q]$.

This conjecture was also independently noted by Kyle Petersen.

Example 4.31.

$$\begin{aligned}
I_2(q,t) =& 1 + tq \\
I_3(q,t) =& (1+tq)(1+tq^2) \\
I_4(q,t) =& (1+tq)(1+tq^2)(1+tq^3) + q^2t(1+tq^2) \\
I_5(q,t) =& (1+tq)(1+tq^2)(1+tq^3)(1+tq^4) + (q^2+q^3)t(1+tq^2)(1+tq^3) \\
&+ (q^4+q^6)t^2
\end{aligned}$$

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APPENDIX A

Tables

TABLE A.1. Baxter objects of order $(k, \ell) = (3, 0)$

$$\Theta_{k,\ell}(q) = 1$$
$$\Theta_{k,\ell}(1) = 1$$
$$\Theta_{k,\ell}(-1) = 1$$

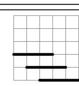
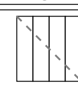
Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
1234 $\circlearrowleft$	1234 $\circlearrowleft$	 $\circlearrowleft$	1 3 6 9 2 5 8 11 4 7 10 12 $\circlearrowleft$	 $\circlearrowleft$	$\emptyset$ $\circlearrowleft$

TABLE A.2. Baxter objects of order $(k, \ell) = (0, 3)$

$$\Theta_{k,\ell}(q) = 1$$
$$\Theta_{k,\ell}(1) = 1$$
$$\Theta_{k,\ell}(-1) = 1$$

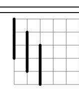
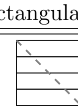
Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
4321 $\circlearrowleft$	4321 $\circlearrowleft$	 $\circlearrowleft$	1 4 7 10 2 5 8 11 3 6 9 12 $\circlearrowleft$	 $\circlearrowleft$	$\emptyset$ $\circlearrowleft$

Table A.3. Baxter objects of order $(k, \ell) = (2, 1)$

$$\Theta_{k,\ell}(q) = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6$$
$$\Theta_{k,\ell}(1) = 10$$
$$\Theta_{k,\ell}(-1) = 2$$

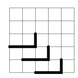
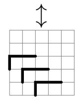
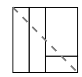
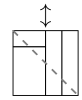
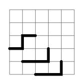
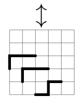
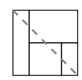
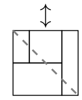
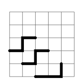
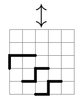
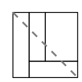
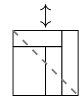
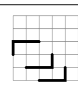
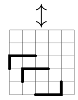
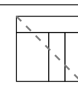
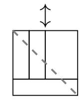
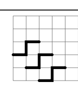
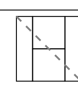
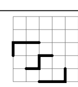
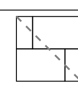
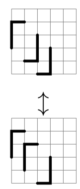
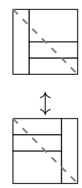
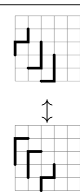
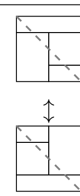
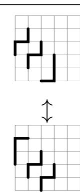
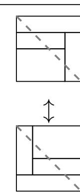
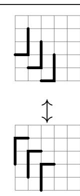
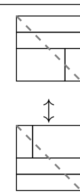
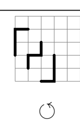
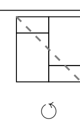
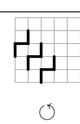
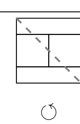
Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
1243 $\updownarrow$ 2134	1243 $\updownarrow$ 2134	 $\updownarrow$ 	1 3 6 9 2 5 7 11 4 8 10 12 $\updownarrow$ 1 3 5 9 2 6 8 11 4 7 10 12	 $\updownarrow$ 	3 3 $\updownarrow$ 0 0
1342 $\updownarrow$ 3124	1342 $\updownarrow$ 3124	 $\updownarrow$ 	1 3 6 9 2 4 8 11 5 7 10 12 $\updownarrow$ 1 3 6 8 2 5 9 11 4 7 10 12	 $\updownarrow$ 	3 2 $\updownarrow$ 1 0
$\updownarrow$ 1423 $\updownarrow$ 2314	$\updownarrow$ 1423 $\updownarrow$ 2314	$\updownarrow$  $\updownarrow$ 	$\updownarrow$ 1 3 5 7 2 6 9 11 4 8 10 12 $\updownarrow$ 1 3 5 9 2 4 7 11 6 8 10 12	$\updownarrow$  $\updownarrow$ 	$\updownarrow$ 3 1 $\updownarrow$ 2 0
2341 $\updownarrow$ 4123	2341 $\updownarrow$ 4123	 $\updownarrow$ 	1 4 6 9 2 5 8 11 3 7 10 12 $\updownarrow$ 1 3 6 10 2 5 8 11 4 7 9 12	 $\updownarrow$ 	2 2 $\updownarrow$ 1 1
1324 $\circlearrowleft$	1324 $\circlearrowleft$	 $\circlearrowleft$	1 3 5 7 2 4 9 11 6 8 10 12 $\circlearrowleft$	 $\circlearrowleft$	3 0 $\circlearrowleft$
3412 $\circlearrowleft$	3142 $\circlearrowleft$	 $\circlearrowleft$	1 3 7 9 2 5 8 11 4 6 10 12 $\circlearrowleft$	 $\circlearrowleft$	2 1 $\circlearrowleft$

Table A.4. Baxter objects of order $(k,\ell)=(1,2)$

$$\Theta_{k,\ell}(q) = 1+q+2q^2+2q^3+2q^4+q^5+q^6$$
$$\Theta_{k,\ell}(1) = 10$$
$$\Theta_{k,\ell}(-1) = 2$$

Baxter Permutations	Twisted Baxter Permutations	Baxter Paths	Baxter Tableaux	Diagonal Rectangulations	Baxter Plane Partitions
1432 ↕ 3214	1432 ↕ 3214		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&6&9\\\hline 2&4&7&11\\\hline 5&8&10&12\\\hline\end{array}$ ↕ $\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&5&8\\\hline 2&6&9&11\\\hline 4&7&10&12\\\hline\end{array}$		2 2 ↕ 1 1
2431 ↕ 4213	2431 ↕ 4213		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&4&6&9\\\hline 2&5&7&11\\\hline 3&8&10&12\\\hline\end{array}$ ↕ $\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&5&10\\\hline 2&6&8&11\\\hline 4&7&9&12\\\hline\end{array}$		3 2 ↕ 1 0
3241 ↕ 4132	3241 ↕ 4132		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&4&6&8\\\hline 2&5&9&11\\\hline 3&7&10&12\\\hline\end{array}$ ↕ $\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&6&10\\\hline 2&4&8&11\\\hline 5&7&9&12\\\hline\end{array}$		3 1 ↕ 2 0
3421 ↕ 4312	3421 ↕ 4312		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&4&7&9\\\hline 2&5&8&11\\\hline 3&6&10&12\\\hline\end{array}$ ↕ $\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&7&10\\\hline 2&5&8&11\\\hline 4&6&9&12\\\hline\end{array}$		3 3 ↕ 0 0
2143 ↺	2143 ↺		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&3&6&8\\\hline 2&4&9&11\\\hline 5&7&10&12\\\hline\end{array}$ ↺		2 1 ↺
4231 ↺	4231 ↺		$\begin{array}{\|c\|c\|c\|c\|}\hline 1&4&6&10\\\hline 2&5&8&11\\\hline 3&7&9&12\\\hline\end{array}$ ↺		3 0 ↺