Random Walks, Electrical Networks, and Perfect Squares

Patrick John Floryance

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Abstract. This thesis will use Dirichlet's problem and harmonic functions to show a connection between random walks on a graph and electric networks. Additionally, we will show that the probabilities of exiting a graph using a random walk are equivalent to the voltages of the subsequent electrical network modeling the same system. Finally, we show electrical networks with certain properties can be used to model and construct perfect squared squares.

Contents

1. Introduction

This paper is an overview of what the author has learned about the topics of random walks, electrical networks, and perfect squares and the connections between these systems. We discuss how the probabilities of exiting a graph when starting at specific points within the graph are equivalent to the voltages at the corresponding nodes in an electrical network associated to that graph. This holds in both the 1 and 2-dimensional cases and, with appropriate definitions, in general. This is shown using a discrete application of the Dirichlet problem and harmonic functions. The work by Doyle is a useful reference on this topic. Furthermore, we look at how properties of electrical networks can be instrumental in finding perfect squares. This paper uses many principles of circuits including Ohm's Law and Kirchoff's Laws to formulate its conclusions.

2. Random Walks

A random walk is defined as a sequence of moves or steps taken in a finite graph with directional probabilities of steps at each stage. The graph will contain a finite number of vertices and a finite number of edges. Some of these vertices will be interior points in which a walker will leave this point at the next iteration of the walk. The exterior points will be trap points in which, once the walker enters, he will remain there for all future iterations of the walk. From a probability standpoint, we study the probability of reaching one or a set of these trap points given that we start at one of the interior points of the graph.

2.1. **Random Walk in $\mathbb{R}$.** A random walk is sometimes referred to as a drunkard's walk. In this example, we look at the probability of a drunk man reaching home before reaching the bar given that he starts at a certain intersection or interior point. The man will have a given probability of moving in each direction at each vertex. This is a fairly simple problem when we look solely at a linear graph where all interior vertices have degree 2 and the endpoints have degree 1. Figure 1 is the graph of this system.

Here we have the vertices $x = \{1, 2, \cdots, 7\}$ where $x = 1$ represents the bar and $x = 7$ represents his house. We define a function $p(x)$ to be the probability that the man, starting at x, reaches his house before he returns to the bar. In this simple example, there are two disjoint events, H and B, such that $A \cap B = \emptyset$ and $A \cup B = S$ where H is the event that the man moves toward his house, and B is the event that he moves toward the bar. In a completely random model, these events have probability $P(H) = 1/2$ and $P(B) = 1/2$.

Proposition 2.1.

$$p(x) = \begin{cases} 0 & \textit{if } x = 1 \\ 1 & \textit{if } x = 7 \\ \frac{p(x-1)}{2} + \frac{p(x+1)}{2} & \textit{if } x = \{2,3,4,5,6\} \end{cases} \tag{2.1}$$

Proof. That $p(1) = 0$ and $p(7) = 1$ follow from the definition of this random walk. These points are called trap points. Once the walker

FIGURE 1

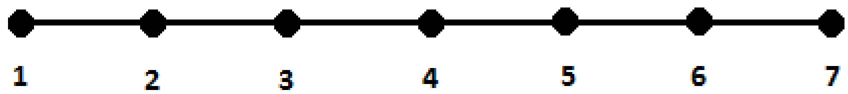
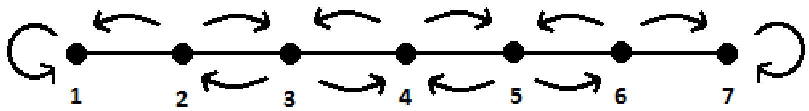


FIGURE 2



enters a trap point, they remain there indefinitely. So if the walker is at $x = 1$, it is impossible for him to reach $x = 7$ because he is stuck at the bar. Figure 2 demonstrates the dynamics of the system.

It also follows that $p(x) = \frac{p(x-1)}{2} + \frac{p(x+1)}{2}$ from basic probability theory, as we explain now. We have two disjoint events H and B where $H \cup B = S$ where S is our entire sample space. Probability theory states that the probability of an event, E, is defined as

$$P(E) = P(B)P(E|B) + P(H)P(E|H) \text{ [3].} \tag{2.2}$$

In our case, E is defined to be the event of reaching home before reaching the bar, $P(B) = P(H) = 1/2$, $P(E|B) = p(x-1)$, and $P(E|H) = p(x+1)$. □

Now we can solve for $p(x)$. A simple way to solve is by linear algebra. Equation 2.3 is of the matrix form shown here:

$$\begin{bmatrix} -1 & 1/2 & 0 & 0 & 0 \\ 1/2 & -1 & 1/2 & 0 & 0 \\ 0 & 1/2 & -1 & 1/2 & 0 \\ 0 & 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 0 & 1/2 & -1 \end{bmatrix} \begin{bmatrix} \mathrm{p}(2) \\ \mathrm{p}(3) \\ \mathrm{p}(4) \\ \mathrm{p}(5) \\ \mathrm{p}(6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1/2 \end{bmatrix} \tag{2.3}$$

Therefore,

$$[p(X)] = \begin{bmatrix} p(2) \\ p(3) \\ p(4) \\ p(5) \\ p(6) \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \\ 2/3 \\ 5/6 \end{bmatrix} \tag{2.4}$$

For the linear case of random walk, we can actually generalize our equation. A linear graph on $n = N$ vertices will have the probability function

$$p(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = N \\ \frac{p(x-1)}{2} + \frac{p(x+1)}{2} & \text{if } x = \{1, 2, \cdots, N-1\}. \end{cases} \tag{2.5}$$

The function $p(x)$ is a harmonic function in which all interior points satisfy $p(x) = \frac{p(x-1)+p(x+1)}{2}$. The fact that $p(x)$ is harmonic will be important later.

So far we have assumed that the probability of the drunkard moving toward the bar and his house was equivalent, i.e. with probability 1/2. What if this wasn't the case? What if the man had a bias to moving toward his house or the bar? To keep with the analogy, what if he wasn't that drunk and therefore headed toward home two-thirds of the time? How would we change our formula $p(x)$ to take this into account? If we go back to equation 2.2, we would have to change the values $P(B)$ and $P(H)$ to our new probabilities of turning left and turning right respectively. Also, the values of $P(E|B)$ and $P(E|H)$ would need to be adjusted to satisfy our new values of $P(B)$ and $P(H)$. However, it is much simpler to rewrite $p(x)$ as

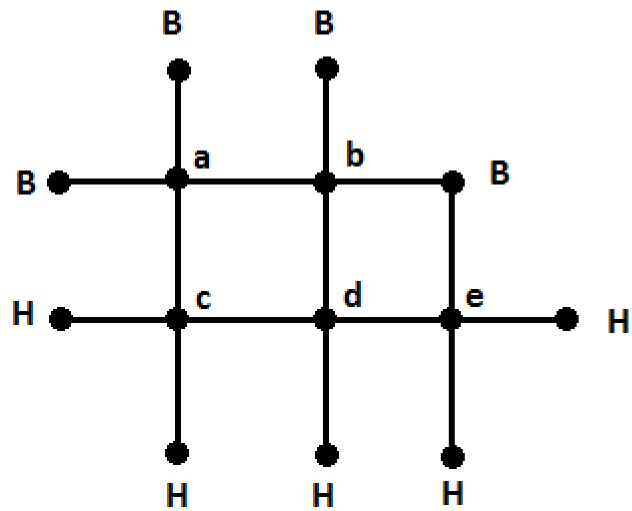
$$p(x) = P(B)p(x-1) + P(H)p(x+1). \tag{2.6}$$

This follows from the same reasoning as before.

2.2. **Random Walk in $\mathbb{R}^2$.** Like the random walk in $\mathbb{R}$, a random walk in $\mathbb{R}^2$ is based on probability theory. Now, the walk gets laid out in a grid or city street system. We must change our notation a little to accommodate for the added dimension. For all interior points $x = (a, b)$, there exists four neighboring points that the walker can move to. These are $(a, b-1)$, $(a, b+1)$, $(a-1, b)$, and $(a+1, b)$. The probability of moving in each direction is 1/4 if the walker chooses a direction at random with no bias. Therefore, we derive our function

$$p(a, b) = \frac{p(a, b-1) + p(a, b+1) + p(a-1, b) + p(a+1, b)}{4} \tag{2.7}$$

FIGURE 3



for all interior points of our system.

In the 2-dimensional case, we will have more than our 2 exterior (trap) points in our system. Some of these points we will define to be bars, and the others will be defined as houses. We can choose which points we wish to be houses and which will be bars in any way that we want. In general, we will solve for the probability of reaching a subset of exterior points before reaching the complementary set of exterior points, and we can choose which points are in each set. We will now use an example to demonstrate this.

Example 2.2. Let's use the graph shown in Figure 3. When we reach a point labeled B or H we finish our random walk because these are trap points. Now let $p(x) = 0$ if $x \in \{B\}$ and $p(x) = 1$ if $x \in \{H\}$. Now we can solve for the probability of the interior points by using linear algebra again. Answers are rounded to the nearest thousandth.

$$\begin{bmatrix} -1 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & -1 & 0 & 1/4 & 0 \\ 1/4 & 0 & -1 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & -1 & 1/4 \\ 0 & 0 & 0 & 1/4 & -1 \end{bmatrix} \begin{bmatrix} p(x_a) \\ p(x_b) \\ p(x_c) \\ p(x_d) \\ p(x_e) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ -1/4 \\ -1/2 \end{bmatrix} \tag{2.8}$$

Therefore,

$$[p(X)] = \begin{bmatrix} p(x_a) \\ p(x_b) \\ p(x_c) \\ p(x_d) \\ p(x_e) \end{bmatrix} = \begin{bmatrix} .236 \\ .223 \\ .722 \\ .652 \\ .663 \end{bmatrix} \tag{2.9}$$

Like with the 1-dimensional random walk, the probability function $p(a, b)$ for a 2-dimensional random walk satisfies the properties of a harmonic function. Here, it satisfies that all interior points follow the equation $p(a, b) = \frac{p(a,b-1)+p(a,b+1)+p(a-1,b)+p(a+1,b)}{4}$. Again, we will discuss the importance of this later on in Section 4.

Again, we can look at what would happen if there was a bias in the direction the walker chooses. Now, adjust $p(a, b)$ accordingly to account for the bias. Define the event $(b - 1)$ as the event of a motion in the direction of vertex $x = (a, b-1)$ from vertex $x = (a, b)$. Similarly, define events $(a-1)$, $(a+1)$, and $(b+1)$ in the same fashion. A motion in each direction would be defined to have probability $P(a - 1), P(a + 1), P(b - 1)$, and $P(b + 1)$ such that

$$P(a - 1) + P(a + 1) + P(b - 1) + P(b + 1) = 1,$$

and the probability of each event in greater than or equal to 0. This results in our revised function

$$\begin{aligned} p(a, b) = {} & P(b - 1)p(a, b - 1) + P(b + 1)p(a, b + 1) \\ & + P(a - 1)p(a - 1, b) + P(a + 1)p(a + 1, b) \end{aligned} \tag{2.10}$$

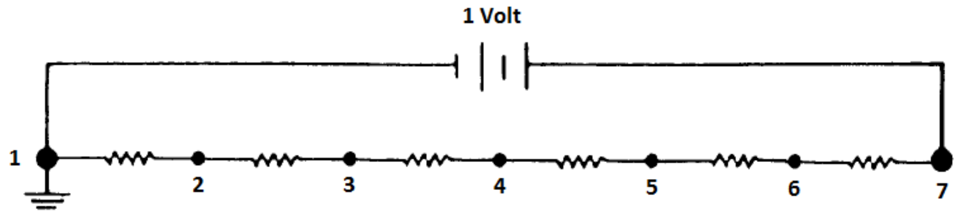
3. Corresponding Electrical Networks

The networks we will be using are circuits consisting solely of resistors. The first set of circuits will be linear circuits where the resistors will be in series. The second type will be circuits laid out in a grid. For convenience, the circuits we will be using will be systems identical to the graphs we used in sections 2.1 and 2.2 when we were discussing random walks. By doing so, we hope to illuminate the connection between the two systems.

3.1. **Resistors in Series.** Electrical circuits follow laws just like any aspect of physics. The laws we need to apply are Ohm's Law and Kirchhoff's Laws. Ohm's Law states that the change in voltage for a fixed current across a resistor is proportional to resistance. The formula for the change of voltage is

$$\Delta V = RI \iff R = \frac{\Delta V}{I} \iff I = \frac{\Delta V}{R} \tag{3.1}$$

FIGURE 4



where V is voltage, R is resistance, and I is current [5]. Ohm's Law gives us the relationship between resistance, voltage, and current.

Kirchhoff's Laws give rules to how the current and voltage interact in a circuit. Kirchhoff's Laws state:

- The sum of currents entering any junction is equal to the sum of the currents leaving the junction, and
- The sum of the potential difference across each element around any closed circuit must be 0 [5].

We will be looking to identify the voltage at junction points in between resistors. As mentioned earlier, our example circuit will mirror the earlier random walk. Let $v(x)$ be the voltage at junction x. Our circuit will have 7 junction points that are between 6 resistors of equal resistance as seen in Figure 4. There will be a 1 volt power supply going in junction 7 and leaving at junction 1. Junction 1 is also grounded. Therefore, $v(1) = 0$ and $v(7) = 1$.

Now we need to determine the voltage for the other points in the circuit. By Ohm's Law we can conclude that

$$i_{ab} = \frac{v(a) - v(b)}{R}. \tag{3.2}$$

Then we can apply Kirchhoff's Law to get

$$\frac{v(a-1) - v(a)}{R} = \frac{v(a) - v(a+1)}{R}$$

$$\frac{v(a-1) + v(a+1) - 2v(a)}{R} = 0$$

$$v(a) = \frac{v(a-1) + v(a+1)}{2} \tag{3.3}$$

Now we can solve for our voltages at all of our junction points. We will do this again by Linear Algebra.

$$\begin{bmatrix} 1 & -1/2 & 0 & 0 & 0 \\ -1/2 & 1 & -1/2 & 0 & 0 \\ 0 & -1/2 & 1 & -1/2 & 0 \\ 0 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 0 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} v(2) \\ v(3) \\ v(4) \\ v(5) \\ v(6) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

$$[v(X)] = \begin{bmatrix} v(2) \\ v(3) \\ v(4) \\ v(5) \\ v(6) \end{bmatrix} = \begin{bmatrix} 1/6 \\ 1/3 \\ 1/2 \\ 2/3 \\ 5/6 \end{bmatrix} \tag{3.4}$$

If we look back to equation 2.4 from the first section, we notice that $[p(X)]$=$[v(X)]$. Similarly, $v(x)$ is a harmonic function just like $p(x)$. The implications of this will be discussed later.

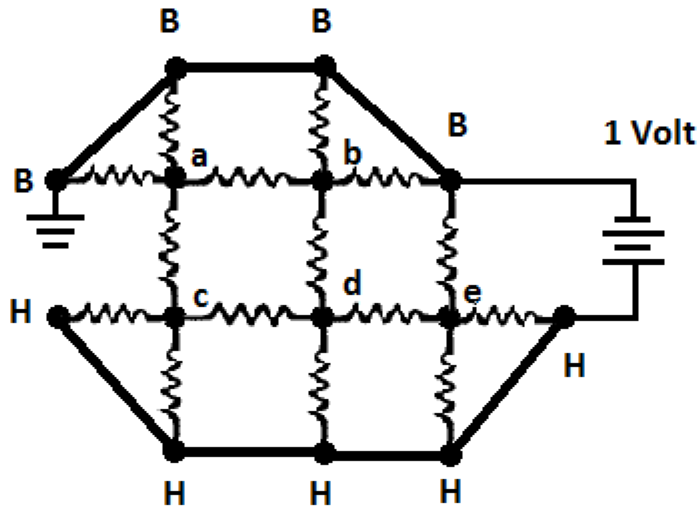
3.2. **Uniform Resistors in a Grid.** Now we will take the electrical network to be a 2-dimensional grid. The example we will be working with will parallel the 2-dimensional random walk problem from earlier. In this electrical network, we will have an inflow of 1V going into our houses, and the bars will be grounded. Since there are no resistors along the wires between the junction points connecting the bars to bars and houses to houses, there will theoretically be no drop in voltage between these points. However, in reality even a wire has some resistance, but it is usually negligible on small scales.

Now we derive equations for $v(x)$. Since the power supply is producing 1V of electrical potential, $v(x) = 1$, for all $x \in H$ where H is the set of boundary points labeled H in figure 5. Similarly, $v(x) = 0$, for all $x \in B$ where B is the set of points labeled as B in figure 5 that are grounded in our network. The hard part is to determine the voltages for the interior points of our system.

All of the resistors in the electrical network have the same resistance $R = 1\Omega$ in this example. Again, Ohm's Law and Kirchhoff's Law are essential to the derivation of our equation $v(x)$. From equation 2.2 and the application of Kirchhoff, we get that

$$\frac{v(a-1,b)-v(a,b)}{R} + \frac{v(a+1,b)-v(a,b)}{R} + \frac{v(a,b-1)-v(a,b)}{R} + \frac{v(a,b+1)-v(a,b)}{R} = 0,$$

FIGURE 5



and it follows that

$$v(a,b) = \frac{v(a-1,b) + v(a+1,b) + v(a,b-1) + v(a,b+1)}{4}. \tag{3.5}$$

Note that $v(a,b)$ is a harmonic function in two dimensions as defined in section 2.2. Now that we have equations for our function $v(x)$, we can begin to solve for voltage at our interior points using linear algebra.

$$\begin{bmatrix} -1 & 1/4 & 1/4 & 0 & 0 \\ 1/4 & -1 & 0 & 1/4 & 0 \\ 1/4 & 0 & -1 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & -1 & 1/4 \\ 0 & 0 & 0 & 1/4 & -1 \end{bmatrix} \begin{bmatrix} v(x_a) \\ v(x_b) \\ v(x_c) \\ v(x_d) \\ v(x_e) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1/2 \\ -1/4 \\ -1/2 \end{bmatrix} \tag{3.6}$$

Therefore, we see that our values for $[v(X)]$ are equal to the values of $[p(X)]$ from section 2.2.

$$[v(X)] = \begin{bmatrix} v(x_a) \\ v(x_b) \\ v(x_c) \\ v(x_d) \\ v(x_e) \end{bmatrix} = \begin{bmatrix} .236 \\ .223 \\ .722 \\ .652 \\ .663 \end{bmatrix} \tag{3.7}$$

4. Harmonic Functions and The Dirichlet Problem

So far we have looked at random walks and electrical networks independently. In fact, they are one and the same. This section will go on to prove that electrical networks and random walks are the same by using the uniqueness principle of the Dirichlet problem and the fact that our functions $p(x)$ and $v(x)$ are harmonic functions.

Theorem 4.1. *(Dirichlet Problem) A harmonic function is determined by the values it takes on the boundary* [3].

This theorem implies that by solely knowing the values and positions of the boundary points, there exists only one solution as to what the interior points can take given the function is harmonic. Hence, any two functions having the same boundary values and positions must be identical. We saw how this was true with our examples of the random walks and electrical networks in the previous sections. For our example, we will be using a discrete form of the Dirichlet problem instead of the original continuous version.

In order to use Theorem 4.1, we must show uniqueness and existence. Existence of a solution is reasonable in our physical case. A solution exists when the function $f(x)$ is regular in a domain with a sufficiently smooth boundary [7]. We will focus primarily on the uniqueness of the solution. To do this, we need the Maximum principle.

Theorem 4.2. *(Maximum Principle; see* [3, pp.7]*) A harmonic function f(x) defined on S takes on its maximum value M and its minimum value m on the boundary.*

Proof. (Theorem 4.2: 1-Dimensional Case) Assume that M is the maximum value of a harmonic function $f(x) = \frac{f(x-1)+f(x+1)}{2}$. Therefore, there exists an $x \in S$ such that $f(x) = M$. If x is a boundary point, then the statement in trivially true. If x is not a boundary point, then $f(x) = M$ and $f(x-1) = f(x+1) = M$ since M is maximal on S. By similar logic, $f(x-2) = M$. This process can be repeated until $f(0) = M$ at the boundary point $x = 0$. By similar argument, we can prove that minimum value m is located on the boundary. □

Theorem 4.3. *(Uniqueness Principle of Dirichlet; see* [3, pp.7]*) If $f(x)$ and $g(x)$ are harmonic functions of S such that $f(x) = g(x)$ on the boundary, then $f(x) = g(x)$.*

Proof. Let $h(x) = f(x) - g(x)$.Then $h(x)$ is a harmonic function, and $h(x) = \frac{h(x-1)+h(x+1)}{2}$, since

$$\frac{h(x-1)+h(x+1)}{2} = \frac{f(x-1)+f(x+1)}{2} - \frac{g(x-1)+g(x+1)}{2}.$$

Since $f(x) = g(x)$ when x is a boundary point, $h(x) = 0$ for all x on the boundary. By Theorem 4.2, $h(x) = 0$ for all $x \in S$. □

Therefore since our functions $p(x)$ and $v(x)$ took on the same values at the boundary points, we can conclude that $p(x) = v(x)$. This can also be shown for the 2-dimensional examples as well.

Proof. (Theorem 4.2: 2-Dimensional Case) Let M be the maximal value on S. Let $f(a,b) = M$. If $x = (a,b)$ is located on the boundary, then this is trivially true. Assume not. Then $f(a-1,b) = f(a+1,b) = f(a,b-1) = f(a,b+1) = M$ because M is maximal on S. Similarly, $f(a-2,b) = M$. In this fashion $f(0,b) = M$, and $x = (0,b)$ is located on the boundary. Therefore the maximal value of $f(a,b)$ occurs on the boundary. □

We can prove Theorem 4.3 by a similar argument as before by letting $h(a,b) = f(a,b) - g(a,b)$ such that $h(a,b)$ is the harmonic function

$$h(x) = \frac{h(a-1,b) + h(a+1,b) + h(a,b-1) + h(a,b+1)}{4}.$$

The same argument holds for the 2-dimensional case. Therefore, we conclude that $p(a,b) = v(a,b)$ from the 2-dimensional random walk and electrical network. This completes the argument that computing the probabilities in a random walk and the voltages in an electrical network are the same problem.

5. General Electrical Networks

This section section will focus on general circuits involving non-uniform resistors set in parallel and series. We will show how to calculate voltages in these specific systems using properties of these circuits. These electrical networks are more complex than the networks mentioned previously so the calculations of voltages will be more difficult, but these networks still follow Ohm's and Kirchhoff's laws in their behavior. We now introduce a couple of definitions and theorems necessary to our calculations.

Definition 5.1. Equivalent resistance is the equivalent single resistance that has the same effect on the circuit i.e. it does not affect the current of the circuit [5].

Theorem 5.2. *Equivalent resistance is equal to the sum of the resistances if the resistors are in series.* $R_{eq} = R_1 + R_2 + \cdots + R_n$.

Definition 5.3. Conductance is the inverse of resistance. $C_i = 1/R_i$

Definition 5.4. Equivalent conductance is the equivalent single conductance that has the same effect on the circuit i.e. it does not affect the current of the circuit [5].

Figure 6. Resistance

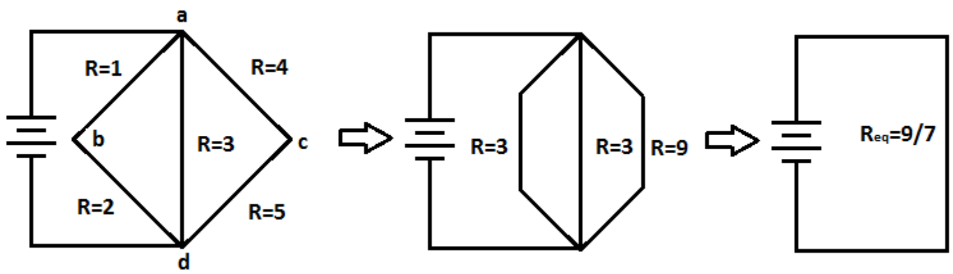
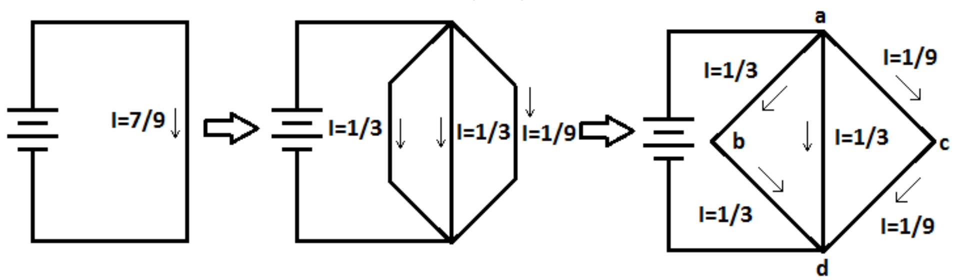


Figure 7. Current



Theorem 5.5. *Equivalent conductance is equal to the sum of the conductances if the resistors are in parallel.* $C_{eq} = C_1 + C_2 + \cdots + C_n$.

These two formulas are key in simplifying a complex network into a simpler network by replacing multiple resistors with one or more equivalent resistors such that the current of the entire circuit is unaffected.

Example 5.6. Illustrated by Figure 6, we show how to use formulas in Theorems 5.2 and 5.5.

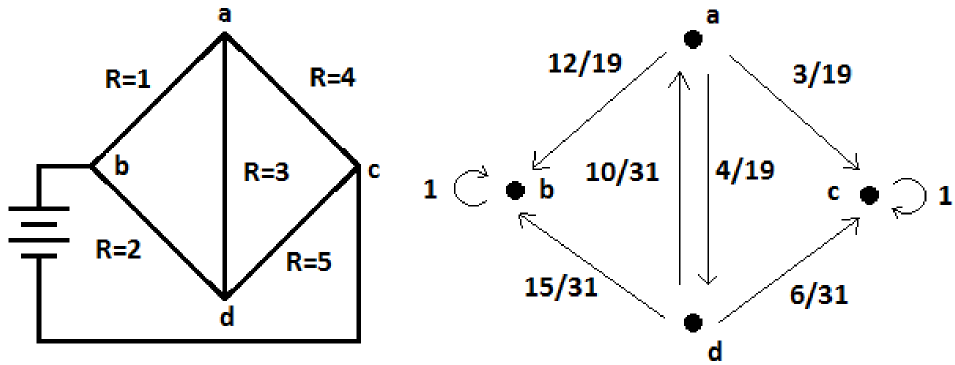
$$\begin{aligned} &R_{abd} = R_1 + R_2 = 1 + 2 = 3 \\ &R_{acd} = R_4 + R_5 = 4 + 5 = 9 \\ &C_{eq} = C_{abd} + C_{ad} + C_{acd} = 1/3 + 1/3 + 1/9 = 7/9 \\ &\Rightarrow R_{eq} = 9/7 \end{aligned}$$

Using R_{eq}, apply Ohm's Law to determine the current flowing through the circuit. Assume that the power supply is 1V. Thus, $I = 1/(9/7) = 7/9$. Therefore the current entering and exiting the network is equal to 7/9. The next step is to calculate the current through each resistor.

$I_{abd} = 1/R_{abd} = 1/3, I_{ad} = 1/R_{ad} = 1/3, I_{acd} = 1/R_{acd} = 1/3.$

$I_{ab} = I_{bd} = 1/3$ and $I_{ac} = I_{cd} = 1/3$ by Kirchhoff's Law.

FIGURE 8



This is diagrammed in Figure 7. Now the currents and resistances are known which leaves only the voltages as unknowns. Again, this is calculated by Ohm's Law. Having done most of the calculations already, it is easy to see that $v(a) = 1$, $v(d) = 0$, $v(b) = 2/3$, and $v(c) = 5/9$.

Example 5.7. This can also be viewed from a probabilistic frame work involving a random walk. We will, however, change our network such that the 1V is entering at junction b and grounded at junction c. Figure 8 is the diagram of our new electrical network and the probabilistic directional graph of this network.

We can apply our knowledge of probability to determine the voltages using formula (1.2). Let E be the event of reaching c before reaching b. There are now more than 2 other events that can occur but these events are disjoint and their probabilities sum to 1 so the formula is still valid. We determine the probability at each node by solving the linear system

$$\begin{cases} p_1 + p_2 + \cdots + p_n = 1 \\ p_1 R_1 = p_2 R_2 \\ \vdots \\ p_1 R_1 = p_n R_n \end{cases}$$

where p_i is the probability of taking the path across resistor R_i starting at a given node. For this example, the linear system will be:

$$\text{from node } a \begin{cases} p_b + p_c + p_d = 1 \\ p_b = 4p_c \\ p_b = 3p_d \end{cases}$$

$$\text{and from node } d \begin{cases} p_a + p_b + p_c = 1 \\ 3p_a = 2p_b \\ 3p_a = 5p_c \end{cases}$$

Solving these systems of linear equations produces the values seen in Figure 8. From there, plug the respective p_i into our voltage equation and solve. For our example, the system of equations become

$$\begin{cases} v(b) = 1 \\ v(c) = 0 \\ v(a) = \frac{12}{19}v(b) + \frac{4}{19}v(d) + \frac{3}{19}v(c) \\ v(d) = \frac{15}{31}v(b) + \frac{10}{31}v(a) + \frac{6}{31}v(c) \end{cases}$$

Solving this system implies $v(a) = 48/61$ and $v(d) = 45/61$ which is equivalent to the probabilities of reaching b before reaching c given that a random walk in this network is started at the corresponding nodes.

6. Perfect Squares

Perfect squared squares and perfect squared rectangles are squares and rectangles that can be divided into two or more distinct perfect square regions such that

(1) these square regions touch at only the edges,
(2) their union completely covers the large square or rectangle, and
(3) no two squares have the same side lengths.

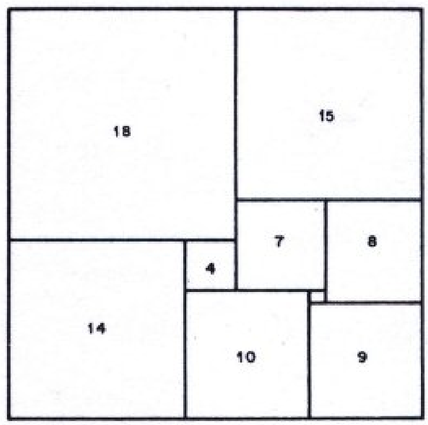
It can be a difficult process to find a square or rectangle in which all three properties hold.

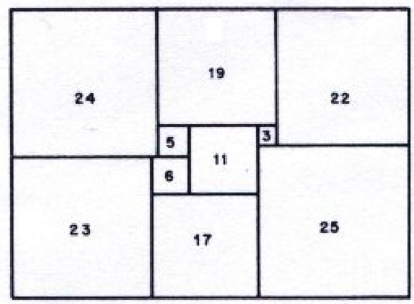
Perfect rectangles are easier to find than perfect squares. The first perfect rectangle was discovered by Zbigniew Moroń in his 1925 paper "O Rozkladach Prostokatow Na Kwadraty" meaning "On the Dissection of a Rectangle into Squares." The first rectangle he presented was a 32x33 rectangle subdivided into 9 distinct squares. In the same paper, he also showed the dissection of a 65x47 rectangle into 10 distinct squares [4]. Though he showed that it was possible to to dissect these rectangles into squares, he never explained his process of doing so. These can be seen in Figure 9.

Perfect squares, being more difficult, were originally thought to be impossible. This was proven wrong in 1939 by Roland Sprague who discovered a perfect square of order 55 [6]. That means the square was subdivided into 55 smaller squares.

So what does this have to do with random walks and electrical networks? Let's try to model Moroń's second perfect rectangle of order

FIGURE 9. Moroń's Perfect Square Rectangles





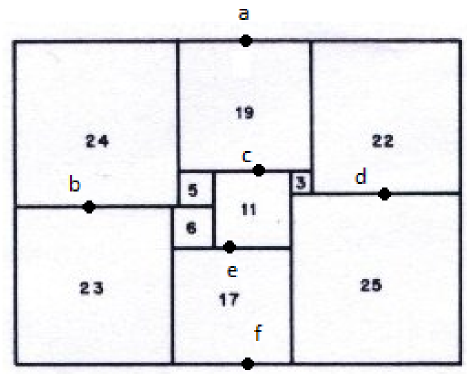
10. Start by taking a piece of sheet metal with low conductivity cut with dimensions 47x65 to match the rectangle. Place a rod with high conductivity at every horizontal edge of the the sub-squares. Connect a power supply of 47V to the rod at the top of the rectangle and ground the rod at the bottom of the rectangle such that the voltage would be 0V. At every vertical edge of the sub-squares, make a cut such that the squares do not touch and therefore do not pass current [2].

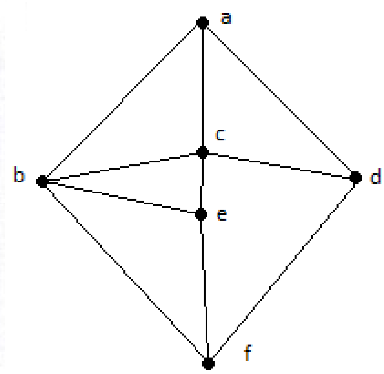
Proposition 6.1. *The resistance of a rectangle is proportional to the length and width of the rectangle*

Proof. A sheet of metal has dimensions length (L), width (W), and thickness (T). The metal will have some coefficient of resistivity ($\rho = \Omega$/meter). The current in a sheet flows along the length of the sheet not perpendicular to it like a river flows along its length and not its width. The current must flow through an area that is equivalent to $W \cdot T$. From Equation 3.1, $R = \frac{\Delta V}{I}$. The current is proportional to the cross-sectional area, and the change in voltage is proportional to the resistivity times the length. Thus, $R \propto \rho L/TW$ and $R \propto L/W$. (In actuality, $R = \rho L/TW$ with ρ/T defined as the sheet resistance [1].) □

Proposition 6.1 implies that every square section has the same resistance because the length and width are equivalent in a square. The equivalent resistance of the system will therefore be the ratio of height to width of the large rectangle. In our case, $R_{eq} = 47/65$. The voltage at any point of this system is equivalent to the distance off the bottom to the point because the current is uniformly traveling from the 47V

FIGURE 10





at the top to the 0V at the bottom. Using these properties, it becomes simple to create an electrical network for a system like this.

Figure 10 outlines how to construct the electrical network from the perfect rectangle. In this system we obtain voltages $v(a) = 47$, $v(b) = 23$, $v(c) = 28$, $v(d) = 25$, $v(e) = 17$, and $v(f) = 0$. One thing to note, the perfect squared rectangle had order 10. The electrical network corresponding to the perfect rectangle has 10 edges. This will always be the case that the order of the perfect rectangle will be equal to the number of edges in the electrical network.

Now that we have seen how to construct an electrical network from a perfect squared rectangle, we can reverse the process in order to find perfect squares. It is already known that perfect squared squares exist by the work of Sprague.

The first thing we can do is take a connected planar graph. Turn this graph into an electrical network by assigning resistance value of 1Ω to each edge of the network. Then calculate the equivalent resistance of the network. If this is equal to 1, then it is possible that the network corresponds to a squared square. If the change in voltage through every resistor is different, then each square that corresponds to that resistor/edge is of a different size. If all of these properties hold, then we have found a perfect squared square [2].

Although the process of finding perfect squared squares is not straightforward, it is a way of finding them. The process involves much guessing and checking. Therefore, it is very time consuming. Computers are commonly used to perform the computations of these systems as a time saving process.

7. Concluding Remarks

This paper began by addressing random walks in a graph of dimensions 1 and 2. Then, it discussed the physics of electrical networks and drew a parallel to random walks. It proved that random walks are linked to electrical networks by applying the Dirichlet problem of harmonic functions. Finally, it touched on perfect squared squares and rectangles and their connection to electrical networks. Hopefully, the reader can see how all these topics are interconnected; how when we talk about electrical networks, we might as well be talking about random walks.

This paper went into the beginnings of these topics and introduced the concepts of interconnectedness. More research is being conducted, and interesting conclusions have resulted. One of these results is Polya's recurrence problem involving infinite random walks on n-dimensional lattices of infinite size. Hopefully, this paper is intriguing enough to spark some interest in random walks and electrical networks, so that you, the reader, will continue to advance your knowledge of the subject.

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