Karlsruher Institut für Technologie

Master Thesis

Poincaré series for Finite and Affine Coxeter Groups

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Statement of Authorship

I hereby declare that this document has been composed by myself and describes my own work, unless otherwise acknowledged in the text.

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1 Introduction

Given a Coxeter system (W, S), one can define the *Poincaré series* $W(q) = \sum_{w \in W} q^{\ell(w)}$. It measures the growth of the Coxeter group W if we look at all elements shorter than an increasing length. Combinatorial methods can be used to calculate the Poincaré series (see [2, chapter 7]).

On the other hand, we can consider the following algebraic-geometric object: We examine the geometrical action of a finite Coxeter group W on its Coxeter complex. This action gives rise to an action on the symmetric algebra $\mathrm{Sym}(V^*)$. Analysing the set of invariant elements S^W, one can define the *Hilbert series*: $\mathrm{Hilb}(S^W, q) = \sum_{d \geq 0} \dim(S^W_d) q^d$.

The following theorems relate the combinatorial and the geometric series:

Theorem 4.1. *Let W be a finite Coxeter group. Then*

$$W(q) = \prod_{i=1}^{n} (1 + q + q^2 + \ldots + q^{d_i - 1}) = \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q} = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(S^W, q)}$$

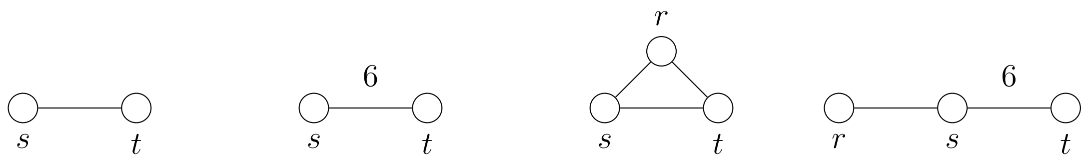
Theorem 5.1. *Let $\tilde{W}$ be an affine Coxeter group with associated finite irreducible Coxeter group W. Then*

$$\tilde{W}(q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q^{d_i - 1}} = \mathrm{Hilb}(S, q) \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q^{d_i - 1}} = W(q) \prod_{i=1}^{n} \frac{1}{1 - q^{d_i - 1}}$$

The first theorem dates back to Claude Chevalley, who showed it in the context of Lie groups. Louis Solomon gave a uniform proof for all finite Coxeter groups in [9]. The second theorem was proven by Raoul Bott [1]. Richard Steinberg [10] provided an alternative proof with combinatorial arguments, based on a variant of Solomon's proof.

The content and structure of this thesis is based on the notes of Victor Reiner[6], which feature the proof of Solomon and Steinberg. Several examples have been added and the proofs have been deepend on several spots. Some images have been added to illustrate the proofs.

In the beginning of Section 2, the most important definitions are stated and we gather some facts on Coxeter groups and the Coxeter complex. In Section 3, we derive a recursive formula for the Poincaré series which will be useful for the finite and the affine case. Then, in Section 4, we develop a similar formula for the Hilbert series. A comparison of both formulae shows Theorem 4.1. This is done by deducing a relationship between the combinatorial structure of the Coxeter complex and its homology with the Euler-Poincaré principle (Theorem 4.5). Finally, we apply the Euler-Poincaré principle to the affine situation in Section 5. We analyse the exterior algebra and its invariants and use the tools of Section 4 to derive Theorem 5.1.



(a) A_2 (b) G_2 (c) $\tilde{A}_2$ (d) $\tilde{G}_2$

Figure 1: The Coxeter diagram of...

2 Preliminaries

2.1 Coxeter Groups

Definition 2.1. Let $(m_{ij}) = M \in (\mathbb{Z} \cup \{\infty\})^{n \times n}$ be a symmetrical matrix with $m_{ii} = 1$ for all i and $m_{ij} \geq 2$ for $i \neq j$. We call such a matrix a *Coxeter matrix.* Then the group

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

is called the associated *Coxeter group* to M, where $(s_i s_j)^\infty = 1$ means that there is no relation between the generators. We denote the generating set by $S = \{s_1, \ldots, s_n\}$ and call (W, S) a *Coxeter system.* We say that W is of rank n, where n is the number of generators.

A *reflection* of W is an element wsw^{-1} for $s \in S$ and $w \in W$. In particular, the elements of the generating set S are reflections. We call those *simple reflections.*

Often, one denotes a given Coxeter group W by its *Coxeter diagram* Γ_W. This is a graph where every generator is represented by a vertex. Two generators s_i and s_j are connected, if $m_{ij} \geq 3$ and the connecting edge is labeled with m_{ij} if $m_{ij} \geq 4$.

A very important class of Coxeter groups are the so-called *spherical Coxeter groups*, which are exactly the finite Coxeter groups. Every spherical Coxeter group can be represented as a reflection group on an $(n-1)$-sphere.

Example 2.2. The Coxeter diagram in Fig. 1a corresponds to the Coxeter group

$$A_2 = \langle s, t \mid s^2 = t^2 = ststst = 1 \rangle$$

A_2 is finite. It acts on S^1 via reflections on two lines through the origin, with an angle of 60° (see Fig. 2).

Example 2.3. The Coxeter diagram in Fig. 1b corresponds to the Coxeter group

$$G_2 = \langle s, t \mid s^2 = t^2 = (st)^6 = 1 \rangle$$

G_2 is finite. It acts on S^1 via reflections on two lines through the origin, with an angle of 30° (see Fig. 2).

The so-called parabolic subgroups of a Coxeter group play an important role in the study of Coxeter groups:

Definition 2.4. Let $J \subseteq S$. Then the subgroup $W_J := \langle J \rangle$ is called a *(standard) parabolic subgroup* of W. It is itself a Coxeter group and its Coxeter graph is obtained by deleting all vertices of Γ_W which do not belong to J.

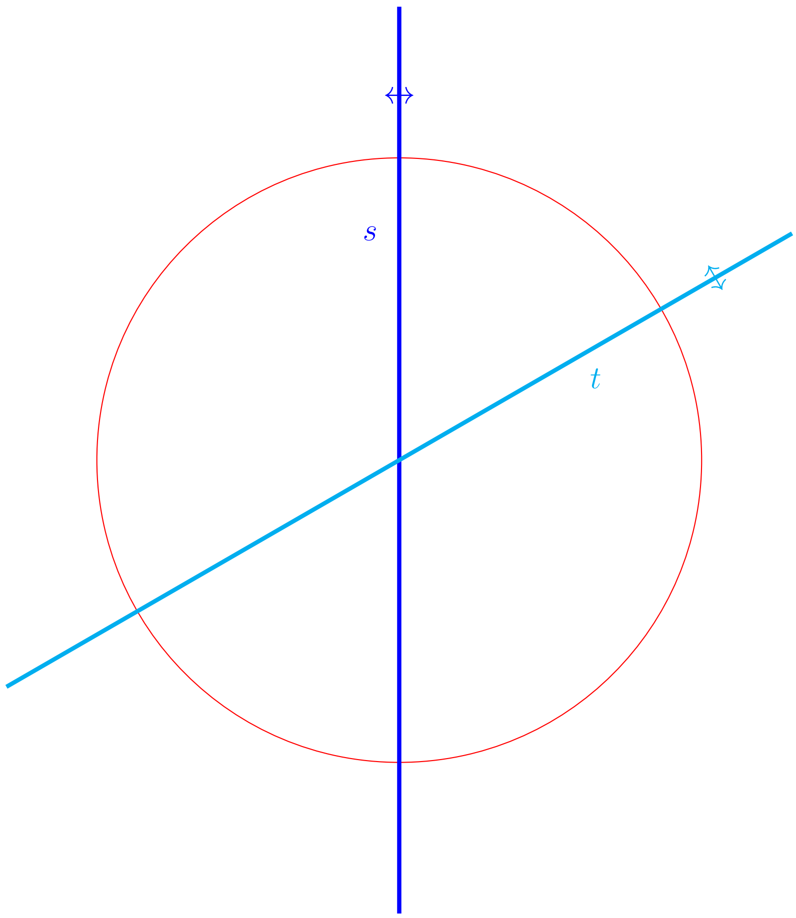


Figure 2: The action of A_2 on S^1.

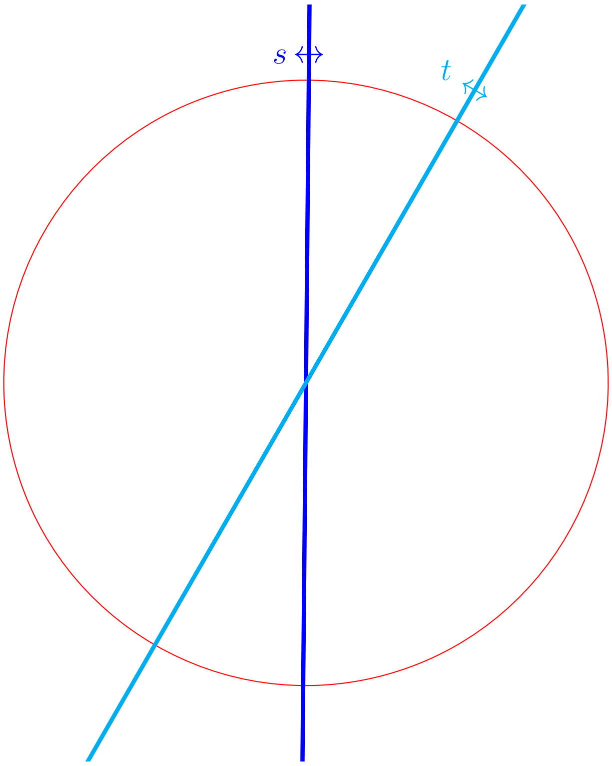


Figure 3: The action of G_2 on S^1.

2.1.1 Root Systems

Definition 2.5. Let V be an Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$. A finite set Φ of non-zero vectors, called *roots*, is called a *root system*, if

- the roots span V, i.e.: $\langle \Phi \rangle = V$,
- the only multiples of a root belonging to Φ are exactly the root itself and its negative: $\lambda x \in \Phi \Leftrightarrow \lambda = \pm 1$ for $x \in \Phi$ and $\lambda \in \mathbb{R}$,
- for $\alpha \in \Phi$, the set Φ is closed under reflection on the hyperplane orthogonal to α. So for $\beta \in \Phi$, it holds that $s_\alpha(\beta) \in \Phi$.

where $s_\alpha(v) := v - \frac{2\langle v, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha$ is the reflection on the euclidean hyperplane orthogonal to α.

Every reflection in a finite Coxeter group W determines a reflection hyperplane through the origin. The collection of all vectors orthonormal to such hyperplanes forms a root system. On the other hand, given a root system, the group generated by the reflections on hyperplanes orthogonal to the roots is a finite Coxeter group.

Given a root system Φ, we can always partition $\Phi = \Phi^+ \dot\cup \Phi^-$ such that

- for $\alpha \in \Phi$, either α or $-\alpha$ belongs to Φ^+,
- if the sum of two roots $\alpha, \beta \in \Phi^+$ is a root, $\alpha + \beta \in \Phi^+$.

This partition will obviously not be unique in general. We call Φ^+ the set of *positive roots*.

Definition 2.6. A subset $\Delta \subset \Phi$ is called a *simple system* if Δ is a vector space basis of V and for each $\alpha \in \Phi$, the coefficients in the linear decomposition of α are all of the same sign.

The following theorem shows that simple systems exist:

Theorem 2.7 ([4, Theorem 1.3])**.** *Every positive system in a root system contains a unique simple system. Likewise, if Δ is a simple system in a root system, then there is a unique positive system containing Δ.*

In particular, every finite Coxeter group has a simple root system. The choice of a simple root system corresponds to the choice of a set of simple reflections S of W.

Definition 2.8. Let Φ be a root system. We say that Φ is *crystallographic*, if

$$\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \in \mathbb{Z} \text{ for all } \alpha, \beta \in \Phi$$

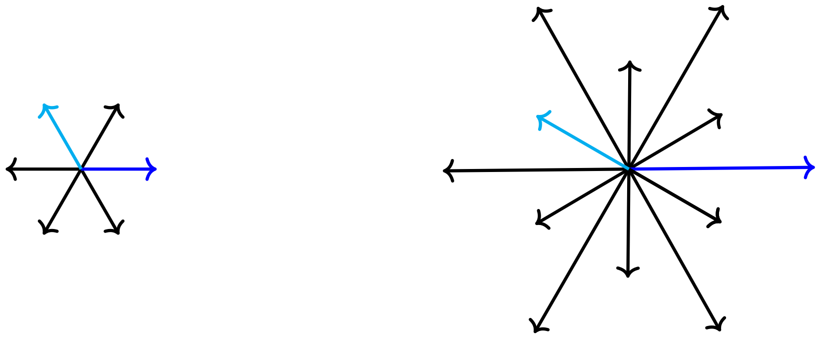
If a spherical Coxeter group admits a crystallographic root system, it stabilizes a lattice in V. This is important in the construction of the affine Coxeter groups. We will however not discuss this process in detail.

2.1.2 Affine Coxeter Groups

Another important class of Coxeter groups are the *affine Coxeter groups.* A Coxeter group is called affine, if it can be represented as a reflection group on $\mathbb{R}^n$ with no common invariant subspace with dimension $< n$. Every affine Coxeter group arises from a finite Coxeter group in the following way: Let Φ be a crystallographic root system for a finite Coxeter group W (which exists exactly for the so-called finite Weyl groups). We define an affine reflection as follows:

$$s_{\alpha,k}(v) := v - (\langle v, \alpha \rangle - k)\frac{2\alpha}{\langle \alpha, \alpha \rangle} \quad \text{for } k \in \mathbb{R}, \alpha \in \Phi$$

This is indeed a reflection on the affine hyperplane $H_{\alpha,k} := \{v \in V \mid \langle v, \alpha \rangle = k\}$. The group generated by all those reflections (which include the reflections $s_\alpha = s_{\alpha,0}$ of W) is denoted by $\tilde{W}$. Furthermore, if we start with an *irreducible* finite Coxeter system (W, S) (meaning that the Coxeter diagram of W is connected), we can find an *irreducible* root system Φ for W, i.e. there is no partition of Φ into root systems such that any two roots in different subsets are orthogonal. Given an irreducible root system, there is a root $\tilde{\alpha}$ such that for any root α, the root $\tilde{\alpha} - \alpha$ is a sum of simple roots. This root is unique and we call it the *highest root.* The affine group $\tilde{W}$ is generated by $\tilde{S} = S \cup \{s_{\tilde{\alpha},1}\}$ and $(\tilde{W}, \tilde{S})$ is the corresponding affine Coxeter system.



(a) A_2 (b) G_2

Figure 4: The root system of ...

Proposition 2.9 ([4, Proposition 4.2]). *Let $\tilde{W}$ be an affine Coxeter group obtained from a crystallographic root system Φ of a finite Coxeter group W. Then $\tilde{W}$ is the semidirect product of W and the translation group L corresponding to the coroot lattice $\Lambda := \Lambda(\Phi^\vee) := \mathbb{Z}\Phi^\vee$, where $\Phi^\vee := \{\frac{2\alpha}{\langle\alpha,\alpha\rangle} \mid \alpha \in \Phi\}$.*

In particular, $\tilde{W}$ fixes the coroot lattice Λ.

All the spherical and affine Coxeter groups are known and are, for example, listed in [2, Appendix A1].

Example 2.10. The group $\tilde{A}_2$, corresponding to the Coxeter diagram in Fig. 1c, is an affine Coxeter group. The associated finite Coxeter group is A_2. The action of A_2 on $\mathbb{R}^2$ is depicted in Fig. 2. The root system of this action is

$$\Phi = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

and, in fact, is crystallographic (see Fig. 4 for a depiction of the roots). The corresponding coroots are $\Phi^\vee = 2\Phi$. They induce the normal translation subgroup inside $\tilde{A}_2$. The induced tiling of $\mathbb{R}^2$ is depicted in Fig. 5. A possible choice of positive roots would be

$$\Phi^+ = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} \right\}$$

Then the highest root is $\tilde{\alpha} = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$ and $r = s_{\tilde{\alpha},1}$ is the new generator of $\tilde{A}_2$.

Example 2.11. The group $\tilde{G}_2$, corresponding to the Coxeter diagram in Fig. 1d, is an affine Coxeter group. The associated finite Coxeter group is G_2. The action of G_2 on $\mathbb{R}^2$

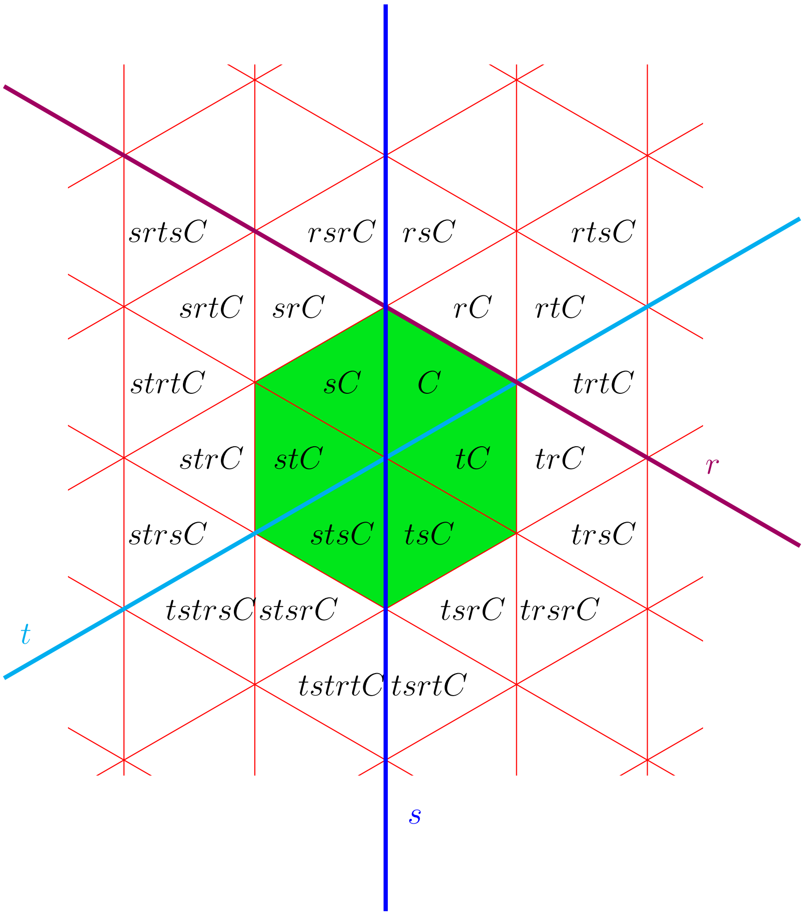


Figure 5: The action of $\tilde{A}_2$ on $\mathbb{R}^2$. The action of the parabolic subgroup A_2 is marked in green and matches the action of Fig. 2. The new basic reflection is r.

is depicted in Fig. 3. The root system of this action is

$$\Phi = \left\{ \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}, \begin{pmatrix} \sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} -\sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} \frac{\sqrt{6}}{2} \\ -\frac{3\sqrt{2}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{2} \\ -\frac{3\sqrt{2}}{2} \end{pmatrix} \right\}$$

and, in fact, is crystallographic. The coroots $\Phi^\vee$ induce the normal translation subgroup inside $\tilde{G}_2$. The induced tiling of $\mathbb{R}^2$ is depicted in Fig. 6. The chosen standard reflections correspond to the simple system

$$\Delta = \left\{ \begin{pmatrix} \sqrt{6} \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2} \end{pmatrix} \right\}$$

Then the highest root is $\tilde{\alpha} = \begin{pmatrix} \frac{\sqrt{6}}{2} \\ \frac{3\sqrt{2}}{2} \end{pmatrix}$ and $r = s_{\tilde{\alpha},1}$ is the new generator of $\tilde{G}_2$.

2.2 Algebras

We will mainly consider the symmetric algebra and the exterior algebra:

Definition 2.12. Let V be an n-dimensional $\mathbb{K}$-vector space. We denote its tensor algebra by $T(V)$. The *symmetric algebra* of V is defined as $\mathrm{Sym}(V) := T(V)/I$, where I is the ideal generated by $u \otimes v - v \otimes u$ for all $u, v \in V$. The symmetric algebra inherits the grading of $T(V)$ and is therefore a graded algebra.

The symmetric algebra of the dual space $S := \mathrm{Sym}(V^*)$ can be naturally identified with polynomials on the space V, i.e. $S \cong \mathbb{K}[X_1, \ldots, X_n]$. Let W be a spherical (i.e. finite) Coxeter group, acting on V. Then W and all subgroups of W act on S via

$$g \bullet s := s \circ g^{-1}$$

The Theorem 4.18 by Chevalley, Shephard and Todd asserts that the invariant ring S^W is a polynomial subalgebra. So S^W inherits the grading of S. We are interested in the so-called *Hilbert series* of this subalgebra S^W:

Definition 2.13. Let $M = \bigoplus_{d \geq 0} M_d$ be a graded $\mathbb{K}$-algebra. Then the *Hilbert series* of M is defined as

$$\mathrm{Hilb}(M, q) := \sum_{d=0}^{\infty} \dim_{\mathbb{K}}(M_d) q^d \in \mathbb{Z}[[q]]$$

Dealing with Coxeter groups, the Hilbert series might as well be expressed in the following way: The polynomial subalgebra $S^W = \mathbb{K}[f_1, \ldots, f_n]$ is generated by homogeneous polynomials $f_1, \ldots, f_n$. Let $d_i := \deg(f_i)$ be their degrees, called the *fundamental degrees.* Then

$$\mathrm{Hilb}(S^W, q) = \prod_{i=1}^{n} \frac{1}{1 - q^{d_i}}$$

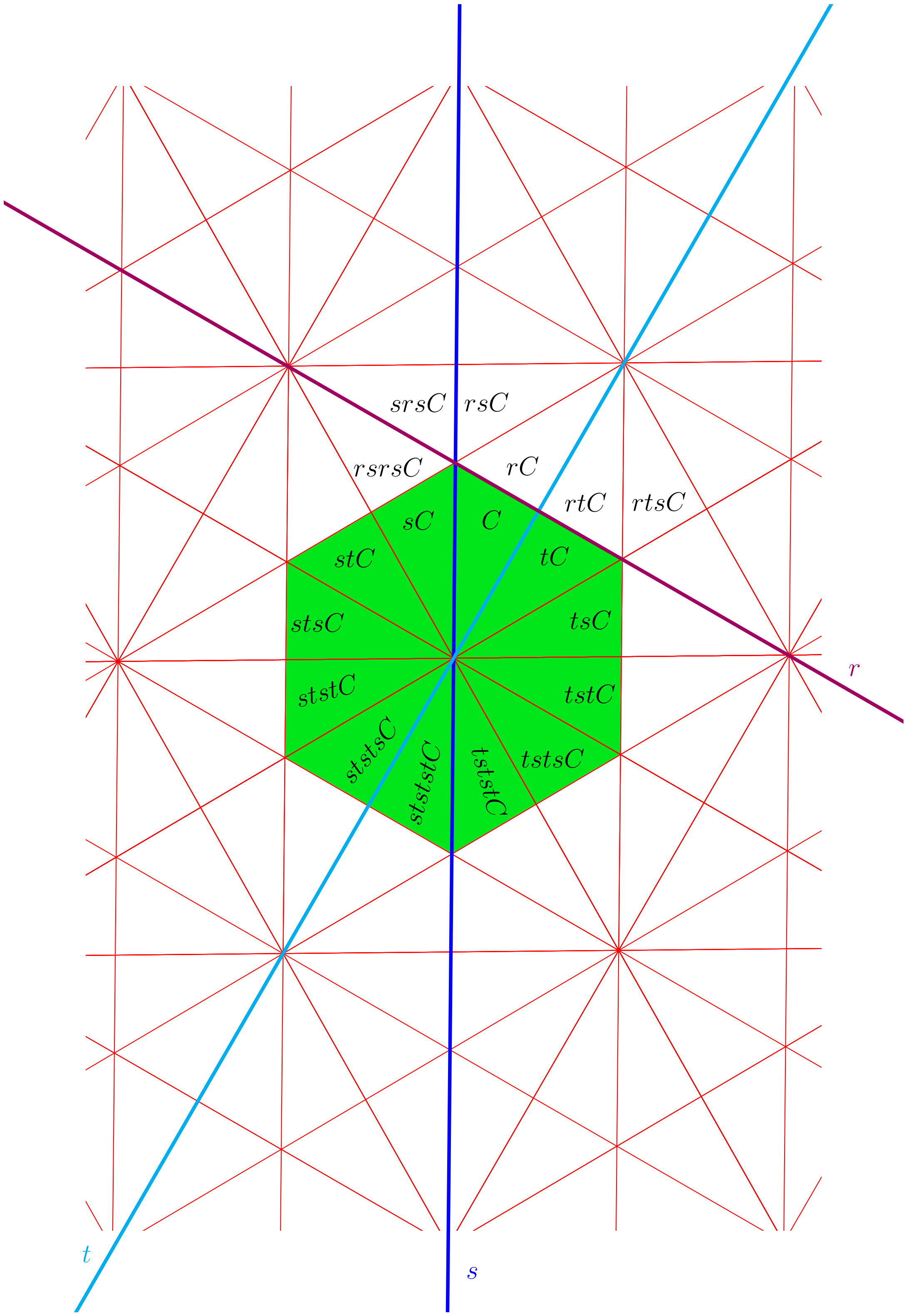


Figure 6: The action of $\tilde{G}_2$ on $\mathbb{R}^2$. The action of the parabolic subgroup G_2 is marked in green and matches the action of Fig. 3. The new basic reflection is r.

holds. We will prove this fact in Proposition 4.24.

The other important algebra is the *exterior algebra*. Its construction is dual to the construction of the symmetric algebra:

Definition 2.14. Let V be an n-dimensional $\mathbb{K}$-vector space. The *exterior algebra* of V is defined as

$$\Lambda(V) := T(V)/I$$

where I is the ideal generated by $v \otimes v$ for $v \in V$. The exterior algebra inherits the grading of $T(V)$ and is therefore a graded algebra $\Lambda(V) = \bigoplus_{p \geq 0} \Lambda^p(V)$. The product of the algebra is denoted by $\wedge$. If $\{b_1, \ldots, b_n\}$ is a basis of V, then $\{b_{i_1} \wedge \ldots \wedge b_{i_p} \mid i_1 < \ldots < i_p\}$ is a basis of $\Lambda^p(V)$. In particular, $\Lambda^0(V) = \mathbb{K}$ and $\Lambda^1(V) = V$.

2.3 The Coxeter Complex

In our case, the above vectorspace arises from the action of W on its *Coxeter complex* $\Delta := \Delta(W, S)$. It is constructed as follows: Consider all cosets wW_J of all parabolic subgroups, where $w \in W$. They form a partially ordered set as sets with inclusion. Let $(P, <)$ be the reversed ordering on this set.

The Coxeter complex is the simplicial complex associated to $(P, <)$, the so-called *order complex*. As a set, the whole group $W = W_S$ is maximal. Therefore it induces a minimal simplex, which is contained in every other simplex. This is the empty simplex $\emptyset$.

If we take $J = \emptyset$, the induced parabolic subgroup is the trivial group $W_\emptyset = \{1\}$. Its cosets are precisely the elements of W. In the complex Δ they correspond to maximal simplices. For an example of a Coxeter Complex, see Fig. 8 in Section 5.

We call a simplex $C = wW_J$ to be of *type* J.

In the case of a finite Coxeter group of rank n, the Coxeter complex is homeomorphic to an $(n-1)$-sphere. In the case of an irreducible affine Coxeter group of rank n, the Coxeter complex is a tessellation of $\mathbb{R}^n$.

In both cases, W acts on the Coxeter complex by left multiplication. This induces an action on an $\mathbb{R}$-vector space. However, we will consider W acting on the $\mathbb{C}$-vector space coming from the natural embedding of $\mathbb{R}^n \hookrightarrow \mathbb{C}^n = V$.

3 Descent Induction

In this chapter, we derive a recursive formula for the Poincaré series of a Coxeter system. We follow the arguments of [6].

Given a Coxeter system (W, S), we may write an element $w \in W$ as a product of generators in many ways. However, all minimal presentations will have the same length, which we will denote by $\ell(w) := \min\{n \in \mathbb{N} \mid \exists\, s_1, \dots, s_n \in S \colon w = s_1 \cdot \ldots \cdot s_n\}$. This is the *length* of w with respect to S.

A basic fact about Coxeter groups tells us that multiplying an element $w \in W$ by a generator $s \in S$ from the left or from the right either decreases or increases the length of w by 1.

Definition 3.1. Let $w \in W$ for a given Coxeter system (W, S). Then

$$D_R(w) := \{s \in S \mid \ell(ws) < \ell(w)\}$$

is called the *right descent set* of w and the elements of $D_R(w)$ are called the *(right) descents.*

Given a parabolic subgroup W_J of W for a subset $J \subseteq S$, we consider the quotient group W/W_J:

Definition 3.2. For a subset $J \subset S$ we call

$$W^J := \{w \in W \mid D_R(w) \subseteq S \setminus J\}$$

the *quotient* for J.

The elements of W^J are the (unique) minimum length coset representatives for W/W_J. This can be seen by the following easy proposition:

Proposition 3.3 ([2, Proposition 2.4.4])**.** *Let $J \subseteq S$. Then, the following holds:*

- *Every $w \in W$ has a unique factorization $w = w^J \cdot w_J$ such that $w^J \in W^J$ and $w_J \in W_J$.*
- *For this factorization the length of w is $\ell(w) = \ell(w^J) + \ell(w_J)$.*

In the following, we will compare the Hilbert series of W with the Poincaré series $W(q)$, which "measures" the growth of elements in W:

Definition 3.4. Let (W, S) be a Coxeter system and $A \subseteq W$ a subset, the formal series given by

$$A(q) := \sum_{w \in A} q^{\ell(w)}$$

is called the *Poincaré series* (or, if A is finite, *Poincaré polynomial*) of A.

With Proposition 3.3, we can split the Poincaré series into smaller parts:

$$W(q) = \sum_{w \in W} q^{\ell(w)} = \sum_{w \in W} q^{\ell(w^J)} q^{\ell(w_J)} = \sum_{w^J \in W^J} \sum_{w_J \in W_J} q^{\ell(w^J)} q^{\ell(w_J)} = W^J(q) W_J(q)$$

Therefore, we get

$$\frac{W(q)}{W_J(q)} = W^J(q) = \sum_{\substack{w \in W \\ D_R(w) \subseteq S \setminus J}} q^{\ell(w)}$$

We compute the alternating sum $\sum_{J \subseteq S} (-1)^{|J|}$ of this equality. First, for $J = \emptyset$, we get the set $\{w \in W \mid D_R(w) \subseteq S\}$, which is the whole group W. Then we subtract all terms where $|J| = 1$. This excludes everything besides $\{w \in W \mid D_R(w) = S\}$, but there are elements which we excluded too often. Those are the elements in $\{w \in W \mid D_R(w) \subseteq S \setminus J \text{ for } |J| = 2\}$. So we include those elements again by adding the $|J| = 2$ term. Now we end up with surplus elements $\{w \in W \mid D_R(w) \subseteq S \setminus J \text{ for } |J| = 3\}$. We continue this process until we corrected all terms. We end up with

$$\sum_{J \subseteq S} (-1)^{|J|} \frac{W(q)}{W_J(q)} = \sum_{\substack{w \in W \\ D_R(w) = S}} q^{\ell(w)} = \begin{cases} q^{\ell(w_0)} & \text{if } W \text{ is finite} \\ 0 & \text{if } W \text{ is infinite} \end{cases} \tag{3.1}$$

Concerning the equality on the right-hand side: If W is finite, there is exactly one element in W with maximal length, which is denoted by w_0. This is the only element that does not get longer by multiplication by a generator, therefore $D_R(w_0) = S$. If W is infinite, there is no maximal element. So the sum above is empty in this case.

Dividing by $W(q)$ and subtracting the $J = S$ term on both sides of Eq. (3.1) leads to

$$\sum_{J \subsetneq S} (-1)^{|J|} \frac{1}{W_J(q)} = f(q) \frac{1}{W(q)}, \tag{3.2}$$

where

$$f(q) := \begin{cases} q^{\ell(w_0)} - (-1)^{|S|} & \text{if } W \text{ is finite} \\ -(-1)^{|S|} & \text{if } W \text{ is infinite} \end{cases}$$

In the following, we will obtain a similar recursive formula for $\mathrm{Hilb}(S, q)$ and therefore linking the two series.

$$\begin{tikzcd} \mathrm{Proj}(R) \arrow[r, "\varphi"] \arrow[d] & H \\ K_0(R) \arrow[ur, dashed, "\exists!\tilde{\varphi}"] & \end{tikzcd}$$

Diagram 7: The universality of the zeroth K-theory.

4 Poincaré Series for Finite Coxeter Groups

The following theorem is the main theorem of this section:

Theorem 4.1. *Let W be a finite Coxeter group. Then*

$$W(q) = \prod_{i=1}^{n}(1 + q + q^2 + \ldots + q^{d_i - 1}) = \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q} = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(S^W, q)}$$

We will prove this theorem by developing an inductive formula for $\frac{\mathrm{Hilb}(S^W,q)}{\mathrm{Hilb}(S,q)}$ similar to Eq. (3.2).

4.1 The Euler-Poincaré Principle

In the following, we use the "Euler-Poincaré Principle", which relates the cellular structure of the Coxeter complex with its homology. It yields an equation of elements of algebraic K-theory:

Definition 4.2 ([7, Definition 1.1.5])**.** Let R be a ring (with unit). Then $K_0(R)$ is the Grothendieck group of the semigroup $\mathrm{Proj}(R)$ of isomorphism classes of finitely generated projective modules over R.

Remark 4.3. K-theory is universal in the following sense: For each abelian group H and monoid homomorphism $\varphi\colon \mathrm{Proj}(R) \to H$, there is a unique $\tilde{\varphi}\colon K_0(R) \to H$ such that diagram 7 commutes. The vertical arrow is the canonical inclusion of a monoid into its Grothendieck group. If P is a finitely generated projective R-module, we denote its class in $K_0(R)$ by $[P]$.

However, we will only consider K-theory over $\mathbb{C}[W]$ or the complex numbers $\mathbb{C}$. In the latter case, $K_0(\mathbb{C}) \cong \mathbb{Z}$, since all finitely generated vector spaces are free and uniquely classified by dimension up to isomorphism.

Definition 4.4. Given a chain complex (C_*, d) of finite type, where all R-modules C_i are projective, we call

$$\chi(C_*) := \sum_{j=-\infty}^{\infty} (-1)^j [C_j]$$

the *Euler characteristic* of C_*, where the sum above is taken in $K_0(R)$.

The Euler-Poincaré Principle is a more abstract variant of the Euler-Poincaré formula. The formula states that the alternating sum of the Betti numbers and the Euler characteristic of a finite CW-complex coincide.

Theorem 4.5 (Euler-Poincaré Principle,[7, Proposition 1.7.10])**.** *Let (C_*, d) be a chain complex of finite type of projective R-modules. If all its homology modules are projective, then*

$$\chi(C_*) = \sum_{j=-\infty}^{\infty} (-1)^j [H_j(C_*)]$$

holds, where the sum is taken in $K_0(R)$.

Proof. Let (C_*, d) be a chain complex of finite type of projective R-modules, such that all its homology modules are projective. We denote the cycles and boundaries of C_* by $Z_j := \ker(\mathrm{d}_j)$ and $B_j := \mathrm{im}(\mathrm{d}_{j+1})$. There are two short exact sequences:

$$0 \to Z_{j+1} \to C_{j+1} \to B_j \to 0 \tag{4.1}$$

$$0 \to B_j \to Z_j \to H_j(C_*) \to 0 \tag{4.2}$$

By assumption, $H_j(C_*)$ is projective, so seq. (4.2) splits and we obtain $Z_j \cong B_j \oplus H_j(C_*)$. By induction, Z_j and B_j are projective for all $j \in \mathbb{Z}$: Since C_* is of finite type, there is $N \in \mathbb{Z}$ with $C_j = 0$ for $j \leq N$. Then $Z_j \subseteq C_j$ and $B_j \subseteq C_j$ are trivial and therefore projective. This is the induction base.

As for the induction step, assume that Z_j and B_j are projective for all $j \leq J$. Since B_J is projective, seq. (4.1) splits for $j = J$ and we obtain $C_{J+1} \cong B_J \oplus Z_{J+1}$. Hence Z_{J+1} is projective as a factor of a projective module. But since $Z_{J+1} \cong B_{J+1} \oplus H_{J+1}(C_*)$, B_{J+1} is projective, too.

Therefore, the seqs. (4.1) and (4.2) split for all $j \in \mathbb{Z}$. If we express the splitting-isomorphisms in terms of $K_0(R)$, we obtain:

$$\begin{aligned} [C_{j+1}] &= [Z_{j+1}] + [B_j] \\ [Z_j] &= [B_j] + [H_j(C_*)] \end{aligned}$$

We now derive the desired formula by substituting with the above equations:

$$\begin{aligned} \chi(C_*) &= \sum_{j=-\infty}^{\infty} (-1)^j [C_j] \\ &= \sum_{j=-\infty}^{\infty} (-1)^j ([Z_j] + [B_{j-1}]) \\ &= \sum_{j=-\infty}^{\infty} (-1)^j ([B_j] + [H_j(C_*)] + [B_{j-1}]) \\ &= \sum_{j=-\infty}^{\infty} (-1)^j [H_j(C_*)] \end{aligned}$$

□

Let $C_* := C_*(\Delta, \mathbb{C})$ be the simplicial chain complex of the Coxeter complex Δ with coefficients in the complex numbers. We want to apply the Euler-Poincaré principle to C_*. However, we will consider the chain complex endowed with the action of W, which arises from the action of W on the coxeter complex Δ. Therefore, we are dealing with $\mathbb{C}[W]$-modules. If we were to consider the modules as $\mathbb{C}$-modules, they would automatically be projective, as they are vector spaces. However, at first it is not clear, why homology modules and complex modules are projective in our case. This is the subject of Maschke's theorem. The proof is from [5, Theorem 8.1].

Theorem 4.6 (Maschke's theorem)**.** *Let G be a finite group. Then all $\mathbb{C}[G]$-modules are projective.*

Proof. Let $0 \to U \to V \to W \to 0$ be a short exact sequence of $\mathbb{C}[G]$-modules. We have to show that this sequence is split. We may consider U as a $\mathbb{C}[G]$-submodule of V since $U \hookrightarrow V$ is injective.

First, we will show that there is a $\mathbb{C}[G]$-submodule $U^\perp < V$ with $V = U \oplus U^\perp$. Let $\tilde{W}$ be any vector space complement of U in V, i.e. $V = U \oplus \tilde{W}$ as vector spaces. Consider the projection $\mathrm{pr}\colon V \to V, v = u + w \mapsto u$ from V to U. We will modify this projection to obtain a $\mathbb{C}[G]$-homomorphism. We define the averaging operator

$$\pi\colon V \to V, v \mapsto \frac{1}{|G|}\sum_{g\in G}(g^{-1} \circ \mathrm{pr} \circ g)(v)$$

Obviously, π is $\mathbb{C}$-linear and $\mathrm{im}(\pi) \subseteq U$, since U is a $\mathbb{C}[G]$-module. Furthermore, π is a $\mathbb{C}[G]$-homomorphism:

$$\begin{aligned}
\pi(h(v)) &= \frac{1}{|G|}\sum_{g\in G}(g^{-1} \circ \mathrm{pr} \circ g \circ h)(v) \\
&= \frac{1}{|G|}\sum_{f\in G}(h \circ f^{-1} \circ \mathrm{pr} \circ f)(v) && \text{substitute } f = g \circ h \\
&= (h \circ \pi)(v)
\end{aligned}$$

Now we show that π is a projection by showing that $\pi^2 = \pi$. Take $u \in U$:

$$\begin{aligned}
\pi(u) &= \frac{1}{|G|}\sum_{g\in G}(g^{-1} \circ \mathrm{pr} \circ g)(u) \\
&= \frac{1}{|G|}\sum_{g\in G} g^{-1}gu && \text{since } gu \in U \\
&= \frac{1}{|G|}\sum_{g\in G} u \\
&= u
\end{aligned}$$

So π is indeed a $\mathbb{C}[G]$-linear projection onto U. Therefore, $V = U \oplus \ker(\pi)$ as $\mathbb{C}[G]$-modules.

Thereby, we get the splitting on the left: $U \hookrightarrow V \cong U \oplus \ker(\pi) \xrightarrow{\pi} U = \mathrm{id}_U$.

□

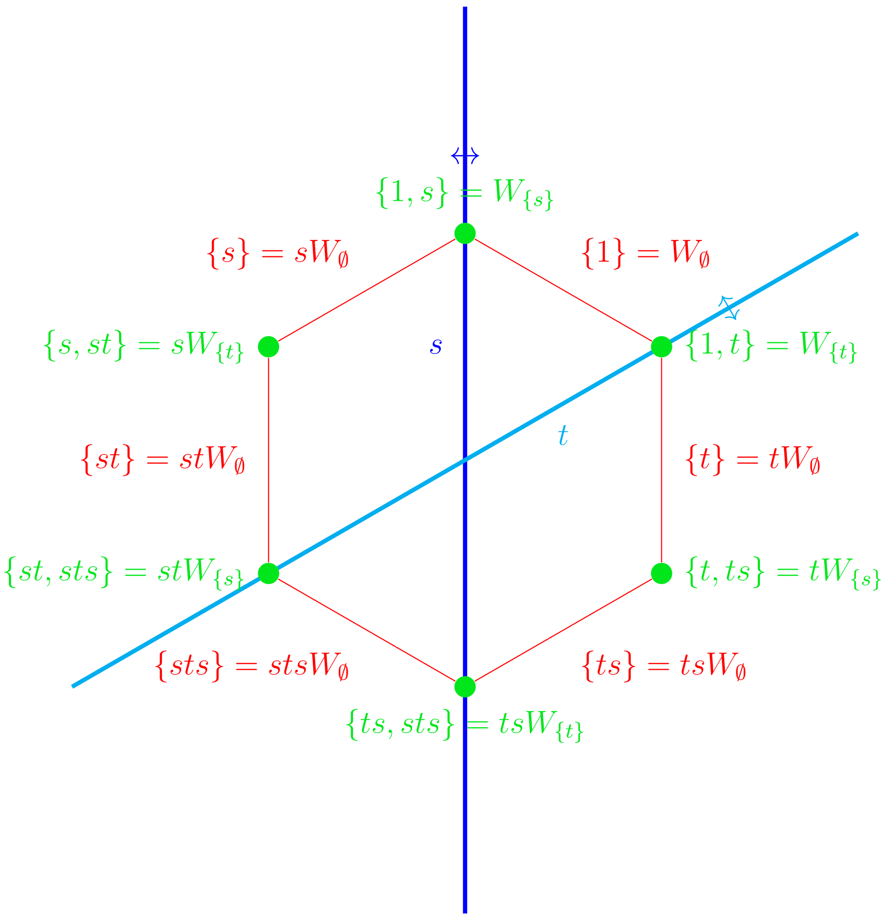


Figure 8: The Coxeter complex of A_2.

Therefore, we can apply the Euler-Poincaré principle to the simplicial chain complex. Let us consider the group action on the simplicial chain complex. The vertices of Δ correspond to the cosets of parabolic subgroups, generated by $|S|-1$ elements (hence the subgroup generated by $|S|$ elements is W itself, corresponding to the empty simplex $\emptyset$). Higher dimensional simplices correspond to cosets of parabolic subgroups, generated by fewer elements (for an example, look at Fig. 8). If we pick representatives for the cosets, we may denote

$$C_i \cong \bigoplus_{\substack{J \subsetneq S \\ |J|=n-i-1 \\ w^J \in W^J}} w^J W_J \mathbb{C}$$

as $\mathbb{C}$-vector spaces. W acts on the simplices by left-multiplication. In the following, we examine the $\mathbb{C}[W]$-structure of C_*.

Lemma 4.7. *Let $C = wW_J$ be a simplex of Δ. Then the orbit of W decomposes as*

$$W \bullet C = \dot{\bigcup_{w^J \in W^J}} w^J W_J$$

Proof. We show that for any other simplex $D = vW_I$ with $D = gC$, the simplex D has the same type as C. Without loss of generality, let $D = W_I$. Therefore, $1 \in D = gwW_J$, so $(gw)^{-1} \in W_J$. But then $gw \in W_J$, so gw acts trivially on W_J and $D = W_J$. Obviously, all simplices of type J lie in the orbit of W_J. By Proposition 3.3, every simplex vW_J can be written as $vW_J = v^J v_J W_J = v^J W_J$. If $v^J W_J = w^J W_J$ for $v^J, w^J \in W^J$, then there are $v_j, w_J \in W_J$ with $w^J w_J = v^J v_J$. By Proposition 3.3, $w^J = v^J$, so the decomposition of the orbit is disjoint. □

We will describe the cell complex by the following representations:

Definition 4.8. Let $H < G$ be a subgroup of a finite group G. Let $\rho\colon H \to \mathrm{GL}(V)$ be a representation on a vector space V. Let $g_1, \dots, g_n$ be representatives of the cosets in G/H. For $g \in G$, there is a permutation $\sigma_g \in \mathrm{Sym}(n)$ such that $g(g_i H) = g_{\sigma_g(i)} H$. Then G acts on $\bigoplus_{i=1}^n g_i V$ via

$$g \bullet \sum_{i=1}^n g_i v_i := \sum_{i=1}^n g_{\sigma_g(i)} \rho(h_i)(v_i)$$

where $gg_i = g_{\sigma_g(i)} h_i$. We call this representation the *induced representation of ρ from H to G* and denote it by $\mathrm{Ind}_H^G \rho$.

Let $\rho\colon G \to \mathrm{GL}(W)$ be an action on a vector space W. Then the action

$$H \to \mathrm{GL}(W), h \mapsto \rho(h)$$

of H is called the *restriction* of ρ. We denote it by $\mathrm{Res}_H^G \rho$.

Since the subgroup W_J acts trivially on the simplex W_J, we can describe each orbit as the induced representation of the trivial representation:

$$C_i > W \bullet 1W_J\mathbb{C} = \mathrm{Ind}_{W_J}^W \mathrm{id} = \bigoplus_{w^J \in W^J} w^J W_J \mathbb{C}$$

If we add up all standard simplices in the same dimension, we obtain

$$C_i = \bigoplus_{\substack{J \subseteq S \\ |J| = n-1-i}} \mathrm{Ind}_{W_J}^W \mathrm{id} \qquad \text{for } i \geq 0$$

There is just one flaw in this description: the empty simplex $\emptyset$ corresponding to the parabolic subgroup $W_S = W$ does not add anything to the cellular chain complex. We can fix this by considering reduced homology. The cellular chain complex for reduced homology is obtained by adding $\mathbb{C}$ in degree -1. To sum up:

$$\tilde{C}_*(\Delta, \mathbb{C}) = \bigoplus_{J \subseteq S} \mathrm{Ind}_{W_J}^W \mathrm{id}$$

as $\mathbb{C}[W]$-modules, where $\mathrm{Ind}_{W_J}^W \mathrm{id}$ has degree $|S| - |J| - 1$.

Example 4.9. Let's calculate $C_0(A_2, \mathbb{C})$. A_2 has two parabolic rank-1 subgroups, namely $W_{\{s\}}$ and $W_{\{t\}}$. By the above formula,

$$C_0(A_2, \mathbb{C}) = \mathrm{Ind}_{W_{\{s\}}}^{W} \mathrm{id} \oplus \mathrm{Ind}_{W_{\{t\}}}^{W} \mathrm{id} = \bigoplus_{w^s \in W^{\{s\}}} w^s W_{\{s\}}\mathbb{C} \bigoplus_{w^t \in W^{\{t\}}} w^t W_{\{t\}}\mathbb{C}$$

We see that $W^{\{s\}} = \{1, t, st\}$ and $W^{\{t\}} = \{1, s, ts\}$, so we obtain

$$C_0(A_2, \mathbb{C}) \quad = \quad 1W_{\{s\}}\mathbb{C} \oplus tW_{\{s\}}\mathbb{C} \oplus stW_{\{s\}}\mathbb{C} \quad \oplus \quad 1W_{\{t\}}\mathbb{C} \oplus sW_{\{t\}}\mathbb{C} \oplus tsW_{\{t\}}\mathbb{C}$$

Obviously, this coincides with the $\mathbb{C}$-vectorspace over the vertices of the Coxeter complex seen in Fig. 8. Now we examine the action of st on $C_0(A_2, \mathbb{C})$. The element st acts on the induced representation $\mathrm{Ind}_{W_J}^{W} \mathrm{id}$ simply by multiplying $w^J W_J$ by st. So if we denote the canonical basis of $C_0(A_2, \mathbb{C})$ by $\{b_1, b_2, b_3, c_1, c_2, c_3\}$, we get

$$st \cdot b_1 = b_3, \quad st \cdot b_2 = b_1, \quad st \cdot b_3 = b_2, \quad st \cdot c_1 = c_2, \quad st \cdot c_2 = c_3, \quad st \cdot c_3 = c_1$$

On the Coxeter complex this corresponds to a rotation by $\frac{2\pi}{3}$ in positive direction.

As for the right-hand side of the Euler-Poincaré-Principle, the homology of Δ is a lot easier: Δ is homeomorphic to a $|S| - 1$-sphere, so

$$H(\tilde{C}_*(\Delta, \mathbb{C})) = \tilde{H}_*(\Delta, \mathbb{C}) \cong H_{|S|-1}(\Delta, \mathbb{C}) \cong \mathbb{C}$$

.

Lemma 4.10. *The action of W on $\tilde{H}_*(\Delta)$ is the determinant action, i.e. $g \cdot z = \det(g)z$ for all $g \in W$.*

Proof. It is enough to show that $s \cdot z = \det(s)z$ holds for the generators $s \in S$. Since s is a reflection, $\det(s) = -1$. We assume that

$$s \cdot (x_1, x_2 \ldots, x_n) = (-x_1, x_2, \ldots, x_n)$$

without loss of generality, by change of basis. We will proceed with an induction over $n = |S| - 1 = \dim(\Delta)$. Let $n = 0$. We can decompose $S^0 = \{-1, 1\}$ into a disjoint sum of two points. Then s induces maps between the pushouts:

$$\begin{tikzcd}
\emptyset \arrow[rr] \arrow[dr] \arrow[dd] & & \{1\} \arrow[dr] \arrow[dd] & \\
& \emptyset \arrow[rr] \arrow[dd] & & \{-1\} \arrow[dd] \\
\{-1\} \arrow[rr] \arrow[dr] & & S^0 \arrow[dr, "s"] & \\
& \{1\} \arrow[rr] & & S^0
\end{tikzcd}$$

The maps between the pushouts induce maps between the Mayer-Vietoris-sequence. Identifying the terms in the sequence by $H_*(\emptyset) = 0$ and applying the dimension axiom, we obtain isomorphisms and can identify $H_0(s)$:

$$\begin{tikzcd}
(a,b) \arrow[d, mapsto] & \in & \mathbb{C} \oplus \mathbb{C} \arrow[d] & H_0(\{-1\}) \oplus H_0(\{1\}) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & H_0(S^0) \arrow[d, "H_0(s)"] \\
(b,a) & \in & \mathbb{C} \oplus \mathbb{C} & H_0(\{1\}) \oplus H_0(\{-1\}) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H_0(S^0)
\end{tikzcd}$$

We are interested in the reduced homology, which can be identified inside the normal homology:

$\tilde{H}_0(S^0) = \ker(H_0(S_0) \to H_0(*)) = \ker(\mathbb{C}^2 \to \mathbb{C}, (x,y) \mapsto x+y) = \{(x,y) \in \mathbb{C}^2 \mid x = -y\}$

We compare $H_0(s) \circ (\tilde{H}_0(S^0) \hookrightarrow H_0(S_0))$ to $(\tilde{H}_0(S^0) \hookrightarrow H_0(S_0)) \circ \tilde{H}_0(s)$:

$$(-x, x) = H_0(s)(x, -x) = (\tilde{H}_0(s)(x), -\tilde{H}_0(s)(x))$$

We realize that $\tilde{H}_0(s) = -\mathrm{id}$.

For the induction step, we write S^n as a pushout of an S^{n-1} and the two hemispheres $D^n_\pm := \{(x_0, x_1, \dots x_n) \mid \pm x_0 \geq 0\} \subseteq S^n$:

$$\begin{tikzcd}
S^{n-1} \arrow[rr] \arrow[dd] \arrow[dr] & & D^n_+ \arrow[dd] \arrow[dr] & \\
& S^{n-1} \arrow[rr, crossing over] & & D^n_- \arrow[dd] \\
D^n_- \arrow[rr] \arrow[dr] & & S^n \arrow[dr, "s"] & \\
& D^n_+ \arrow[rr] \arrow[from=uu, crossing over] & & S^n
\end{tikzcd}$$

The Mayer-Vietoris sequence for this pushout splits again for $n \geq 2$, since $D^n_\pm \simeq *$ so its homology vanishes except in degree 0. By induction hypothesis $\tilde{H}_{n-1}(s) = -\mathrm{id}$, since $H_n(X) \cong \tilde{H}_n(X)$ for $n \geq 1$.

For $n = 1$, there is homology for the hemispheres: $H_0(D^0_\pm) \cong \mathbb{C}$, so the map $H_1(S^1) \hookrightarrow H_0(S^0)$ is only injective, not surjective. By exactness,

$$\begin{aligned}
H_1(S^1) &\cong \ker(H_0(S_0) \to H_0(D^1_+) \oplus H_0(D^1_-)) \\
&\cong \ker(H_0(S^0) \to H_0(D^1_+)) \cap \ker(H_0(S^0) \to H_0(D^1_-)) \\
&\cong \ker(H_0(S_0) \to H_0(*)) = \tilde{H}_0(S_0)
\end{aligned}$$

So we obtain diagrams

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cong"] \arrow[d, "H_n(s)"] & H_{n-1}(S^{n-1}) \arrow[d, "H_{n-1}(s)"] \\
H_n(S^n) \arrow[r, "\cong"] & H_{n-1}(S^{n-1})
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1(S^1) \arrow[r, "\cong"] \arrow[d, "H_1(s)"] & \tilde{H}_0(S^0) \arrow[d, "\tilde{H}_0(s)"] \\
H_1(S^1) \arrow[r, "\cong"] & \tilde{H}_0(S^0)
\end{tikzcd}$$

Altogether, we obtain $\tilde{H}_n(s) = -\mathrm{id}$ for $n \geq 0$. $\square$

Remark 4.11. Note that in this context, id should not be understood as a map, but rather a $\mathbb{C}[W]$-module: the one-dimensional $\mathbb{C}$-vector space endowed with the trivial W-action. Accordingly, det is a one-dimensional $\mathbb{C}$-vector space endowed with the determinant action.

We remind ourselves of our setting: W is a finite Coxeter group acting on its Coxeter complex Δ, which is imbedded in a complex vector space V. The action of W on V induces an action on the symmetric algebra S.

In the rest of Section 4, we provide a proof for Theorem 4.1. Applying the Euler-Poincaré-Principle to the chain complex of reduced simplicial homology, we obtain the equality

$$\sum_{J \subseteq S} (-1)^{|J|} \operatorname{Ind}_{W_J}^W \mathrm{id} = \det \tag{4.3}$$

by multiplying both sides by $(-1)^{|S|-1}$.

We consider the functor $F\colon \underline{\mathbb{C}[W]-\mathrm{Mod}} \to \underline{\mathbb{C}-\mathrm{Vec}}$ via $V \mapsto \operatorname{Hom}_{\mathbb{C}[W]}(V, S_d)$, where S_d is a graded component of S. This functor respects the sums in both categories:

$$F(V_1 \oplus V_2) = \operatorname{Hom}_{\mathbb{C}[W]}(V_1 \oplus V_2, S_d) \cong \operatorname{Hom}_{\mathbb{C}[W]}(V_1, S_d) \oplus \operatorname{Hom}_{\mathbb{C}[W]}(V_2, S_d) = F(V_1) \oplus F(V_2)$$

so it descends to a monoid homomorphism into $K_0(\mathbb{C})$ if we consider isomorphism classes. Therefore, by universality of the K_0-functor (see Remark 4.3), we get a group homomorphism:

$$K_0(F)\colon K_0(\mathbb{C}[W]) \to K_0(\mathbb{C}) \overset{\dim}{\cong} \mathbb{Z}$$

Applying $K_0(F)$ to Eq. (4.3), we obtain

$$\sum_{J \subseteq S} (-1)^{|J|} \dim \operatorname{Hom}_{\mathbb{C}[W]}(\operatorname{Ind}_{W_J}^W \mathrm{id}, S_d) \cong \dim \operatorname{Hom}_{\mathbb{C}[W]}(\det, S_d) \tag{4.4}$$

In the following, we identify the left-hand and the right-hand side of the above equation.

Definition 4.12. Let G be a group acting on a vector space M. We call

$$M^G := \{m \in M \mid g \cdot m = m \ \forall g \in G\}$$

the *G-invariants* and

$$M^{G,\det} := \{m \in M \mid g \cdot m = \det(g) m \ \forall g \in G\}$$

the *skew-invariants* of M.

Working with representations, we will use the following lemma:

Lemma 4.13 (Frobenius reciprocity, [2, Proposition 2.4.4])**.** *Let H be a subgroup of a finite group G. Let ρ be a representation of H and η be a representation of G over a field $\mathbb{K}$ whose characteristic does not divide $|G|$. Then Ind_H^G is a left-adjoint to the functor Res_H^G, meaning*

$$\operatorname{Hom}_{\mathbb{K}[G]}(\operatorname{Ind}_H^G \rho, \eta) \cong \operatorname{Hom}_{\mathbb{K}[H]}(\rho, \operatorname{Res}_H^G \eta)$$

We identify the left-hand side:

Lemma 4.14.
$$\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Ind}_{W_J}^{W}\mathrm{id}, S_d) \cong S_d^{W_J}$$

Proof. By Lemma 4.13,
$$\mathrm{Hom}_{\mathbb{C}[W]}(\mathrm{Ind}_{W_J}^{W}\mathrm{id}, S_d) \cong \mathrm{Hom}_{\mathbb{C}[W_J]}(\mathrm{id}, \mathrm{Res}_{W_J}^{W} S_d)$$
holds. Elements of $\mathrm{Hom}_{\mathbb{C}[W_J]}(\mathrm{id}, \mathrm{Res}_{W_J}^{W} S_d)$ are W_J-equivariant $\mathbb{C}$-linear maps $\varphi\colon \mathbb{C} \to S_d$. Such a map φ is uniquely determined by the image of $1 \in \mathbb{C}$. Since W_J acts trivially on $\mathbb{C}$, we also get $\varphi(1) = \varphi(g1) = g\varphi(1)$ for any $g \in W_J$. So $\varphi(1) \in S_d^{W_J}$ and $\mathrm{Hom}_{\mathbb{C}[W_J]}(\mathrm{id}, \mathrm{Res}_{W_J}^{W} S_d) \to S_d^{W_J}, \varphi \mapsto \varphi(1)$ is an isomorphism. □

We identify the right-hand side:

Lemma 4.15.
$$\mathrm{Hom}_{\mathbb{C}[W]}(\det, S_d) \cong S_d^{W,\det}$$

Proof. Let $\varphi \in \mathrm{Hom}_{\mathbb{C}[W]}(\det, S_d)$. Then φ is a $\mathbb{C}$-linear map $\mathbb{C} \to S_d$ which is W-equivariant. As in Lemma 4.14, φ is uniquely determined by the image of $1 \in \mathbb{C}$. Considering the action of W on $\mathbb{C}$, we obtain $\det(g)\varphi(1) = \varphi(g1) = g\varphi(1)$, so $\varphi(1) \in S_d^{W,\det}$. Therefore, $\mathrm{Hom}_{\mathbb{C}[W]}(\det, S_d) \to S_d^{W,\det}, \varphi \mapsto \varphi(1)$ is an isomorphism. □

To turn Eq. (4.4) into an equation of the Hilbert series, we just have to multiply the equation by q^d for each $d \geq 0$ and sum all equations up. We obtain
$$\sum_{J\subseteq S} (-1)^{|J|}\,\mathrm{Hilb}(S^{W_J}, q) = \mathrm{Hilb}(S^{W,\det}, q) \tag{4.5}$$

Remark 4.16. Note that we did not use that W_J is a parabolic subgroup of W in Lemma 4.14 and Lemma 4.15. It is enough for W_J to be a subgroup of W.

Next, we examine the Hilbert series of $S^{W,\det}$. For this, we first have to understand S^W better.

4.2 The Invariant Subalgebra

The study of invariant polynomials dates back over 150 years. In this subchapter, we collect some results on invariant subspaces. We will prove some and skip the proofs on others. Let us remind ourselves of our setting first: Our group W acts on a $\mathbb{C}$-vector space V with symmetric algebra $S = \mathrm{Sym}(V^*)$. If we fix a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ of V, we can identify S with the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, where $x_i\colon V \to \mathbb{C}, x_i(b_j) := \delta_{ij}$. The grading of S is the standard grading of a polynomial ring. The action of W induces an action on S: $(g \bullet f)(v) = (f \circ g^{-1})(v)$. Note that the action of W preserves the grading of S, i.e. $\deg(g \bullet f) = \deg(f)$ for all $g \in W, f \in S$.

Definition 4.17. An algebraically independent set $\{f_1, \ldots f_r\} \subset S^W$ is called a set of *fundamental invariants* for S^W, if $1, f_1, \ldots, f_r$ generate S^W as a $\mathbb{C}$-algebra.

At first, it is unclear how big the set of invariants is and whether fundamental invariants exist. However, the following theorem by Chevalley (which we will not prove) answers this question:

Theorem 4.18 ([4, Theorem 3.5]). *Let W be a finite Coxeter group acting on V, which is an n-dimensional $\mathbb{C}$-vector space. Then there exist n homogeneous, algebraically independent elements of positive degree which are fundamental invariants for S^W.*

The fundamental invariants in the theorem are not unique in general. But with the following theorem (which we will not prove either), we see that their degrees are independent of the choice of generators:

Theorem 4.19 ([4, Proposition 3.7]). *Let $f_1, \dots, f_n$ and $g_1, \dots, g_n$ be two sets of fundamental invariants like those in Theorem 4.18. Let $d_i := \deg(f_i)$ and $e_i := \deg(g_i)$. Then, after renumbering one of the sets, $d_i = e_i$ holds.*

We call those degrees *fundamental degrees* of S^W. Those degrees will play an important role in the following. The main result of this subsection is the following theorem:

Theorem 4.20. *Let $d_1, \dots, d_n$ be the fundamental degrees of S^W. Let N be the number of reflections in W. Then*

$$\sum_{i=1}^{n} (d_i - 1) = N$$

Example 4.21. Let's calculate the fundamental degrees of A_2 (see 2 in Section 2 for the action). The transformation matrices of the generating reflections written in the standard basis $\mathcal{E}$ are

$$D_{\mathcal{E}\mathcal{E}}(s) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_{\mathcal{E}\mathcal{E}}(t) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

To show that a polynomial in $\mathbb{C}[X, Y]$ is invariant, it is enough to show invariance under s and t. Obviously, all constant polynomials are invariant. The basic reflection s maps X to $-X$ and leaves Y invariant, while t maps X to $\frac{1}{2}X + \frac{\sqrt{3}}{2}Y$ and Y to $\frac{\sqrt{3}}{2}X - \frac{1}{2}Y$.

Let's consider an invariant polynomial. By Chevalley's theorem, S^{A_2} is generated by homogenous elements. So we can search for invariant polynomials in each degree separately. In particular, an invariant polynomial is invariant under s. So all coefficients of monomials where X appears with uneven degree have to be zero. Hence all invariant polynomials of degree 1 are of the form aY. But those are obviously not invariant under t. So there are no linear invariant polynomials. Let $f := aX^2 + bY^2$ be an invariant polynomial of degree 2. We obtain

$$f = s_\beta \cdot f = (\frac{1}{4}a + \frac{3}{4}b)X^2 + (\frac{\sqrt{3}}{2}a - \frac{\sqrt{3}}{2}b)XY + (\frac{3}{4}a + \frac{1}{4}b)Y^2$$

By solving the corresponding linear equation system, we see that $f = X^2 + Y^2$ is invariant. For an invariant $g := aX^2Y + bY^3$ of degree 3, we obtain the equation

$$g = (\frac{\sqrt{3}}{8}a + \frac{3\sqrt{3}}{8}b)X^3 + (\frac{\sqrt{3}}{8}a + \frac{3\sqrt{3}}{8}b)XY^2 + (\frac{5}{8}a - \frac{9}{8}b)X^2Y + (-\frac{3}{8}a - \frac{1}{8}b)Y^3$$

with the nontrivial solution $a = 3$ and $b = -1$. So $g = 3X^2Y - Y^3$ is another invariant. Since f and g are algebraically independent, they are fundamental invariants by Theorem 4.18. So $S^{A_2} \cong \mathbb{C}[f, g]$ and the fundamental degrees of A_2 are 2 and 3. A reality-check with Theorem 4.20 confirms this: $1 + 2 = 3$, the number of reflections in A_2.

In the following, let $\rho\colon W \to \mathrm{GL}_n(\mathbb{C})$ denote the action of W on V.

Lemma 4.22. *Let $g \in W$. Then $\rho(g)$ is diagonalizable and all eigenvalues of $\rho(g)$ have absolute value* 1*.*

Proof. Since W is finite, there is an $m \in \mathbb{N}$ with $g^m = 1$. Therefore $\rho(g)^m = I_n$. Since $X^m - 1$ cancels $\rho(g)$, the minimal polynomial m_g of $\rho(g)$ divides $X^m - 1$. If $\lambda \in \mathbb{C}$ is an eigenvalue of $\rho(g)$, it is a root of m_g, hence $\lambda^m - 1 = 0$. So $|\lambda| = 1$. Furthermore, m_g has no multiple roots, since $X^m - 1$ has none. But since the multiplicity of the eigenvalues in m_g corresponds to the size of the biggest Jordan block in the Jordan normal form of $\rho(g)$, all blocks have size 1. So the matrix $\rho(g)$ is already diagonalizable. □

Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\rho(g)$. We consider the following polynomial which looks a bit like the characteristic polynomial:

$$H_g(t) := \det(I_n - \rho(g)t) = \lambda_1 \cdot \ldots \cdot \lambda_n \cdot \mathrm{CP}_{\rho(g^{-1})}(t)$$

Then

$$H_g(t)^{-1} = \prod_{j=1}^{n} \frac{1}{1 - \lambda_j t} = \sum_{i=0}^{\infty} \prod_{j=1}^{n} \lambda_j^i t^i \in \mathbb{C}[\![t]\!] \tag{4.6}$$

since $(1 - \lambda_j t)(1 + \lambda_j t + \lambda_j^2 t^2 + \ldots) = 1$. By expanding this product and ordering it by degree, we obtain

$$H_g(t)^{-1} = \sum_{i=0}^{\infty} t^i \left(\sum_{a_1 + \ldots + a_n = i} \lambda_1^{a_1} \cdots \lambda_n^{a_n} \right)$$

We now may obtain an alternative description for the Hilbert series:

Proposition 4.23. *The Hilbert series decomposes as*

$$\mathrm{Hilb}(S^W, t) = \frac{1}{|W|} \sum_{g \in W} \frac{1}{H_g(t)}$$

The proof is from [3, Chapter 5, Lemma 2&3].

Proof. Let $g \in W$ and $\mathcal{B}$ be a basis of eigenvectors of $\rho(g)$. Then the dual basis $\mathcal{B}^* = \{\beta_1, \ldots, \beta_n\}$ of $\mathcal{B}$ is a basis for S_1, the polynomials of first degree. We consider g as an automorphism of S. If b_j is the j-th vector of the basis $\mathcal{B}$, we get

$$(g \bullet \beta_i)(b_j) = (\beta_i \circ \rho(g)^{-1})(b_j) = \beta_i(\frac{1}{\lambda_j} b_j) = \frac{1}{\lambda_j} \delta_{ij}$$

So g acts on S_1 just by multiplication with $\rho(g)^{-1}$. For $i \geq 0$, $\{\beta_1^{a_1} \cdots \beta_n^{a_n} \mid a_1 + \ldots + a_n = i\}$ is a basis of S_i. By plugging in the action of g on each β_i, we obtain:

$$g \bullet (\beta_1^{a_1} \cdots \beta_n^{a_n}) = \frac{1}{\lambda_1^{a_1} \cdots \lambda_n^{a_n}} \beta_1^{a_1} \cdots \beta_n^{a_n}$$

Now we add up these eigenvalues to obtain the trace of g:

$$\operatorname{tr}(g|_{S_i}) = \sum_{a_1+\ldots+a_n=i} \frac{1}{\lambda_1^{a_1} \cdots \lambda_n^{a_n}}$$

So now we can identify the right-hand side of the claim with Eq. (4.6). Note that the fraction disappears because we sum up all group elements $g \in W$ and the eigenvalues of g and g^{-1} are exactly inverse.

$$\frac{1}{|W|} \sum_{g \in W} \frac{1}{H_g(t)} = \frac{1}{|W|} \sum_{g \in W} \sum_{i=0}^{\infty} t^i \cdot \operatorname{tr}(g|_{S_i})$$

On the right-hand side, the averaging operator we saw in the proof of Theorem 4.6 appears:

$$\pi|_{S_i} \colon S_i \to S_i^W \subset S_i, f \mapsto \frac{1}{|W|} \sum_{g \in W} g \bullet f$$

We can write $\pi|_{S_i} = \frac{1}{|W|} \sum g|_{S_i}$, and by the linearity of the trace we get

$$\sum_{i=0}^{\infty} t^i \sum_{g \in W} \frac{1}{|W|} \operatorname{tr}(g|_{S_i}) = \sum_{i=0}^{\infty} t^i \cdot \operatorname{tr}(\pi|_{S_i})$$

In Theorem 4.6 we saw that $\pi|_{S_i}$ is a projection from S_i onto S_i^W. Therefore,

$$\dim(S_i^W) = \operatorname{rk}(\pi|_{S_i}) = \operatorname{tr}(\pi|_{S_i})$$

To sum up:

$$\frac{1}{|W|} \sum_{g \in W} \frac{1}{\mathrm{CP}_{\rho(g)}(t)} = \sum_{i=0}^{\infty} t^i \dim(S_i^W)$$

The coefficient of the Hilbert series in degree i is exactly the dimension of S_i^W. This concludes the proof. □

Proposition 4.24. *The Hilbert series factors as*

$$\operatorname{Hilb}(S^W, t) = \prod_{j=1}^{n} \frac{1}{1 - t^{d_j}}$$

Proof. Suppose that there is just one fundamental invariant with fundamental degree d, so $S^W = \mathbb{C}[f]$ with $f \in S_d$. Then S_{rd}^W is generated by f^r as a $\mathbb{C}$-vector space, so

$$\operatorname{Hilb}(\mathbb{C}[f], t) = 1 + t^d + t^{2d} + t^{3d} + \ldots = \frac{1}{1 - t^d}$$

In general, if we consider graded $\mathbb{C}$-algebras V and U, then

$$(V \otimes U)_d = \bigoplus_{e+f=d} V_e \otimes U_f$$

So $\dim((V \otimes U)_d) = \sum_{i=0}^{d} \dim(V_i) \cdot \dim(U_{d-i})$, therefore

$$\mathrm{Hilb}(V \otimes U, t) = \mathrm{Hilb}(V, t) \cdot \mathrm{Hilb}(U, t)$$

If S^W has fundamental invariants $f_1, \ldots, f_n$, we may factorize

$$S^W = \mathbb{C}[f_1, \ldots, f_{n-1}] \otimes \mathbb{C}[f_n]$$

By induction over the number of fundamental invariants, we obtain the wanted formula. $\square$

Remark 4.25. Note that if we consider the trivial group, $S^{\{1\}} = S = \mathbb{C}[x_1, \ldots, x_n]$. Therefore, by Proposition 4.24, $\mathrm{Hilb}(S, q) = \mathrm{Hilb}(S^{\{1\}}, q) = \frac{1}{(1-q)^n}$. This shows the two equalities on the right-hand side of Theorem 4.1:

$$\prod_{i=1}^{n} (1 + q + \ldots + q^{d_i - 1}) = \prod_{i=1}^{n} \frac{1 - q^{d_i}}{1 - q} = \frac{\mathrm{Hilb}(S, q)}{\mathrm{Hilb}(S^W, q)}$$

We are now ready to prove the theorem.

Theorem 4.20. *Let $d_1, \ldots, d_n$ be the fundamental degrees of S^W. Let N be the number of reflections in W. Then*

$$\sum_{i=1}^{n} (d_i - 1) = N$$

The proof is from [3, Chapter 5, Lemma 3].

Proof. Let us examine the polynomial $H_g(t) = \det(1 - \rho(g)t)$ of an element $g \in W$ again. For a reflection s_α, we have $H_{s_\alpha}(t) = (1-t)^{n-1}(1+t)$. Obviously, we have $H_{\mathrm{id}}(t) = (1-t)^n$ for the identity. Let $g \in W$ have $(n-1)$-times the eigenvalue 1. The action of g on the $\mathbb{C}$-vector space V is induced by an action on an $\mathbb{R}$-vector space. Therefore, the trace of g is real. By Lemma 4.22 all eigenvalues have absolute value 1. So, if $g \neq \mathrm{id}$, we obtain $H_g(t) = (1-t)^{n-1}(1+t)$. But then g is a reflection along the eigenvector of eigenvalue -1.

If we rewrite H_g with that information, we obtain the following equality from the two new descriptions of the Hilbert series:

$$\frac{1}{|W|}\left(\frac{1}{(1-t)^n} + \frac{N}{(1-t)^{n-1}(1+t)} + \sum_{\substack{g \in W \setminus \{1\} \\ g \text{ no reflection}}} \frac{1}{H_g(t)} \right) = \prod_{i=1}^{n} \frac{1}{1 - t^{d_i}} \tag{4.7}$$

where the polynomials $H_g(t)$ on the left side are not divisible by $(1-t)^{n-1}$.

Claim: $d_1 \cdots d_n = |W|$
Proof: By multiplying both sides of Eq. (4.7) by $(1-t)^n$, we obtain

$$\frac{1}{|W|}\left(1 + N\frac{1-t}{(1+t)} + (1-t)^2 \cdot h(t)\right) = \prod_{i=1}^{n} \frac{1-t}{1-t^{d_i}} = \prod_{i=1}^{n} \frac{1}{1+t+t^2+\ldots+t^{d_i-1}} \tag{4.8}$$

where $h(t)$ is a rational function with a denominator not divisible by $(1-t)$. By plugging $t=1$ into the equation, we get

$$\frac{1}{|W|} = \frac{1}{d_1 \cdots d_n}$$

■

We apply $\frac{\partial}{\partial t}$ to Eq. (4.8):

$$\frac{1}{|W|}\left(\frac{-2N}{(1+t)^2} + (1-t)^2 h'(t) - 2(1-t)h(t)\right)$$
$$= \left(\prod_{i=1}^{n} \frac{1}{1+t+\ldots+t^{d_i-1}}\right)\left(\sum_{i=1}^{n} -\frac{1+2t+\ldots+(d_i-1)t^{d_i-2}}{1+t+\ldots+t^{d_i-1}}\right)$$

Again, we plug in $t=1$:

$$\frac{1}{|W|}\frac{-2N}{4} = \left(\prod_{i=1}^{n} \frac{1}{d_i}\right)\left(\sum_{i=1}^{n} -\frac{\sum_{j=1}^{d_i-1} j}{d_i}\right)$$
$$\Leftrightarrow \qquad -\frac{N}{2|W|} = \frac{1}{d_1 \cdots d_n}\left(\sum_{i=1}^{n} -\frac{d_i(d_i-1)}{2d_i}\right)$$
$$\Leftrightarrow \qquad N = \sum_{i=1}^{n}(d_i - 1)$$

□

4.3 Skew-Invariants

We want to understand the structure of the skew-invariants $S^{W,\det}$. They are deeply related with the Jacobian:

Definition 4.26. Let $f_1, \ldots, f_n$ be fundamental invariants for S^W. Let M be the matrix with

$$M_{i,j} := \frac{\partial f_i}{\partial x_j}$$

Then we call $J := J(f_1, \ldots f_n) := \det(M)$ the *Jacobian determinant.*

Let α be a root of W. There is a unique hyperplane H_α orthogonal to α. We denote by l_α a linear polynomial which has H_α as zero set. Note that l_α is only defined up to a nonzero scalar and that $l_\alpha = l_\beta$ if and only if $\langle\alpha\rangle = \langle\beta\rangle$, i.e. α and β are linearly dependent.

One might use the Jacobian determinant to check the algebraic independence of polynomials:

Lemma 4.27 (Jacobian Criterion, [4, Proposition 3.10])**.** *Let $f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ be algebraically independent polynomials. Then $J(f_1, \dots f_n) \neq 0$.*

Proof. Fix $1 \leq i \leq n$ and consider the polynomials $x_i, f_1, \dots, f_n$. Since $\mathbb{C}(x_1, \dots, x_n)$ has transcendence degree n over $\mathbb{C}$, there is a nontrivial polynomial $h_i \in \mathbb{C}[y_0, \dots, y_n]$ with $h_i(x_i, f_1, \dots, f_n) = 0$. We can chose h_i to be of minimal (total) degree among all those polynomials. We differentiate the above equation with respect to x_k and use the chain rule to obtain:

$$0 = \frac{\partial h_i}{\partial x_k}(x_i, f_1, \dots, f_n) = \sum_{j=1}^{n} \frac{\partial h_i}{\partial y_j}(x_i, f_1, \dots, f_n)\frac{\partial f_j}{\partial x_k} + \frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n)\delta_{ik}$$

where δ_{ik} is the Kronecker delta. We may express these equalities for $1 \leq i, k \leq n$ by an equation in matrix form:

$$\left(\frac{\partial h_i}{\partial y_k}(x_i, f_1, \dots, f_n)\right)\left(\frac{\partial f_i}{\partial x_k}\right) = \left(-\delta_{ik}\frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n)\right)$$

Note that every term on both sides of the equation is a matrix and we use matrix multiplication. The Jacobian matrix on the left-hand side is invertible if the diagonal matrix on the right-hand side is invertible. So it remains to show that $\frac{\partial h_i}{\partial y_0}(x_i, f_1, \dots, f_n) \neq 0$ for all $1 \leq i \leq n$. Since the $f_1, \dots, f_n$ are algebraically independent, the degree of h_i in y_0 is positive. Otherwise, h_i would annihilate $f_1, \dots, f_n$. Therefore, $\frac{\partial h_i}{\partial y_0}$ is nonzero and its degree is smaller than the degree of h_i. So $\frac{\partial h_i}{\partial y_0}$ cannot annihilate $(x_i, f_1, \dots, f_n)$ since the degree of h_i was minimal. The claim follows. $\square$

Proposition 4.28 ([4, Proposition 3.13])**.** *The following hold:*

i) $J = k \prod_{\alpha \in \Phi^+} l_\alpha$ *for some* $k \in \mathbb{C}$

ii) A polynomial $f \in S$ is skew-invariant if and only if it can be written as $f = Jh$, where $h \in S^W$.

iii) $\deg(J) = |\Phi^+| = \ell(w_0)$.

Proof. ad i): Define a mapping $\varphi \colon \mathbb{C}^n \to \mathbb{C}^n$ by

$$\varphi(a_1, \dots, a_n) := (f_1(a_1, \dots, a_n), \dots, f_n(a_1, \dots, a_n))$$

Let $a = (a_1, \dots, a_n) \in H_\alpha$ lay on a hyperplane for some root α. We assume that J does not vanish at a. Then, by the Inverse Function Theorem, φ maps an open neighborhood of a one-to-one onto some open neighborhood of $\varphi(a)$. Let U be such a neighborhood of a. Since a lies on a reflecting hyperplane, U contains a pair of distinct points b and c for which $s_\alpha b = c$. But then, since f_i is an invariant, $f_i(c) = f_i(s_\alpha b) = (s_\alpha \bullet f_i)(b) = f_i(b)$

for all i. Therefore, $\varphi(b) = \varphi(c)$. So φ cannot be invertible on U. This contradicts the Inverse Function Theorem.

This means that J vanishes at a and, since a was chosen arbitrarily, on H_α for all $\alpha \in \Phi^+$. So the zero set $V(l_\alpha) \subset V(J)$. By Hilbert's Nullstellensatz, all l_α divide J. Since the l_α are linear and distinct, they are coprime. So $\prod_{\alpha\in\Phi^+} l_\alpha$ divides J. But the product on the left side has degree $|\Phi^+|$ and by *iii)*, $\deg(J) = |\Phi^+|$. This shows i).

ad ii): Since $\mathbb{C} \cdot S^W = S^W$, we may assume that $J = \prod_{\alpha\in\Phi^+} l_\alpha$. We first show that J is skew-invariant. Let β be a simple root. Then $s_\beta \bullet l_\alpha$ is still linear and nullifies the hyperplane $s_\beta H_\alpha = H_{s_\beta\alpha}$. So $s_\beta \bullet l_\alpha = l_{s_\beta\alpha}$. By the definition of a root system, s_β maps β to its negative and permutes the other roots. Therefore, we obtain

$$s_\beta \bullet \prod_{\alpha\in\Phi^+} l_\alpha = -\prod_{\alpha\in\Phi^+} l_\alpha$$

We can write $g \in W$ as a product of simple reflections: $g = s_{i_1} \dots s_{i_m}$. Then we obtain $g \bullet J = (-1)^m J = \det(g)J$. So J is skew-invariant. In particular, hJ is skew-invariant for every $h \in S^W$.

Now let $f \in S^{W,\det}$ be any skew-invariant polynomial. Let $a \in H_\alpha$. Then

$$f(a) = f(s_\alpha(a)) = (s_\alpha \bullet f)(a) = -f(a)$$

So f vanishes on the zero set $V(l_\alpha)$. Again, by Hilbert's Nullstellensatz, l_α divides f for every $\alpha \in \Phi^+$. Therefore $\prod l_\alpha = J$ divides f. So there is a $h \in S$ with $f = hJ$. Let $g \in W$. Then

$$(g \bullet h)J = \frac{(g \bullet h)(g \bullet J)}{\det(w)} = \frac{g \bullet (hJ)}{\det(w)} = \frac{\det(g) \bullet (hJ)}{\det(g)} = hJ$$

Therefore $g \bullet h = h$. This is the desired factorization.

ad iii): We apply the Leibniz formula to the Jacobian determinant:

$$J = \sum_{\sigma\in\mathrm{Sym}_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} \frac{\partial f_i}{\partial x_{\sigma(i)}} \right)$$

Since the Jacobian is nonzero by Lemma 4.27, at least one of the summands is nonzero. The terms in every product are of degree $d_1 - 1, \dots, d_n - 1$, so $\deg(J) = \sum(d_i - 1) = N$ by Theorem 4.20. The length of the longest element $\ell(w_0) = N$ by [2, Proposition 2.3.2]. $\square$

By applying Proposition 4.28, we may calculate the Hilbert series of the skew invariant subspace:

$$\begin{aligned}\mathrm{Hilb}(S^{W,\det}, q) &= \sum_{d\geq 0} \dim S_d^{W,\det} q^d = \sum_{d\geq 0} \dim S^W_{d-\ell(w_0)} q^d \\ &= q^{\ell(w_0)} \sum_{d\geq 0} \dim S_d^W q^d = q^{\ell(w_0)} \,\mathrm{Hilb}(S^W, q)\end{aligned}$$

We conclude the proof of Theorem 4.1. In Section 4.1, we obtained:

$$\sum_{J\subseteq S}(-1)^{|J|}\operatorname{Hilb}(S^{W_J},q)=\operatorname{Hilb}(S^{W,\mathrm{det}},q) \tag{4.5 revisited}$$

We want to obtain a formula similar to Eq. (3.2):

$$\sum_{J\subsetneq S}(-1)^{|J|}\frac{1}{W_J(q)}=\left(q^{\ell(w_0)}-(-1)^{|S|}\right)\frac{1}{W(q)}$$

To get there, we divide Eq. (4.5) by $\operatorname{Hilb}(S,q)$ and subtract $(-1)^{|J|}\frac{\operatorname{Hilb}(S^W,q)}{\operatorname{Hilb}(S,q)}$ on both sides:

$$\sum_{J\subsetneq S}(-1)^{|J|}\frac{\operatorname{Hilb}(S^{W_J},q)}{\operatorname{Hilb}(S,q)}=\left(q^{\ell(w_0)}-(-1)^{|S|}\right)\frac{\operatorname{Hilb}(S^W,q)}{\operatorname{Hilb}(S,q)}$$

As induction base, if $|S|=0$, the trivial group fixes the whole algebra S, so $\operatorname{Hilb}(S^W,q)=\operatorname{Hilb}(S,q)$. Therefore, both $\frac{1}{W(q)}$ and $\frac{\operatorname{Hilb}(S^W,q)}{\operatorname{Hilb}(S,q)}$ are 1. We saw above that both terms satisfy the same inductive recursion. This means that they are equal. So Theorem 4.1 follows.

5 Poincaré Series for Affine Coxeter Groups

We want to proof the following analogon to Theorem 4.1:

Theorem 5.1. *Let $\tilde{W}$ be an affine Coxeter group with associated finite irreducible Coxeter group W. Then*

$$\tilde{W}(q) = \frac{1}{(1-q)^n} \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q^{d_i-1}} = \mathrm{Hilb}(S,q) \prod_{i=1}^{n} \frac{1-q^{d_i}}{1-q^{d_i-1}} = W(q) \prod_{i=1}^{n} \frac{1}{1-q^{d_i-1}}$$

We will mainly use the same technique as in the affine case to proof this. Dealing with the affine Coxeter group $\tilde{W}$, the Coxeter complex $\Delta(\tilde{W}, \tilde{S})$ is now a tessellation of the affine space $\mathbb{R}^n$. The normal subgroup L of $\tilde{W}$ which corresponds to the coroot lattice Λ translates the lattice in $\mathbb{R}^n$. If we consider the quotient $\Delta := \Delta(\tilde{W}, \tilde{S})/L$, we obtain an n-torus, the so-called Steinberg torus of $\tilde{W}$. The finite subgroup $W < \tilde{W}$ fixes the coroot lattice Λ in $\mathbb{R}^n$. Therefore, its action descends to the quotient Δ. We will apply the Euler-Poincaré Principle to the simplicial homology of the Steinberg torus Δ.

Example 5.2. We already saw the coroots of $\tilde{A}_2$ in Example 2.10:

$$\Phi^\vee = \left\{ \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}, \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix} \right\}$$

If we take the quotient by the action of the induced translation subgroup, we obtain a 2-torus (see Fig. 9).

Proposition 5.3. *The cohomology of Δ is the exterior algebra*

$$H^*(\Delta, \mathbb{C}) = \Lambda(V^*) = \bigoplus_{k=0}^{n} \mathbb{C}^{\binom{n}{k}}$$

and the action of W is the canonical action

$$W \times \Lambda(V^*) \to \Lambda(V^*), \quad (g, f) \mapsto [(v_1, \ldots, v_k) \mapsto f(g^{-1}v_1, \ldots, g^{-1}v_k)]$$

Proof. We may decompose Δ as a product of spheres: $\Delta \cong \bigtimes_{i=1}^{n} S^1$. With the Künneth formula, we obtain by induction that

$$H^*(\Delta, \mathbb{C}) = \sum_{k=0}^{\infty} \sum_{i_1+\ldots+i_n=k} H^{i_1}(S^1, \mathbb{C}) \otimes \ldots \otimes H^{i_n}(S^1, \mathbb{C}) \cong \Lambda[\alpha_1, \ldots, \alpha_n]$$

where $\deg(\alpha_i) = 1$ for all $i = 1, \ldots, n$. Suppose $g \in W$. Then g acts on Δ since g fixes Λ. This induces a natural action on cohomology: $H^*(\Delta, \mathbb{C}) \to H^*(\Delta, \mathbb{C})$ via $\varphi \mapsto \varphi \circ g^{-1}$. □

By the universal coefficient theorem for cohomology and the fact that we are dealing with $\mathbb{C}$-coefficients, the homology and cohomology are isomorphic as vector spaces.

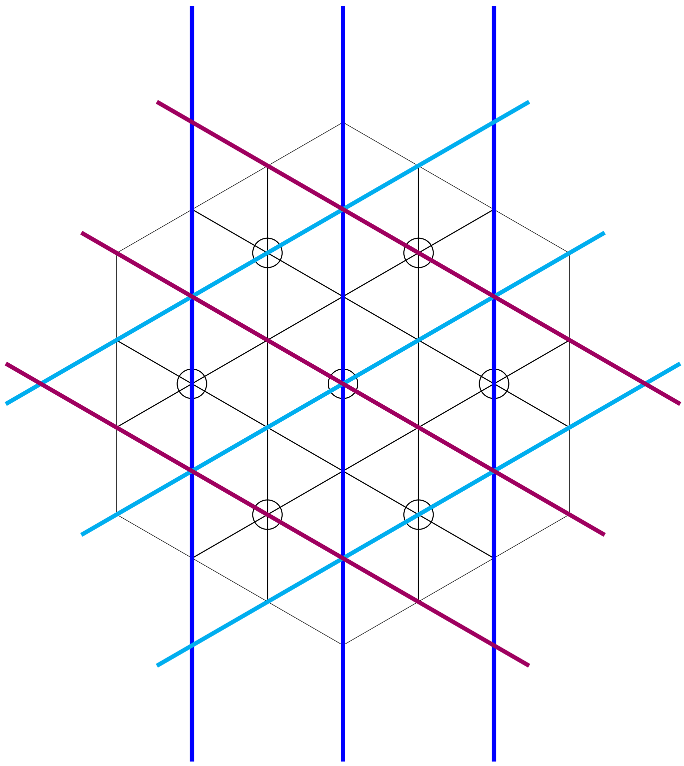


Figure 9: This is the Steinberg torus of $\tilde{A}_2$. The colored reflection-axis are identified under the action of the lattice group.

However, the action of W on homology is dual. We will denote this by $H_*(\Delta, \mathbb{C}) \cong \Lambda(V^*)^*$. So we already identified the right-hand side of the Euler-Poincaré Principle in Theorem 4.5.

As in the finite case, we will the combinatorial structure of Δ. The faces of the Coxeter complex $\Delta(\tilde{W}, \tilde{S})$ correspond to the cosets of the parabolic subgroups $w\tilde{W}_J$ of the affine Coxeter group. We want to understand the action of the finite group W on the quotient.

We consider the projection

$$\pi\colon \tilde{W} \to W$$

arising from $\tilde{W} \cong W \ltimes L$.

Proposition 5.4. *The map π induces an isomorphism between $\tilde{W}_J$ and $\pi(\tilde{W}_J) =: W_J^\pi$. Furthermore,*

$$C_*(\Delta, \mathbb{C}) \cong \bigoplus_{J \subseteq \tilde{S}} \operatorname{Ind}_{W_J^\pi}^W \operatorname{id}$$

Proof. Let's assume that there is $w \in \tilde{W}_J$ such that $\pi(w) = 0$. So $w \in \ker(\pi) = L$. Since L is a lattice, w has infinite order. But then $\tilde{W}_J$ is infinite, which is a contradiction. The first statement follows.

A cell of $\Delta(\tilde{W}, \tilde{S})$ corresponds to a coset in $\tilde{W}/\tilde{W}_J$. The affine group $\tilde{W}$ acts transitively on $\Delta(\tilde{W}, \tilde{S}) \cong \bigcup_{J \subsetneq \tilde{S}} \tilde{W}/\tilde{W}_J$ by left multiplication. If we apply the projection π, we obtain a transitive action of W on $\Delta = \bigcup_{J \subsetneq \tilde{S}} {}_L\backslash \tilde{W}/_{\tilde{W}_J}$. Therefore, we obtain Δ as a quotient $\Delta = W/_{\mathrm{Stab}_W(1)}$. Let $[\tilde{w}] \in \mathrm{Stab}_W(1)$, i.e. $[\tilde{w}][1] = [1]$ in Δ. Then $\tilde{w} \in L\tilde{W}_J$. So there is $\tilde{w}' \in \tilde{W}_J$ such that $[\tilde{w}] = [\tilde{w}']$ in the quotient. But this means $[\tilde{w}] \in W_J^\pi$. Therefore, $\Delta \cong \bigcup_{J \subsetneq \tilde{S}} W/W_J^\pi$. This implies the second statement. □

The new group W_J^π is, in general, not a parabolic subgroup of $\tilde{W}$. But it acts on V as a Coxeter group:

Lemma 5.5. *The group W_J^π acts on V by euclidean reflections.*

Proof. Since $\pi|_{\tilde{W}_J}$ is an isomorphism, W_J^π is generated by the image of J. Let $r \in J$ and $g = \pi(r)$. Then $gr \in \ker(\pi)$, so there is a translation $l \in L$ with $gr = l$. So $g = lr$. Since $W_J^\pi < W$, g fixes the origin. Therefore, $l(v) = v + l(0)$. So l is a translation orthogonal to the hyperplane H_r. But such a composition is itself a reflection. □

This time we apply the Euler-Poincaré Principle to the non-reduced cellular homology. We obtain

$$\sum_{J \subsetneq \tilde{S}} (-1)^{|J|} \operatorname{Ind}_{W_J^\pi}^W(\mathrm{id}) \cong (-1)^n \sum_{p=0}^{n} (-1)^p \Lambda^p(V)$$

The sign correction on the right-hand side again comes from multiplying both sides by $(-1)^{|S|}$. We again apply the homomorphism

$$K_0(F)\colon K_0(\mathbb{C}[W]) \to K_0(\mathbb{C}) \overset{\dim}{\cong} \mathbb{Z}$$

to both sides, multiply by q^d and sum up over all $d \geq 0$. Since we already identified the Hilbert series for finite subgroups before, we obtain

$$\sum_{J\subsetneq\tilde{S}}(-1)^{|J|}\operatorname{Hilb}(S^{W_J^\pi},q)=(-1)^n\sum_{p=0}^{n}(-1)^p\sum_{d\geq 0}\dim\operatorname{Hom}_{\mathbb{C}[W]}(\Lambda^p(V),S_d)q^d \tag{5.1}$$

The following proposition will allow us to examine the right-hand side of Eq. (5.1):

Proposition 5.6. *The $\mathbb{C}$-vectorspaces*

$$\operatorname{Hom}_{\mathbb{C}[W]}(\Lambda^p(V^*)^*,S_d)\cong(S\otimes\Lambda^p(V^*))_d^W$$

are isomorphic, where W acts on $S\otimes\Lambda^p(V^*)$ via $g\bullet(f\otimes\omega)=g\bullet f\otimes g\bullet\omega$.

Proof. W acts on $\operatorname{Hom}_{\mathbb{C}}(\Lambda^p(V^*)^*,S_d)$ via $(g\bullet\phi)(\omega)=g\bullet\Phi(g^{-1}\bullet\omega)$. A linear map $\phi\colon\Lambda^p(V^*)^*\to S_d$ is W-linear if and only if it is a fixed point of the action of W. Therefore, $\operatorname{Hom}_{\mathbb{C}}(\Lambda^p(V^*)^*,S_d)^W=\operatorname{Hom}_{\mathbb{C}[W]}(\Lambda^p(V^*)^*,S_d)$. For $\mathbb{C}$-vector spaces V_1,V_2 with finite dimensional V_1, there is an isomorphism $\operatorname{Hom}_{\mathbb{C}}(V_1,V_2)\cong V_1^*\otimes V_2$: Let $\{b_i\}_{i=1\dots n}$ be a basis for V_1.

$$\Psi\colon V_1^*\otimes V_2\to\operatorname{Hom}_{\mathbb{C}}(V_1,V_2),\quad(\alpha\otimes v)\mapsto\alpha(\_)\cdot v$$

is well defined and an isomorphism since

$$\phi\mapsto\sum_{i=1}^{n}b_i^*\otimes\phi(b_i)$$

is the inverse of the above homomorphism. Let W act on V_1 and V_2. Then Ψ is W-equivariant with respect to the induced actions:

$$\begin{aligned}g\bullet\Psi(\alpha\otimes v)&=g\bullet(\alpha(\_)\cdot v)=g((\alpha\circ g^{-1})(\_)\cdot v)\\&=(\alpha\circ g^{-1})(\_)\cdot g(v)=\Psi(\alpha\circ g^{-1}\otimes g(v))=\Psi(g\bullet(\alpha\otimes v))\end{aligned}$$

for every $\alpha\otimes v\in V_1^*\otimes V_2$ and $g\in W$. So we found a W-equivariant isomorphism which induces an isomorphism on the fixed sets. Therefore, $\operatorname{Hom}_{\mathbb{C}}(\Lambda^p(V^*)^*,S_d)^W\cong(\Lambda^p(V^*)\otimes S_d)^W$. Finally, $(S\otimes\Lambda^p(V^*))_d^W=(S_d\otimes\Lambda^p(V^*))^W$, since $\Lambda^p(V^*)$ admits the trivial grading. This concludes the proof. □

5.1 Invariants of the Exterior Algebra

In the following we examine the structure of the exterior algebra and its invariants. The theorem and proofs are from [8]. We will abbreviate the product $x\wedge y$ by xy.

Definition 5.7. We call the $\mathbb{C}$-linear map $\mathrm{d}\colon S\otimes\Lambda(V^*)\to S\otimes\Lambda(V^*)$ defined by

$$s\otimes(x_{i_1}\dots x_{i_p})\mapsto\sum_{j=1}^{n}\frac{\partial s}{\partial x_j}\otimes x_jx_{i_1}\dots x_{i_p}$$

the *exterior differentiation.*

We can identify the symmetric algebra S with $S \otimes \mathbb{C}$ inside $S \otimes \Lambda(V^*)$. The generators $1 \otimes x_i$ can then be written as $1 \otimes x_i = \mathrm{d}(x_i \otimes 1) = \mathrm{d}x_i$. Therefore, we can write every element of $S \otimes \Lambda^p(V^*)$ in the form

$$\sum_{0 \leq i_1 < \ldots < i_p \leq n} s_{i_1 \ldots i_p} \mathrm{d}x_{i_1} \ldots \mathrm{d}x_{i_p}$$

with $s_{i_1 \ldots i_p} \in S$ since the products of the $\mathrm{d}x_i$ generate $\Lambda^p(V^*)$ and all coefficients can be shifted to the S-term.

Theorem 5.8 ([8, Theorem 3])**.** *Let $f_1, \ldots, f_n$ be the fundamental invariants of S^W. Then every element of $S \otimes \Lambda^p(V^*)$ may be written as*

$$\sum_{0 \leq i_1 < \ldots < i_p \leq n} s_{i_1 \ldots i_p} \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}$$

with $s_{i_1 \ldots i_p} \in S^W$. Therefore, the algebra $(S \otimes \Lambda(V^*))^W$ is an exterior algebra over S^W generated by the differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ together with the unit element 1.

Proof. Let $L = \mathrm{Quot}(S)$ be the quotient field of S.
Then W acts on the L-vector space $L \otimes \Lambda(V^*)$.
First, we show that the the p-forms $\mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}, 1 \leq i_1 < \ldots < i_p \leq n$ are linearly independent. Let $k_{i_1 \ldots i_p} \in L$ for $1 \leq i_1 < \ldots < i_p \leq n$ with

$$\sum_{1 \leq i_1 < \ldots < i_p \leq n} k_{i_1 \ldots i_p} \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p} = 0$$

Claim: $\mathrm{d}f \mathrm{d}f = 0$ for any $f \in S$.
Proof:

$$\begin{aligned}
\mathrm{d}f\mathrm{d}f &= \left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \otimes x_i\right)\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \otimes x_j\right) \quad \text{since } x_i x_i = 0\text{, we obtain} \\
&= \sum_{i<j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \otimes x_i x_j + \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \otimes x_j x_i \\
&= \sum_{i<j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \otimes x_i x_j - \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \otimes x_i x_j = 0
\end{aligned}$$

■

Therefore, if we multiply both sides by $\mathrm{d}f_{p+1} \ldots \mathrm{d}f_n$, all terms involving a $\mathrm{d}f_i$ with $i \leq p$ vanish. We obtain

$$k_{1 \ldots p} \mathrm{d}f_1 \ldots \mathrm{d}f_n = 0$$

We may calculate the product of the $\mathrm{d}f_i$:

$$\begin{aligned}
\mathrm{d}f_1 \ldots \mathrm{d}f_n &= \prod_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} \otimes x_j = \sum_{\sigma \in \mathrm{Sym}(n)} \left(\prod_{i=1}^{n} \frac{\partial f_i}{\partial x_{\sigma(i)}}\right) \otimes \prod_{i=1}^{n} x_{\sigma(i)} \\
&= \left(\sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} \frac{\partial f_i}{\partial x_{\sigma(i)}}\right) \otimes x_1 \ldots x_n = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \otimes x_1 \ldots x_n \\
&= J \mathrm{d}x_1 \ldots \mathrm{d}x_n
\end{aligned}$$

But the product of the $\mathrm{d}x_i$ is non-zero and $J \neq 0$ by Lemma 4.27. Therefore, $k_{1\ldots p} = 0$. Doing the same with the other indices shows that all coefficients are zero and that the p-forms are linearly independent. Since $\dim_L(L \otimes \Lambda^p(V^*)) = \dim_{\mathbb{C}}(\Lambda^p(V^*)) = \binom{n}{p}$, we found a basis of $L \otimes \Lambda^p(V^*)$. Let $\omega \in (S \otimes \Lambda^p(V^*))^W$ be an invariant p-form. Then there are coefficients $k_{i_1\ldots i_p} \in L$ with

$$\omega = \sum_{1 \leq i_1 < \ldots < i_p \leq n} k_{i_1\ldots i_p} \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}$$

Claim: $\mathrm{d}f_i$ is W-invariant. We will proof this later.
If we apply the averaging operator $\pi|_{S\otimes\Lambda^p(V^*)}$ to the equation, we obtain

$$|W|\omega = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \left(\sum_{g \in W} g k_{i_1\ldots i_p} \right) \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}$$

since ω and the $\mathrm{d}f_i$ are invariant.
Rewriting the coefficients and dividing by $|W|$, we obtain

$$\omega = \sum_{1 \leq i_1 < \ldots < i_p \leq n} \frac{r_{i_1\ldots i_p}}{s_{i_1\ldots i_p}} \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}$$

with $r_{i_1\ldots i_p}, s_{i_1\ldots i_p} \in S$ and the quotient being invariant. Like before, we multiply both sides by $\mathrm{d}f_{p+1} \ldots \mathrm{d}f_n$. This yields

$$\omega \mathrm{d}f_{p+1} \ldots \mathrm{d}f_n = \frac{r_{1\ldots p}}{s_{1\ldots p}} \mathrm{d}f_1 \ldots \mathrm{d}f_n = \frac{r_{1\ldots p}}{s_{1\ldots p}} J \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

and we may express $\omega \mathrm{d}f_{p+1} \ldots \mathrm{d}f_n \in S \otimes \Lambda^n(V^*)$ as $\omega \mathrm{d}f_{p+1} \ldots \mathrm{d}f_n = u_{1\ldots p} \mathrm{d}x_1 \ldots \mathrm{d}x_n$ with $u_{1\ldots p} \in S$. Substituting this into the formula, we obtain

$$\frac{u_{1\ldots p}}{J} \mathrm{d}x_1 \ldots \mathrm{d}x_n = \frac{r_{1\ldots p}}{s_{1\ldots p}} \mathrm{d}x_1 \ldots \mathrm{d}x_n$$

Since $\frac{r_{1\ldots p}}{s_{1\ldots p}}$ is invariant, $\frac{u_{1\ldots p}}{J}$ has to be invariant, too. Therefore, $u_{1\ldots p}$ has to be skew-invariant since J is skew-invariant by Proposition 4.28. But then, by Proposition 4.28, J divides $u_{1\ldots p}$ in S, so the quotient is in S. Thus $\frac{r_{1\ldots p}}{s_{1\ldots p}} \in S$ as well and, being invariant, in S^W. Analogously, we can show that all coefficients $\frac{r_{i_1\ldots i_p}}{s_{i_1\ldots i_p}}$ of ω lie in S^W. Overall, $(S \otimes \Lambda(V^*))^W$ is generated as an exterior algebra over S^W by the differentials $\mathrm{d}f_1, \ldots, \mathrm{d}f_n$ together with the unit element 1.

To conclude the proof we show the above claim.
Proof: To proof that for every fundamental invariant f the differential $\mathrm{d}f$ is W-invariant, we show that d and the action of W commute. Then, since f is W-invariant, $\mathrm{d}f$ is invariant, too.
Let $g \in W$. Since g respects the degree of elements of S, we can write $gx_i = \sum_{k=1}^{n} a_k^i x_k$

with $a_k^i \in \mathbb{C}$. Also, $g1 = 1$.

$$\begin{aligned}
\mathrm{d}g(x_i \otimes 1) &= \sum_{j=1}^{n} \frac{\partial g x_i}{\partial x_j} \otimes x_j = \sum_{j=1}^{n} \sum_{k=1}^{n} a_k^i \frac{\partial x_k}{\partial x_j} \otimes x_j \\
&= \sum_{k=1}^{n} a_k^i \otimes x_k = 1 \otimes g x_i = g(1 \otimes x_i) = g\left(\sum_{j=1}^{n} \frac{\partial x_i}{\partial x_j} \otimes x_j\right) \\
&= g\mathrm{d}(x_i \otimes 1)
\end{aligned}$$

Another even easier calculation shows that $\mathrm{d}(st \otimes \omega) = \mathrm{d}(s \otimes 1)(t \otimes 1) + (t \otimes 1)\mathrm{d}(s \otimes 1)$ for $s, t \in S$. Using this, we can show that $\mathrm{d}g(s \otimes 1) = g\mathrm{d}(s \otimes 1)$ for all homogenous $s \in S$ by induction on the degree of s. Since g acts by homomorphisms, this generalizes to arbitrary (non-homogenous) $s \in S$. Note that $\mathrm{d}(s \otimes \omega) = \mathrm{d}((s \otimes 1)(1 \otimes \omega))$ for $s \in S$ and $\omega \in \Lambda^i(V^*)$. Since $\mathrm{d}(1 \otimes \omega) = 0$, we obtain

$$g\mathrm{d}(s \otimes \omega) = g(\mathrm{d}(s \otimes 1)(1 \otimes \omega)) = \mathrm{d}((gs \otimes 1))(1 \otimes g\omega) = \mathrm{d}(gs \otimes g\omega) = \mathrm{d}g(s \otimes \omega)$$

for arbitrary $s \otimes \omega \in S \otimes \Lambda^i(V^*)$. This concludes the proof of the claim. ■ □

5.2 Proof of the Theorem

For the Hilbert series, we now want to calculate the term

$$\sum_{p=0}^{n} (-1)^p \operatorname{Hilb}((S \otimes \Lambda^p(V^*))^W, q)$$

With the above Theorem 5.8 we may describe the tensor product as follows:

$$(S \otimes \Lambda^p(V^*))_d^W = \bigoplus_{1 \leq i_1 < \ldots < i_p \leq n} S^W_{d - \sum_{j=1}^{p} d_{i_j} - p} \cdot \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p} \text{ for } p > 0$$

since $d = \deg(s_{i_1 \ldots i_p} \mathrm{d}f_{i_1} \ldots \mathrm{d}f_{i_p}) = \deg(s_{i_1 \ldots i_p}) + d_{i_1} - 1 + \ldots + d_{i_p} - 1$.

We calculate the Hilbert series for $p > 0$:

$$\begin{aligned}
\operatorname{Hilb}((S \otimes \Lambda^p(V^*))^W, q) &= \sum_{d \geq 0} \dim \bigoplus_{1 \leq i_1 < \ldots < i_p \leq n} S^W_{d - \sum_{j=1}^{p} d_{i_j} - p} \cdot q^d \\
&= \sum_{1 \leq i_1 < \ldots < i_p \leq n} \underbrace{\left(\sum_{d \geq 0} \dim S^W_{d - \sum_{j=1}^{p} d_{i_j} - p} \cdot q^{d - \sum_{j=1}^{p} d_{i_j} - p} \right)}_{= \operatorname{Hilb}(S^W, q)} q^{\sum_{j=1}^{p} d_{i_j} - p}
\end{aligned}$$

For $p = 0$ we simply obtain $(S \otimes \Lambda^0(V^*))_d^W \cong (S \otimes \mathbb{C})_d^W \cong S_d^W$ and we may describe the Hilbert series of S_d^W as in Proposition 4.24. If we now take the sum over all $p \geq 0$, we obtain

$$
\begin{aligned}
&\sum_{p=0}^{n}(-1)^p \operatorname{Hilb}((S\otimes \Lambda^p(V^*))^W_d, q)\\
&\qquad = \left(1+\sum_{p=1}^{n}(-1)^p \sum_{1\le i_1<\ldots<i_p\le n} q^{\sum_{j=1}^{p} d_{i_j}-p}\right)\left(\prod_{k=1}^{n}\frac{1}{1-q^{d_k}}\right)\\
&\qquad = \left(1+\sum_{p=1}^{n}(-1)^p \sum_{1\le i_1<\ldots<i_p\le n} q^{d_{i_1}-1}\cdot\ldots\cdot q^{d_{i_p}-1}\right)\left(\prod_{k=1}^{n}\frac{1}{1-q^{d_k}}\right)\\
&\qquad = \left(\prod_{k=1}^{n}1-q^{d_k-1}\right)\left(\prod_{k=1}^{n}\frac{1}{1-q^{d_k}}\right)\\
&\qquad = \prod_{k=1}^{n}\frac{1-q^{d_k-1}}{1-q^{d_k}}
\end{aligned}
$$

If we apply this to Eq. (5.1) and divide both sides by $\operatorname{Hilb}(S,q)$, we obtain

$$\sum_{J\subsetneq\tilde{S}}(-1)^{|J|}\frac{\operatorname{Hilb}(S^{W_J^\pi},q)}{\operatorname{Hilb}(S,q)} = \frac{(-1)^n}{\operatorname{Hilb}(S,q)}\prod_{k=1}^{n}\frac{1-q^{d_k-1}}{1-q^{d_k}}$$

Since W_J^π is a Coxeter group, we can apply Theorem 4.1 to it. We do this on the right-hand side as well. This leads to:

$$\sum_{J\subsetneq\tilde{S}}(-1)^{|J|}\frac{1}{W_J^\pi(q)} = \frac{(-1)^n}{W(q)}\prod_{k=1}^{n}1-q^{d_k-1} \tag{5.2}$$

By Proposition 5.4, the groups $\tilde{W}_J$ and W_J^π are isomorphic. So their Poincaré polynomials coincide: $\tilde{W}_J(q)=W_J^\pi(q)$. We take a look on Eq. (3.2) again, but this time we apply it to the affine Coxeter group $\tilde{W}$:

$$\sum_{J\subsetneq\tilde{S}}(-1)^{|J|}\frac{1}{\tilde{W}_J(q)} = -(-1)^{|\tilde{S}|}\frac{1}{W(q)} \tag{3.2 revisited}$$

Applying this to Eq. (5.2), we obtain:

$$\frac{-(-1)^{|\tilde{S}|}}{\tilde{W}(q)} = \frac{(-1)^n}{W(q)}\prod_{k=1}^{n}1-q^{d_k-1}$$

We multiply by $(-1)^n$ and take the reciprocal:

$$\tilde{W}(q) = W(q)\prod_{k=1}^{n}\frac{1}{1-q^{d_k-1}}$$

This is the desired equation in Theorem 5.1.

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