CRITICAL GROUPS AND LINE GRAPHS

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1. Introduction

This paper is an overview of what the author has learned about the critical group of a graph, including some new results. In particular we discuss the critical group of a graph in relation to that of its line graph when the original graph is regular. We begin by introducing the critical group from various aspects. We then study the subdivision graph and line graph in relation to the critical group, the latter when the graph is regular. We discribe a homomorphism between the critcal groups of the line graph and the subdivision graph in this situation. Lastly, we conjecture some results about the kernel of this homomorphism and the structure of the critcal group of some particular graphs.

2. The critical group

The critical group of a graph is a subtle isomorphism invariant of the graph which is a finite abelian group whose order is the spanning tree number of the graph. It arises in a number of seemingly unrelated places and hence has a number of nice interpretations in terms of the graph. We take the most immediate interpretation as our starting place. Let $G = (V, E)$ be a directed graph with vertex set V and edge set E which, for the sake of simplicity, we take to be without loops or multiple (unoriented) edges. We define the *Laplacian* to be the $|V| \times |V|$ matrix defined by

$$L(G)_{v,v'} = \begin{cases} \deg_G(v) & \text{if, } v = v', \\ -1 & \text{if the unoriented edge, } vv' \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\overline{L(G)}^{v,v'}$ be the matrix obtained by striking out row v and column v' from $L(G)$. Define the *critical group*$K(G)$ to be $\mathbb{Z}^{|V|-1}/\mathrm{im}(\overline{L(G)}^{v,v'})$ where $\mathrm{im}(\cdot)$ refers to the integer span of the columns of the argument. It is not difficult to see that this definition is independent of the choice of v and v'. The critical group is also known as the Jacobian group and the Picard group as in [3],[1], while in the physics literature it is known as the abelian sandpile group. The matrix-tree theorem of Kirchhoff relates the order of $K(G)$ to the spanning tree number $\tau(G)$.

Theorem 1. *For a loopless graph G we have*

$$\tau(G) = \det(\overline{L(G)}^{v,v'}).$$

Also, if $\lambda_1, \ldots, \lambda_{|V|-1}, 0$ are the eigenvalues of $L(G)$ then

$$\tau(G) = \frac{\lambda_1 \cdots \lambda_{|V|-1}}{|V|}.$$

This is a standard theorem, for its proof see [4, Theorem 5.4]. The first part of the theorem gives the following nice result.

Corollary 2. *The order of the critical group of G is its spanning tree number.*

Proof. The order of the abelian group $\mathbb{Z}^{|V|-1}/\mathrm{im}(\overline{L(G)}^{v,v'})$ is $\det(\overline{L(G)}^{v,v'})$, by Kirchhoff's theorem this is $\tau(G)$. □

We will also take a different approach to the critical group in this paper by defining it in terms of the cycle space of G. To do so we let ∂ be the $|V| \times |E|$ *incidence matrix* of G:

$$\partial_{v,e} = \begin{cases} 1 & \text{if } \ e = v'v, \\ -1 & \text{if } \ e = vv', \\ 0 & \text{otherwise.} \end{cases}$$

We see that ∂ defines a linear transform from the real vector space of real-valued functions on E, $C^0(E)$, to the real vector space of real-valued functions on V, $C^0(V)$. For $\alpha \in C^0(E)$ we see that $(\partial\alpha)(v)$ is, roughly speaking, the net accumulation of α at the vertex v. More precisely

$$(\partial\alpha)(v) = \sum_{e=v'v\in E} \alpha(e) - \sum_{e=vv'\in E} \alpha(e).$$

It is natural to consider the kernel of this map. To do this we consider cycles in G. A *cycle* in G is a list $(v_1, v_2, \ldots, v_k)$, where $v_i \in V$, $v_i = v_j$ if and only if $i = j$ and v_i is adjacent to v_{i+1} when the entries are read modulo $k + 1$. Given a cycle Q in G define its characteristic vector $\zeta_Q : E \to V$ to be the function

$$\zeta_Q(v'v) = \begin{cases} 1 & \text{if the sequence } v', v \text{ occurs in } Q, \\ -1 & \text{if the sequence } v, v' \text{ occurs in } Q, \\ 0 & \text{otherwise.} \end{cases}$$

We will identify edges with their coordinate vectors and the negative of the coordinate vector of $e = vv' \in E$ will be denoted $v'v = -e$. It will be convenient to identify a cycle with its characteristic vector. With this convention it is clear that the characteristic vector of a cycle is the sum of the edges comprising it. We will often pass from lists of edges to sums without mention, as it is a convenient notational device and helps to better understand a given situation.

We claim that $\partial\zeta_Q = 0$. Note that if the vertex v is in a cycle v', v, v'' occurs in the cycle where v, v' and v, v'' are unoriented edges of G. From our above formula for $(\partial\alpha)v$ we get,

$$\zeta_Q(vv') - \zeta_Q(vv'') = 1 - 1.$$

Linear algebra provides the orthogonal decomposition

$$C^0(E) = \ker\partial \oplus (\ker\partial)^\perp$$

where the space $(\ker\partial)^\perp$ is of definite interest as it is the row space of ∂. To see this suppose that $\langle a, \ker\partial\rangle = 0$. If b_v is the vth row of ∂ then supposing that

$$a + \sum_v c_v b_v \neq 0$$

for any choice of the constants c_v gives a contradiction by taking the inner product with $k \in \ker\partial$. The arguement clearly reverses.

If we take a non-empty proper subset U of V and define

$$b_U(v'v) = \begin{cases} 1 & \text{if } v \in U \text{ and } v' \notin U, \\ -1 & \text{if } v' \in U \text{ and } v \notin U, \\ 0 & \text{otherwise} \end{cases}$$

then an argument similar to that for $\zeta_Q \in \ker \partial$ shows that $b_Q \in (\ker \partial)^\perp$. The set of edges having exactly one edge in U is called a *cut* (or sometimes a *bond*) and we see that $b_U(e) \neq 0$ exactly when e is in the cut of U. When $U = v$ we see that b_U is the v-th row of ∂. Hence the space of cuts is spanned by the rows of ∂.

It is a simple exercise to show that the kernel of ∂^T is the set of all functions in $C^0(V)$ that are constant on each connected component of G. This shows that the rank of ∂^T is $|V| - c$ where c is the number of connected component of G. Since the row rank of a matrix is the column rank we get that the rank of ∂ is $|V| - c$. This implies that $\dim(\ker \partial) = |E| - |V| + c$ and hence $\dim((\ker \partial)^\perp) = |V| - c$. For the sake of brevity we define $\beta(G) = |E| - |V| + c$. Fix a spanning forest $T \subset E$ of G. There is a unique path in T connecting the endpoints of any edge $e \in E \setminus T$. If $Q(e)$ denotes the cycle obtained by adding the end points of e to this path then, $\{\zeta_{Q(e)} : e \in E - T\}$ is a basis for $\ker \partial$.

Taking the intersections $\ker \partial \cap \mathbb{Z}^{|E|}$ and $(\ker \partial)^\perp \cap \mathbb{Z}^{|E|}$ yields what are known as the *lattice of integral flows* and *lattice of integral cuts* of G which we denote by $Z(G)$ and $B(G)$ and will refer to more briefly as the *cycle space* and the *bond space* of G. These spaces are, as stated, integer lattices in $\mathbb{R}^{|E|}$ and as such it is natural to consider the lattices dual to them.

Given any lattice $\mathcal{L} \subset \mathbb{R}^N$ we define the *dual lattice*

$$\mathcal{L}^* := \left\{ z \in \mathbb{R}^N : (\forall x \in \mathcal{L}) \, \langle z, x \rangle \in \mathbb{Z}^N \right\}$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^N$. These dual lattices help relate the critical group to the cycle space and bond space of G. To state this relation we will generalize the notion of the critical group in terms of lattices.

Given a lattice $\mathcal{L} \subset \mathbb{R}^n$ consider an integer basis $\{\lambda_1, \dots, \lambda_r,\}$ for $\mathcal{L}$ (which is guaranteed to exist according to the general theory). Letting M be the matrix whose columns are the λ_i we define the *cokernel* of M to be $\operatorname{coker}(M) = \mathbb{Z}^r / \operatorname{im}(M)$. With a bit more thought we deduce the following

Lemma 3. *Let $\mathcal{L} \subset \mathbb{Z}^N$ be a rank r lattice and M be the matrix having rows given by $\{\lambda_1, \dots, \lambda_r\}$, an integer basis for $\mathcal{L}$. Then*

$$\mathcal{L}^* / \mathcal{L} \cong \operatorname{coker}(MM^T).$$

Proof. We have the basis $\{\lambda_1, \dots, \lambda_r,\}$ for $\mathcal{L}$ and claim that an integer basis for $\mathcal{L}^*$ is given by some $\{\lambda_1^*, \dots, \lambda_r^*,\} \subset \mathcal{L}_\mathbb{R}$ where $\langle \lambda_i, \lambda_j^* \rangle = \delta_{ij}$ (here δ_{ij} is the Kronecker delta). For a fixed i we certainly can solve for $\lambda_j^* \in \mathcal{L}_\mathbb{R}$ since $\dim_\mathbb{R} \mathcal{L}_\mathbb{R} = r$. That this is an integer basis for $\mathcal{L}^*$ follows since we have enough vectors and we can pick off the i-th coefficient of $\lambda^* = \sum c_i \lambda_i^*$ by taking the inner product with λ_i which will be an integer when $\lambda^* \in \mathcal{L}^*$.

Since $\lambda_i \in \mathcal{L}^*$ we can write

$$\lambda_i = \sum_{j=1}^{r} c_{ij} \lambda_j^*.$$

Then we see that c_{ij} is given by taking the inner product with λ_j, that is to say, $c_{ij} = \langle \lambda_i, \lambda_j \rangle$ which is the i, j-entry of MM^T.

We now make the natural identification $\mathbb{Z}^r \cong \mathcal{L}^*$ by sending the the j-th standard basis vector to λ_j^*. This sends the integer span of the columns of MM^T isomorphically to $\mathcal{L}$. Taking quotients gives the result. □

To relate our lattices $Z(G)$ and $B(Z)$ to the critical group we consider following basic fact.

Proposition 4. *The incidence matrix and Laplacian of G are related by $L(G) = \partial\partial^T$.*

The proof is left to the reader.

The proposition shows us that $\overline{(L(G))}^{v,v} = \partial_v \partial_v^T$ where ∂_v is the matrix obtained from ∂ by striking out the v-th row. Since the vth row of ∂ is exactly the characteristic vector of the bond associated with v and ∂_v has $n - c$ rows forming a basis for $B(G)$ we have $K(G) \cong B(G)^*/B(G)$ by Lemma 3.

It is a well known result that $Z^*(G)/Z(G) \cong B^*(G)/B(G)$ (see [1, Prop.3]). Hence we have three different approaches to the critical group at our disposal, via the Laplacian, the cycle space, or the bond space. We can add two more approaches since Lemma 3 now gives $K(G)$ as $\operatorname{coker}(M_Z M_Z^T)$ and $\operatorname{coker}(M_B M_B^T)$ where M_Z and M_B are the matrices with lattice basis vectors of $Z(G)$ and $B(G)$ for columns.

In some cases it is reasonable to deal with the Laplacian directly, for instance, when it has a highly symmetric form. If we reduce $L(G)$ to its Smith normal form, that is, the unique diagonal form $\operatorname{diag}(d_1, \ldots, d_{\beta(G)}, 0, \ldots, 0)$ where $d_i | d_{i+1}$, then

$$K(G) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_{\beta(G)}\mathbb{Z}.$$

Taking this approach we find that $K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2}$, which yields a simple proof of Cayley's formula. The d_i are referred to as the *invariant factors* of both $L(G)$ and $K(G)$.

3. The subdivision of a graph

The subdivision of the graph $G = (V, E)$ is the graph obtained by placing a vertex in the center of every edge of G. More formally the *subdivision* of G is the graph $\operatorname{Sd}(G)$ with vertex set $V \cup E$ and edge set

$$E_{\mathrm{Sd}} = \{ve : v \in V, e \in E \text{ and } e = v'v\} \cup \{ev : v \in V, e \in E \text{ and } e = vv'\}.$$

As before we identify edges in E_{Sd} with their coordinate vectors and (e, v) with $-(v, e)$.

We immediately note the natural bijection between cycles in G and those in $\operatorname{Sd}(G)$ and suspect a simple relationship between $K(G)$ and $K(\operatorname{Sd}(G))$. This is indeed the case, for we have

Proposition 5 (Lorenzini [6])**.** *If $K(G)$ has invariant factors $d_1, \ldots, d_{\beta(G)}$ then $K(\operatorname{Sd}(G))$ has invariant factors $2d_1, \ldots, 2d_{\beta(G)}$.*

Proof. As noted, given a lattice basis for $Z(G)$ we may obtain a basis for $Z(\operatorname{Sd}(G))$ by subdividing the basis cycles of $Z(G)$. If M is the matrix with columns given by the basis vectors for $Z(G)$ and N is the same but for $Z(\operatorname{Sd}(G))$, then $MM^T = 2NN^T$ since there are twice as many edges to traverse in $\operatorname{Sd}(G)$. We know that

$K(G) \cong \mathrm{coker}(MM^T)$ and that $K(G) \cong \mathrm{coker}(NN^T)$. There are invertible elementary matrices so that $Q(MM^T)P$ is the diagonal matrix of invariant factors of $K(G)$. Then $Q(MM^T)P = 2Q(NN^T)P$. This gives us the invariant factors of $\mathrm{Sd}(G)$ and indeed they have the desired form. □

Corollary 6. *We have* $\tau(Sd(G)) = 2^{\beta(G)}\tau(G)$.

Proof. This follows directly Proposition 2. We can see this combinatorially. Take a spanning tree of G and subdivide it. Every edge off the spanning tree for G (there are $\beta(G)$ of them) yields two possible edges in $\mathrm{Sd}(G)$ which will extend the subdivided tree to a spanning tree of $\mathrm{Sd}(G)$. It is clear that every spanning tree of $\mathrm{Sd}(G)$ occurs in this way. □

4. Line graphs

Given a graph $G = (V, E)$ we define the *line graph* $\mathrm{line}(G)$ to have vertex set E and $e, e' \in E$ are adjacent in $\mathrm{line}(G)$ if and only if e and e' share an endpoint. Denote the edge set of $\mathrm{line}(G)$ by E_{line}. Every edge in the line graph can be written as a pair of edges $[e, e']$ and as before we will identify $[e, e']$ with its coordinate vector and $[e', e]$ with its additive inverse. We will make the convention that $[\pm e, \pm e']$ (the $\pm$'s vary independently) all describe the same edge $[e, e']$.

In [2] the critical group of $\mathrm{line}(K_{m,n})$ was totally described. This was done via the Laplacian description of the critical group, and the proof consisted of integer row and column operations. In general, given a reasonably complicated graph G understanding the relationship between $K(G)$ and $K(\mathrm{line}(G))$ looks very unpromising. Should we impose the condition that every vertex of the original graph have the same degree, however, things become much simpler.

Proposition 7. *Let G be a d-regular graph. Then the spanning tree numbers of* $\mathrm{line}(G)$ *and* $\mathrm{Sd}(G)$ *are related by*

$$\tau(\mathrm{line}(G)) = d^{\beta(G)-2}\tau(\mathrm{Sd}(G)).$$

Sketch of proof. Consider the unsigned incidence matrix D of G obtained by replacing every entry of ∂ with its absolute value. For d-regular graphs we have the relations

$$\begin{aligned} DD^T &= 2dI - L(G), \\ D^T D &= 2dI - L(\mathrm{line}(G)). \end{aligned}$$

It is a result of linear algebra that DD^T and $D^T D$ have the same non-zero eigenvalues. Applying the second part of Kirchhoff's theorem gives the result. □

Recalling that $\tau(G) = |K(G)|$, Proposition 7 suggests a simple relationship between $K(\mathrm{Sd}(G))$ and $K(\mathrm{line}(G))$ when G is regular. Naively, we hope for the short exact sequence

$$0 \to (\mathbb{Z}/d\mathbb{Z})^{\beta(G)-2} \to K(\mathrm{line}(G)) \to K(\mathrm{Sd}(G)) \to 0$$

This hope is validated in many cases, as was seen by computer trials.

5. The homomorphism from $K(\mathrm{line}(G))$ to $K(\mathrm{Sd}(G))$.

The occurrence of $\beta(G)$ in our relation suggest that we approach $K(G)$ in terms of the cycle space of G. However, before we proceed it is necessary to obtain a more algebraic relationship between G and its line graph.

Recall that every edge $e \in E_{\mathrm{Sd}}$ may be written as an ordered pair comprised of an edge of G and a vertex of G, were the vertex is an endpoint of the edge. Hence every $(e, e') \in E_{\mathrm{line}}$ corresponds in a natural way to exactly two edges of $\mathrm{Sd}(G)$, namely, those two edges of $\mathrm{Sd}(G)$ whose edge component is e and vertex component is the agree. For example, $(vv', v''v) \in E_{\mathrm{line}}$ corresponds to edges (v, vv') and $(v''v, v)$ of $\mathrm{Sd}(G)$. We can extend this identification to the vector space of edges of $\mathrm{line}(G)$ and the vector space of edges of $\mathrm{Sd}(G)$ by stating that if $e \in E_{\mathrm{line}}$ corresponds to ϵ_1 and ϵ_2 in E_{Sd} then $-e$ corresponds to $-\epsilon_1$ and $-\epsilon_2$.

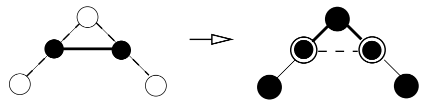

Figure 1. The natural mapping of an edge in $\mathrm{line}(G)$ to its corresponding edges in $\mathrm{Sd}(G)$

We can now make a formal definition.

Definition 8. *Let $f : \mathbb{R}^{|E_{\mathrm{line}}|} \to \mathbb{R}^{|E_{\mathrm{Sd}}|}$ be the linear mapping defined*

$$f((v_1v_2), (v_2v_3)) = (v_1v_2, v_2) + (v_2, v_2v_3).$$

Proposition 9. *The map f restricts to a surjective map $Z(\mathrm{line}(G)) \to Z(\mathrm{Sd}(G))$. When G is regular f restricts to a map $Z^*(\mathrm{line}(G)) \to Z^*(\mathrm{Sd}(G))$ hence we have the induced map $K(\mathrm{line}(G)) \to K(\mathrm{Sd}(G))$.*

When no confusion will arise we will denote all these homomorphisms by f.

Proof. To every vertex in G corresponds a subgraph $K_{d(v)}$ of $\mathrm{line}(G)$ where $d(v)$ is the degree of v. Cycles within this $K_{d(v)}$ will be referred to as *trivial cycles* of G. By definition, every vertex in a trivial cycle is of the form vu where v is fixed an u varies. The reader is invited to check that a trivial cycle is in $\ker f$ by traversing

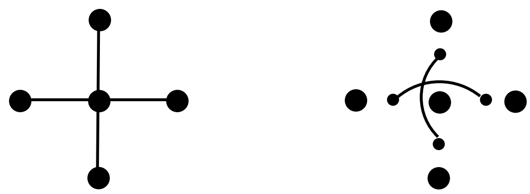

Figure 2. A trivial cycle.

a trivial cycle in $\mathrm{line}(G)$ while simultaneously traversing its image under f. More generally if we have a path that takes place entirely in this $K_{d(v)}$ then we can write it as $\zeta = (vu_1, vu_2) + (vu_2, vu_3) + \cdots + (vu_{k-1}, vu_k)$ so that

$$f(\zeta) = (vu_1, v) + (v, vu_k).$$

We may think of this as saying that the map f only cares where a path in line(G) starts and ends in a given $K_{d(v)}$. Since cycles are closed paths we have shown that f restricts to $Z(\text{line}(G)) \to Z(\text{Sd}(G))$. The reader should convince themself of surjectivity in this case.

To prove the second statement of the proposition we will need the following general result.

Lemma 10. *Suppose that $\phi : \mathbb{R}^m \to \mathbb{R}^{m'}$ and we have the lattices $\mathcal{L} \subset \mathbb{Z}^m$, $\mathcal{L}' \subset \mathbb{Z}^{m'}$. Suppose that ϕ restricts to $\mathcal{L} \to \mathcal{L}'$. Then ϕ restricts to a map $\mathcal{L}^* \to \mathcal{L}'^*$ if and only if $\phi^T : \mathbb{R}^{m'} \to \mathbb{R}^m$ restricts to $\mathcal{L}' \to \mathcal{L}$.*

Proof. We make the observation that $(\mathcal{L}^*)^* = \mathcal{L}$, which follows immediately from the definition. Since $(\phi^T)^T = \phi$ we see that we only need to prove one implication.

Suppose that ϕ^T restricts to $\mathcal{L}' \to \mathcal{L}$. For $\lambda \in \mathcal{L}^*$ we must check that $\langle \phi(\lambda), z \rangle \in \mathbb{Z}$ for every $z \in \mathcal{L}'$. To see this we write

$$\langle \phi(\lambda), z \rangle = \left\langle \lambda, \phi^T(z) \right\rangle \in \mathbb{Z}$$

since $\lambda \in \mathcal{L}^*$ and $\phi^T(z) \in \mathcal{L}$. ☐

We now prove the second restriction of Proposition 9. Suppose that G is d-regular. We have

$$(f^T)(vv', v') = \sum (vv', v'u)$$

where the sum runs over all u which are adjacent to v'. So we take $\xi = (v_1v_2, v_2) + (v_2, v_2v_3)$ for which we get

$$\begin{aligned} (f^T)(\xi) &= \sum (v_1v_2, v_2u) + \sum (v_2u, v_2v_3) \\ &= \hat{\sum}((v_1v_2, v_2u) + (v_2u, v_2v_3) + (v_2v_3, v_1v_2)) - (d-2)(v_1v_2, v_2v_3) \end{aligned}$$

where the hatted sum rums over all u adjacent to v_2 except v_1 and v_3. We now consider a cycle which ξ is on.

$$\xi' = (v_1v_2, v_2) + (v_2, v_2v_3) + (v_2v_3, v_3) + (v_3, v_3v_4) + \cdots + (v_kv_1, v_1) + (v_1, v_1v_2).$$

The regularity hypothesis implies that v_i has $d-2$ neighbors off the cycle and we label these $u_1^i, \ldots, u_{d-2}^i$. Then we can write

$$\begin{aligned} (f^T)(\xi') &= -(d-2)\zeta + \zeta_1 + \cdots + \zeta_{d-2}, \\ \zeta &= (v_1v_2, v_2v_3) + (v_2v_3, v_3v_4) + \cdots + (v_kv_1, v_1v_2) \in Z(\text{line}(G)) \\ \zeta_i &= (v_1v_2, v_2u_i^2) + (v_2u_i^2, v_2v_3) + \ldots + (v_kv_1, v_1u_i^1) + (v_1u_i^1, v_1v_2) \\ &\in Z(\text{line}(G)). \end{aligned}$$

We see that $(f^T)(\xi')$ is an integer combination of cycles in line(G) so that $(f^T)(\xi') \in Z(\text{line}(G))$.

The last statement of the proposition simply follows from $K(G) \cong Z^*(G)/Z(G)$. ☐

It is disappointing to see that the result of Proposition 9 cannot be strengthened: the map $f : Z^*(\text{line}(G)) \to Z^*(\text{Sd}(G))$ is not necessarily a surjective map and appears in many cases not to be.

6. Conjectures

Let G be a d-regular graph. For the investigation of our problem one must resist the temptation to use the case $d = 2$ since then line(G) is isomorphic to G. Hence the simplest examples occur when G is 3-regular. Already this makes things reasonably complicated. We can show that

$$K(\text{line}(K_4)) \cong K(\text{Sd}(K_4)) \oplus \mathbb{Z}/3\mathbb{Z},$$

but to prove this small case requires considerably more effort than it is worth. Instead, we conjecture the following.

Conjecture 11. *For even $n \geq 4$ there is a short exact sequence*

$$0 \to (\mathbb{Z}/(n-1)\mathbb{Z})^{\beta(K_n)-2} \to K(\text{line}(K_n)) \xrightarrow{f} K(\text{Sd}(K_n)) \to 0.$$

For any $n > 4$

$$K(\text{line}(K_n)) \cong (\mathbb{Z}/2n(n-1)\mathbb{Z})^{n-2} \oplus (\mathbb{Z}/2(n-1)\mathbb{Z})^{\beta(K_n)-(n-2)-2} \oplus A_4$$

where $A_4 = (\mathbb{Z}/2\mathbb{Z})^2$ if n is even and $A = \mathbb{Z}/4\mathbb{Z}$ if n is odd.

The method which is used to prove the result for K_4 is the one employed in tackel this problem. The details of a proof are currently being refined. The particularly nice behaivior for even n is though to be a result of the proven fact that f : $Z^*(\text{line}(K_n)) \to Z^*(\text{Sd}(K_n))$ surejects when n is even, but not when n is odd. This general type of result appeared to be typical since many examples done with a computer showed that $K(\text{line}(G))$ was obtained by multiplying d to $\beta(G)-2$ of the invariant factors of $K(\text{Sd}(G))$ and taking the direct sum with A_4. In [2] the critcal group of the line graph of all 3 regular graphs on 10 vertices except two were seen to have the form

$$K(\text{line}(G)) \cong \left(\bigoplus_{i=1}^{r} \mathbb{Z}_{2dd_i} \right) \oplus \mathbb{Z}_{2d}^{\beta(G)-r-2} \oplus A_4,$$

where $d_1, \dots, d_r$ are the non-zero,one invariant factors of $K(G)$. It was conjectured that the complete multipartite graph with parts of equal size (to ensure regularity), had such a form. For the precise conjecture refer to [2]. The determination of A_4 appears to be very sensitive to the structure of the graph and in general one should not hope to determine it. These conjectures suggest the following more general conjecture for an arbitrary d-regular graph.

Conjecture 12. *The kernel of the map $f : K(\text{line}(G)) \to K(\text{Sd}(G))$ is all d-torsion. This is to say, for any $z^* \in Z^*(\text{line}(G))$ such that $f(z^*) \in Z(\text{Sd}(G))$ we have $dz^* \in Z(\text{line}(G))$.*

The answer to all these questions is currently under investigation.

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