Contents

Abstract

For a finite real reflection group W and a W-orbit $\mathcal{O}$ of flats in its reflection arrangement – or equivalently a conjugacy class of its parabolic subgroups – we introduce a statistic $\mathrm{noninv}_{\mathcal{O}}(w)$ on w in W that counts the number of "$\mathcal{O}$-noninversions" of w. This generalizes the classical (non-)inversion statistic for permutations w in the symmetric group $\mathfrak{S}_n$. We then study the operator $\nu_{\mathcal{O}}$ of right-multiplication within the group algebra $\mathbb{C}W$ by the element that has $\mathrm{noninv}_{\mathcal{O}}(w)$ as its coefficient on w.

We reinterpret $\nu_{\mathcal{O}}$ geometrically in terms of the arrangement of reflecting hyperplanes for W, and more generally, for any real arrangement of linear hyperplanes. At this level of generality, one finds that, after appropriate scaling, $\nu_{\mathcal{O}}$ corresponds to a Markov chain on the chambers of the arrangement. We show that $\nu_{\mathcal{O}}$ is self-adjoint and positive semidefinite, via two explicit factorizations into a symmetrized form $\pi^t\pi$. In one such factorization, the matrix π is a generalization of the projection of a simplex onto the *linear ordering polytope* from the theory of social choice.

In the other factorization of $\nu_{\mathcal{O}}$ as $\pi^t\pi$, the matrix π is the transition matrix for one of the well-studied *Bidigare-Hanlon-Rockmore random walks* on the chambers of an arrangement. We study closely the example of the family of operators $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$, corresponding to the case where $\mathcal{O}$ is the conjugacy classes of Young subgroups in $W = \mathfrak{S}_n$ of type $(k, 1^{n-k})$. The $k = n-1$ special case within this family is the operator $\nu_{(n-1,1)}$ corresponding to *random-to-random shuffling*, factoring as $\pi^t\pi$ where π corresponds to *random-to-top shuffling*. We show in a purely enumerative fashion that this family of operators $\{\nu_{(k,1^{n-k})}\}$ pairwise commute. We furthermore conjecture that they have integer spectrum, generalizing a conjecture of Uyemura-Reyes for the case $k = n-1$. Although we do not know their complete simultaneous eigenspace decomposition, we give a coarser block-diagonalization of these operators, along with explicit descriptions of the $\mathbb{C}W$-module structure on each block.

We further use representation theory to show that if $\mathcal{O}$ is a conjugacy class of *rank one* parabolics in W, multiplication by $\nu_{\mathcal{O}}$ has integer spectrum; as a very special case, this holds for the matrix $(\mathrm{inv}(\sigma\tau^{-1}))_{\sigma,\tau\in\mathfrak{S}_n}$. The proof uncovers a fact

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of independent interest. Let W be an irreducible finite reflection group and s any reflection in W, with reflecting hyperplane H. Then the $\{\pm 1\}$-valued character χ of the centralizer subgroup $Z_W(s)$ given by its action on the line $H^\perp$ has the property that χ is multiplicity-free when induced up to W. In other words, $(W, Z_W(s), \chi)$ forms a *twisted Gelfand pair*.

We also closely study the example of the family of operators

$$\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,\lfloor \frac{n}{2} \rfloor}$$

corresponding to the case where $\mathcal{O}$ is the conjugacy classes of Young subgroups in $W = \mathfrak{S}_n$ of type $(2^k, 1^{n-2k})$. Here the construction of a *Gelfand model* for $\mathfrak{S}_n$ shows both that these operators pairwise commute, and that they have integer spectrum.

We conjecture that, apart from these two commuting families $\{\nu_{(k,1^{n-k})}\}$ and $\{\nu_{(2^k,1^{n-2k})}\}$ and trivial cases, no other pair of operators of the form $\nu_{\mathcal{O}}$ commutes for $W = \mathfrak{S}_n$.

CHAPTER I

Introduction

This work grew from the desire to understand why a certain family of combinatorial matrices were pairwise-commuting and had only integer eigenvalues. We start by describing them.

1. The original family of matrices

The matrices are constructed from certain statistics on the symmetric group $W = \mathfrak{S}_n$ on n letters. Given a permutation w in W, define the *k-noninversion number* [1] $\mathrm{noninv}_k(w)$ to be the number of k-element subsets $\{i_1, \ldots, i_k\}$ with $1 \leq i_1 < \cdots < i_k \leq n$ for which $w_{i_1} < \cdots < w_{i_k}$. In the literature on *permutation patterns*, one might call $\mathrm{noninv}_k(w)$ the number of occurrences of the permutation pattern $12 \cdots k$. Alternately, $\mathrm{noninv}_k(w)$ is the number of *increasing subsequences* of length k occurring in the word $w = w_1 w_2 \cdots w_n$.

From this statistic $\mathrm{noninv}_k(-)$ on the group $W = \mathfrak{S}_n$, create a matrix $\nu_{(k,1^{n-k})}$ in $\mathbb{Z}^{|W| \times |W|}$, having rows and columns indexed by the permutations w in W, and whose (u, v)-entry is $\mathrm{noninv}_k(v^{-1}u)$. One of the original mysteries that began this project was the following result, now proven in Chapter VI.

THEOREM 1.1. *The operators from the family $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$ pairwise commute.*

It is not hard to see (and will be shown in Proposition II.2.2) that one can factor each of these matrices $\nu_{(k,1^{n-k})} = \pi^T \pi$ for certain other integer (even 0/1) matrices π. Therefore, each $\nu_{(k,1^{n-k})}$ is symmetric positive semidefinite, and hence diagonalizable with only real non-negative eigenvalues. Theorem I.1.1 asserts that they form a commuting family, and hence can be simultaneously diagonalized. The following conjecture also motivated this project, but has seen only partial progress here.

CONJECTURE 1.2. *The operators $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$ have only integer eigenvalues.*

In the special case $k = n - 1$, this matrix $\nu_{(n-1,1)}$ was studied already in the Stanford University PhD thesis of Jay-Calvin Uyemura-Reyes [**76**]. Uyemura-Reyes examined a certain random walk on W called the *random-to-random shuffling* operator, whose Markov matrix is a rescaling of $\nu_{(n-1,1)}$. He was interested in its eigenvalues in order to investigate the rate of convergence of this random walk to the uniform distribution on W. He was surprised to discover empirically, and

[1] The terminology comes from the case $k = 2$, where $\mathrm{noninv}_2(w)$ counts the pairs (i, j) with $1 \leq i < j \leq n$ that index a *noninversion* in a permutation w in W, meaning that $w_i < w_j$.

conjectured, that $\nu_{(n-1,1)}$ has only integer eigenvalues[2]. This was one of many unexpected connections encountered during the work on this project, since a question from computer science (see §III.5) independently led to our Theorem I.1.1 and Conjecture I.1.2.

2. Using the W-action

One can readily check that the matrix $\nu_{(k,1^{n-k})}$ in $\mathbb{Z}^{|W|\times|W|}$ represents multiplication on the right within the group algebra $\mathbb{Z}W$ by the following element of $\mathbb{Z}W$ (also denoted $\nu_{(k,1^{n-k})}$, by an abuse of notation):

$$\nu_{(k,1^{n-k})} := \sum_{w\in W} \mathrm{noninv}_k(w)\cdot w.$$

Consequently, the action of $\nu_{(k,1^{n-k})}$ commutes with the left-regular action of $\mathbb{R}W$ on itself, and the (simultaneous) eigenspaces of the matrices $\nu_{(k,1^{n-k})}$ are all representations of W. This extra structure will prove to be extremely useful in the rest of the work.

In fact, Uyemura-Reyes [**76**] conjectured descriptions for the $\mathbb{R}W$-irreducible decompositions of certain of the eigenspaces of $\nu_{(n-1,1)}$, and was able to prove some of these conjectures in special cases. Furthermore, he reported [**76**, §5.2.3] an observation of R. Stong noting that one of the factorizations of $\nu_{(n-1,1)} = \pi^T\pi$ mentioned earlier can be obtained by letting π be the well-studied *random-to-top* shuffling operator on W. These operators are one example from a family of very well-behaved random walks on W that were introduced by Bidigare, Hanlon, and Rockmore, BHR for short, in [**10**] and [**11**]. These authors showed that the BHR *random walks* have very simply predictable integer eigenvalues, and the W-action on their eigenspaces are also well-described.

We exploit this connection further, as follows. First, we will show (in Proposition II.6.1 and Corollary IV.2.2) that more generally one has a factorization $\nu_{(k,1^{n-k})} = \pi^T\pi$ in which π is another family of BHR random walks. Second, we will use the fact that this implies $\ker \nu_{(k,1^{n-k})} = \ker \pi$, along with Theorem I.1.1, to obtain a W-equivariant filtration of $\mathbb{R}W$ that is preserved by each $\nu_{(k,1^{n-k})}$, with a complete description of the $\mathbb{R}W$-structure on each filtration factor. This has consequences (see e.g. §VI.12) for the $\mathbb{R}W$-module structure on the simultaneous eigenspaces of the commuting family of $\nu_{(k,1^{n-k})}$.

3. An eigenvalue integrality principle

Another way in which we will exploit the W-action comes from a simple but powerful *eigenvalue integrality principle* for combinatorial operators. We record it here, as we will use it extensively later.

To state it, recall that for a finite group W, when one considers representations of W over fields $\mathbb{K}$ of characteristic zero, any finite-dimensional $\mathbb{K}W$-module U is semisimple, that is, it can be decomposed as a direct sum of simple $\mathbb{K}W$-modules.

[2] In addition, the thesis [**76**, p 152-153] mentions other shuffling operators that have "*eigenvalues with surprising structure*". We have been informed by Persi Diaconis, the advisor of Uyemura-Reyes, that among others this refers to computational experiments on shuffling operators that are convex combinations with rational coefficients of the shuffling operators corresponding to $\nu_{(k,1^{n-k})}$. Uyemura-Reyes observed integral spectrum for small n after suitable scaling. Clearly, using Theorem I.1.1 this fact for general n is implied by Conjecture I.1.2.

When considering field extensions $\mathbb{K}' \supset \mathbb{K}$, the simple $\mathbb{K}W$-modules may or may not split further when extended to $\mathbb{K}'W$-modules; one says that a simple $\mathbb{K}W$-module is *absolutely irreducible* if it remains irreducible as a $\mathbb{K}'W$-module for any extension $\mathbb{K}'$ of $\mathbb{K}$. Given any finite group W, a *splitting field* (see [**20**, Chapter X]) for W over $\mathbb{Q}$ is a field extension $\mathbb{K}$ of $\mathbb{Q}$ such that every simple $\mathbb{K}W$-module is absolutely irreducible. Equivalently, $\mathbb{K}$ is a splitting field of W over $\mathbb{Q}$ if and only if every irreducible matrix representation of W over $\mathbb{Q}$ is realizable with entries in $\mathbb{K}$ [**20**, Theorem 70.3]. For such a field $\mathbb{K}$, the simple $\mathbb{K}W$-modules biject with the simple $\mathbb{C}W$-modules, that is, the set of simple $\mathbb{K}W$-modules when extended to $\mathbb{C}W$-modules gives exactly the set of simple $\mathbb{C}W$-modules corresponding to the complex irreducible W-characters χ. For finite W the splitting field $\mathbb{K}$ over $\mathbb{Q}$ can always be chosen to be a finite, and hence algebraic, extension of $\mathbb{Q}$ [**20**, Theorem 70.23]. If W is a reflection group, then there is a unique minimal extension $\mathbb{K}$ of $\mathbb{Q}$ such that $\mathbb{K}$ is a splitting field for W in characteristic 0 (see [**9**, Theorem 0.2], [**6**, Theorem 1], and [**39**, §1.7]).

Denote by $\mathfrak{o}$ the ring of integers within the unique minimal splitting field $\mathbb{K}$ for the reflection group W in characteristic 0; that is, the elements of $\mathbb{K}$ that are roots of monic polynomials with coefficients in $\mathbb{Z}$. An important example occurs when W is a crystallographic reflection group or equivalently a Weyl group. Here it is known that one can take as a splitting field $\mathbb{K} = \mathbb{Q}$ itself (see [**62**, Corollary 1.15]), and hence that $\mathfrak{o} = \mathbb{Z}$.

Proposition 3.1 (Eigenvalue integrality principle). *Let W be a finite group acting in a $\mathbb{Z}$-linear fashion on $\mathbb{Z}^n$ and let $\mathbb{K}$ be a splitting field of W in characteristic 0. Further let $A : \mathbb{Z}^n \to \mathbb{Z}^n$ be a $\mathbb{Z}$-linear operator that commutes with the action W. Extend the action of A and of W to $\mathbb{K}^n$*

Then for any subspace $U \subseteq \mathbb{K}^n$ which is stable under both A and W, and on which W acts without multiplicity (that is, each simple $\mathbb{K}W$-module occurs at most once), all eigenvalues of the restriction of A to U lie in the ring of integers $\mathfrak{o}$ of $\mathbb{K}$.

In particular, if W is a Weyl group these eigenvalues of A lie in $\mathbb{Z}$.

Proof. An eigenvalue of A is a root of its characteristic polynomial $\det(t \cdot \mathrm{I}_{\mathbb{K}^n} - A)$, a monic polynomial with $\mathbb{Z}$ coefficients. As usual $\mathrm{I}_{\mathbb{K}^n}$ denotes the identity matrix. Hence, it is enough to show that the eigenvalues of A acting on the $\mathbb{K}$-subspace U all lie in $\mathbb{K}$.

Because $\mathbb{K}$ is a splitting field for W, one has an isotypic $\mathbb{K}W$-module decomposition $U = \bigoplus_\chi U^\chi$ in which the sum is over the irreducible characters χ of W. Since A commutes with the W-action, it preserves this decomposition. The assumption that U is multiplicity-free says each U^χ is a single copy of a simple $\mathbb{K}W$-module. Schur's Lemma asserts that, on extending $\mathbb{K}$ to its algebraic closure, A must act on each U^χ by some scalar λ_χ. However, λ_χ must lie in $\mathbb{K}$ since A acts $\mathbb{K}$-linearly. Thus, the isotypic decomposition diagonalizes the action of A on U, and all its eigenvalues lie in $\mathbb{K}$ (and hence in $\mathfrak{o}$). $\square$

4. A broader context, with more surprises

Some of the initial surprises led us to consider a more general family of operators, in the context of *real reflection groups* W, leading to even more surprises. We describe some of these briefly and informally here, indicating where they are discussed later.

Let W be a finite real reflection group, acting on an $\mathbb{R}$-vector space V, with set of reflecting hyperplanes $\mathcal{A}$, and and let $\mathcal{L}$ be the (partially-ordered) set of subspaces X that arise as intersections of hyperplanes from some subset of $\mathcal{A}$. The hyperplanes in $\mathcal{A}$ dissect V into connected components called *chambers*, and the set $\mathcal{C}$ of all chambers carries a simply-transitive action of W. Thus, if $\mathbb{1}$ denotes the identity element of W, then one can choose an identity chamber $c_{\mathbb{1}}$ and an indexing of the chambers $\mathcal{C} = \{c_w := w(c_{\mathbb{1}})\}_{w \in W}$.

Given a W-orbit $\mathcal{O}$ of intersection subspaces, define $\mathrm{noninv}_{\mathcal{O}}(w)$ to be the number of subspaces X in $\mathcal{O}$ for which the two chambers c_w and $c_{\mathbb{1}}$ lie on the same side of every hyperplane $H \supseteq X$. In the case where $W = \mathfrak{S}_n$ acts on $V = \mathbb{R}^n$ by permuting coordinates, if one takes $\mathcal{O}$ to be the W-orbit of intersection subspaces of the form $x_{i_1} = \cdots = x_{i_k}$, one finds that $\mathrm{noninv}_{\mathcal{O}}(w) = \mathrm{noninv}_k(w)$.

Consider the operator $\nu_{\mathcal{O}}$ representing multiplication by $\sum_{w \in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot w$ within $\mathbb{Z}W$ or $\mathbb{R}W$. As before, one can show that $\nu_{\mathcal{O}} = \pi^T \pi$ for certain integer matrices π, and again one such choice of a matrix π is the transition matrix for a BHR random walk on W. In this general context, but when $\mathcal{O}$ is taken to be a W-orbit of codimension one subspaces (that is, *hyperplanes*) one encounters the following surprise, proven in §III.3.

THEOREM 4.1. *For any finite irreducible real reflection group W, and any (transitive) W-orbit $\mathcal{O}$ of hyperplanes, the matrix $\nu_{\mathcal{O}}$ has all its eigenvalues within the ring of integers of the unique minimal splitting field for W. In particular, when W is crystallographic, these eigenvalues all lie in $\mathbb{Z}$.*

This result will follow from applying the integrality principle (Proposition I.3.1) together with the discovery of the following (apparently) new family of twisted Gelfand pairs. This is proven in §III.2.

THEOREM 4.2. *Let W be a finite irreducible real reflection group and let H be the reflecting hyperplane for a reflection $s \in W$.*

Then the linear character χ of the W-centralizer $Z_W(s)$ given by its action on the line V/H or $H^\perp$ has a multiplicity-free induced W-representation $\mathrm{Ind}_{Z_W(s)}^W \chi$.

We mention a further surprise proven via Proposition I.3.1 and some standard representation theory of the symmetric group. With $W = \mathfrak{S}_n$ acting on $V = \mathbb{R}^n$ by permuting coordinates, for each $k = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ consider the W-orbit $\mathcal{O}$ of codimension k intersection subspaces of the form

$$\{x_{i_1} = x_{i_2}\} \cap \{x_{i_3} = x_{i_4}\} \cap \cdots \cap \{x_{i_{2k-1}} = x_{i_{2k}}\},$$

where $\{i_1, i_2\}, \ldots, \{i_{2k-1}, i_{2k}\}$ are k pairwise disjoint sets of cardinality two. Let $\nu_{(2^k, 1^{n-2k})}$ denote the operator $\nu_{\mathcal{O}}$ for this orbit $\mathcal{O}$.

THEOREM 4.3. *The operators from the family $\{\nu_{(2^k,1^{n-2k})}\}_{k=1,2\ldots,\lfloor \frac{n}{2} \rfloor}$ pairwise commute, and have only integer eigenvalues.*

Interestingly, the proof of this given in Chapter V tells us that the non-kernel eigenspaces V_λ in the simultaneous eigenspace decomposition for $\{\nu_{(2^k,1^{n-2k})}\}$ should be indexed by all number partitions λ of n, and that V_λ carries the irreducible $\mathbb{R}\mathfrak{S}_n$-module indexed by λ, but it tells us very little about the integer eigenvalue for each $\nu_{(2^k,1^{n-2k})}$ acting on V_λ.

More generally, we can define an operator ν_λ for each partition $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of n by considering the $\mathfrak{S}_n$-orbit of the subspace

$$\{x_1 = x_2 = \cdots = x_{\lambda_1}\} \cap \{x_{\lambda_1+1} = x_{\lambda_1+2} = \cdots = x_{\lambda_1+\lambda_2}\} \\ \cap \{x_{\lambda_1+\lambda_2+1} = x_{\lambda_1+\lambda_2+2} = \cdots = x_{\lambda_1+\lambda_2+\lambda_3}\} \cap \cdots .$$

In light of Theorem I.1.1 and Theorem I.4.3, it is natural to ask whether these operators commute and have integer eigenvalues. Our computer explorations led us to conjecture the following, which we verified for $1 \leq n \leq 6$.

Conjecture 4.4. *Let λ and γ be distinct partitions of n, both different from (1^n) and (n). The operators ν_λ and ν_γ commute if and only if they* both *belong to $\left\{\nu_{(k,1^{n-k})} : 1 < k < n\right\}$ or $\left\{\nu_{(2^k,1^{n-2k})} : 0 < k \leq \lfloor \frac{n}{2} \rfloor\right\}$. Furthermore, ν_λ has integer eigenvalues if and only if ν_λ belongs to one of these two families.*

5. Outline of the paper

We will define and study the operators $\nu_\mathcal{O}$ at various levels of generality.

(H) For hyperplane arrangements $\mathcal{A}$ (see §II.1).
(L) For hyperplane arrangements invariant under a (linear) action of a finite group W (see §II.3).
(R) For reflection arrangements corresponding to a real reflection group W (see §II.5).
(W) For crystallographic reflection groups or, equivalently, Weyl groups W.
(S) For the symmetric group $\mathfrak{S}_n$ (see Chapter V and Chapter VI).

Different properties of the operators $\nu_\mathcal{O}$ manifest themselves at different levels of generality.

In Chapter II we define $\nu_\mathcal{O}$ as in (H) for all hyperplane arrangements $\mathcal{A}$, and prove semidefiniteness by exhibiting a "square root" π for which $\nu_\mathcal{O} = \pi^T\pi$. We also explain how $\nu_\mathcal{O}$ interacts with any finite group W acting on $\mathcal{A}$ as in (L). We then particularize to case (R), and exhibit a second square root π that will turn out to be the transition matrix for a certain BHR random walk. The rest of this chapter contains some general reductions and principles, including a reduction to eigenspaces and an analysis of the Perron-Frobenius eigenspace.

In Chapter III, we discuss and prove Theorem I.4.2 and deduce from it Theorem I.4.1. We also discuss some interesting conjectures that it suggests, and a relation to *linear ordering polytopes*.

In Chapter IV we review some of the theory of BHR random walks, with features at different levels of generality. In particular, some of the W-equivariant theory of the BHR random walks presented here have neither been stated nor proven in the literature in the generality required for the later results, so these are discussed in full detail here. This equivariant theory extends to a commuting $\mathbb{Z}_2$-action coming from the scalar multiplication operator -1. Whenever W does not already contain this scalar -1, the $W \times \mathbb{Z}_2$-equivariant picture provides extra structure in analyzing the eigenspaces of $\nu_\mathcal{O}$. This chapter concludes with some useful reformulations of the representations that make up the eigenspaces, which are closely related to *Whitney cohomology*, *free Lie algebras* and *higher Lie characters*.

The remainder of the paper focuses on the case (S), that is, reflection arrangements of type A_{n-1}, where $W = \mathfrak{S}_n$.

In Chapter V we discuss $\nu_{(2^k,1^{n-2k})}$ and prove Theorem I.4.3. As mentioned earlier, although the proof predicts the $\mathbb{R}W$-module structure on the simultaneous eigenspaces, it does not predict the eigenvalues themselves.

In Chapter VI we discuss the original family of matrices $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$, starting with a proof of Theorem I.1.1. We then proceed to examine their simultaneous eigenspaces. Here one can take advantage of a block-diagonalization that comes from a certain W-equivariant filtration respected by these operators. One can also fully analyze the irreducible decomposition of the filtration factors using a close connection with *derangements*, *desarrangements* and the homology of the *complex of injective words*. We review this material, including some unpublished results [**50**] of the first author and M. Wachs, and extend this to the $W \times \mathbb{Z}_2$-equivariant picture mentioned earlier. Some of this is used to piggyback on Uyemura-Reyes's construction of the eigenvectors of $\nu_{(n-1,1)}$ within a certain isotypic component; we show with no extra work that these are simultaneous eigenvectors for all of the $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$.

CHAPTER II

Defining the operators

1. Hyperplane arrangements and definition of $\nu_{\mathcal{O}}$

We review here some standard notions for arrangements of hyperplanes; good references are [**48**] and [**66**].

A (*central*) *hyperplane arrangement* $\mathcal{A}$ in a d-dimension real vector space V will here mean a finite collection $\{H\}_{H\in\mathcal{A}}$ of codimension one $\mathbb{R}$-linear subspaces, that is, *hyperplanes* passing through the origin.

An intersection $X = H_{i_1} \cap \cdots \cap H_{i_m}$ of some subset of the hyperplanes will be called an *intersection subspace*. The collection of all intersection subspaces, partially ordered by reverse inclusion, is called the *intersection lattice* $\mathcal{L} = \mathcal{L}(\mathcal{A})$. This turns out to be a *geometric lattice* (= atomic, upper semimodular lattice), ranked by the rank function $r(x) = \dim V/X$ with bottom element $\hat{0} = V := \bigcap_{H\in\emptyset} H$, and a top element $\hat{1} = \bigcap_{H\in\mathcal{A}} H$ (see [**77**, Section 3.2.3]). We will sometimes assume that $\mathcal{A}$ is *essential*, meaning that $\bigcap_{H\in\mathcal{A}} H = \{0\}$, so that $\mathcal{L}$ has rank $d = \dim(V)$.

For each X in $\mathcal{L}$, we will consider the *localized arrangement*

$$\mathcal{A}/X := \{H/X : H \in \mathcal{A}, H \supset X\}$$

inside the quotient space V/X, having intersection lattice $\mathcal{L}(\mathcal{A}/X) \cong [V, X]$. Here for elements $U_1, U_2 \in \mathcal{L}$ we denote by $[U_1, U_2]$ the closed interval $\{U \in \mathcal{L} \mid U_1 \leq U \leq U_2\}$. The *complement* $V \setminus \bigcup_{H\in\mathcal{A}} H$ decomposes into connected components which are open polyhedral cones c, called *chambers*; the set of all chambers will be denoted $\mathcal{C} = \mathcal{C}(\mathcal{A})$.

Given any chamber c in $\mathcal{C}$ and any intersection subspace X, there is a unique chamber c/X in V/X for the localized arrangement $\mathcal{A}/X$ for which the quotient map $q : V \twoheadrightarrow V/X$ has $q^{-1}(c/X) \supseteq c$ (see Figure 1).

We can now define our main object of study.

Definition 1.1. Given two chambers c, c' in $\mathcal{C}$, and an intersection subspace X in $\mathcal{L}$, say that X is a *noninversion subspace* for $\{c, c'\}$ if $c/X = c'/X$.

Given any subset $\mathcal{O} \subseteq \mathcal{L}$, define a statistic on (unordered) pairs $\{c, c'\}$ of chambers

$$\mathrm{noninv}_{\mathcal{O}}(c, c') := \mathrm{noninv}_{\mathcal{O}}(c', c) := \Big|\{X \in \mathcal{O} : c/X = c'/X\}\Big|.$$

Define the matrix $\nu_{\mathcal{O}}$ in $\mathbb{Z}^{\mathcal{C}\times\mathcal{C}}$ whose (c, c')-entry equals $\mathrm{noninv}_{\mathcal{O}}(c, c')$. Alternatively, identify $\nu_{\mathcal{O}}$ with the following $\mathbb{Z}$-linear operator on the free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{C}$ that has basis indexed by the chambers $\mathcal{C}$:

$$\begin{array}{ccc} \mathbb{Z}\mathcal{C} & \xrightarrow{\nu_{\mathcal{O}}} & \mathbb{Z}\mathcal{C} \\ c' & \longmapsto & \sum_{c\in\mathcal{C}} \mathrm{noninv}_{\mathcal{O}}(c, c') \cdot c. \end{array} \tag{1}$$

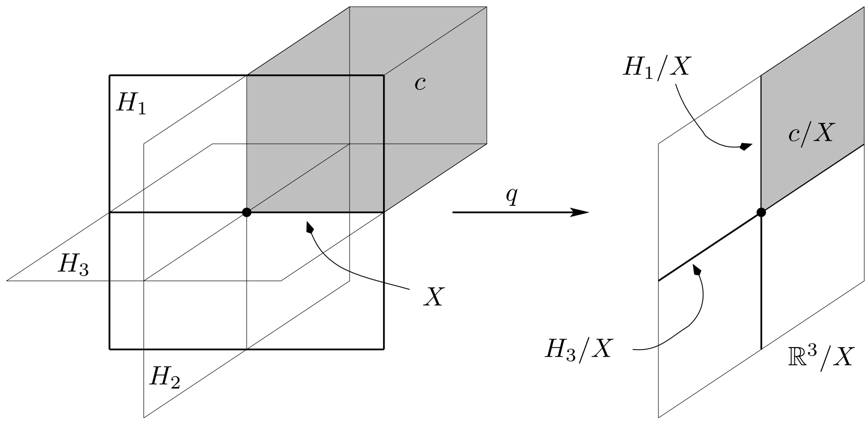

FIGURE 1. Arrangement and its localization

Note that since by definition $\mathrm{noninv}_{\mathcal{O}}(c, c') = \mathrm{noninv}_{\mathcal{O}}(c', c)$ it follows that $\nu_{\mathcal{O}}$ is a symmetric matrix.

EXAMPLE 1.2. We consider the arrangement $\mathcal{A} = \{H_1, H_2, H_3\}$ of the coordinate hyperplanes in $\mathbb{R}^3$ from Figure 1. Chambers $\mathcal{C}$ are in bijection with $\{+1, -1\}^3$, where the image of the chamber is the sign pattern $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3)$ of any of its points.

If $X = H_1 \cap H_3$, then $\mathcal{A}/X = \{H_1/X, H_3/X\}$. For $c = (+1, +1, +1)$, the chamber c/X in $\mathbb{R}^3/X \cong \mathbb{R}^2$ can again be seen as the positive quadrant. The only other chamber $c' \in \mathcal{C}$ for which $c/X = c'/X$ is the image $c' = (+1, -1, +1)$ of c reflection through H_2.

EXAMPLE 1.3. Let $V = \mathbb{R}^n$ and $\mathcal{A}$ the reflection arrangement of type A_{n-1}, whose hyperplanes are $H_{ij} = \{x_i = x_j\}_{1 \le i < j \le n}$ with the action of $W = \mathfrak{S}_n$ permuting coordinates.

Intersection subspaces X, such as the subspace $\{x_1 = x_3 = x_4, x_2 = x_7\}$ inside $V = \mathbb{R}^7$, correspond to set partitions of the coordinates $[n] := \{1, 2, \ldots, n\}$ into blocks $[n] = \bigsqcup_i B_i$ which indicate which coordinates are equal; in this example, this set partition is

$$[7] = \{1, 3, 4\} \sqcup \{2, 7\} \sqcup \{5\} \sqcup \{6\}.$$

The intersection lattice $\mathcal{L}$ is therefore isomorphic to the lattice of set-partitions of $[n]$, ordered by refinement, having the discrete (all singleton) partition as $\hat{0}$, and the trivial partition with one block as $\hat{1}$.

Chambers in the reflection arrangement of type A_{n-1} are the collections of vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$ for which $x_{w_1} < x_{w_2} < \cdots < x_{w_n}$ given a fixed $w \in \mathfrak{S}_n$, where $w_i = w(i)$ for $i \in [n]$. We will denote the chamber corresponding to a fixed w by c_w.

Given an intersection subspace X, corresponding to the partition $[n] = \bigsqcup_i B_i$, and a chamber c_w, the information contained in the chamber c_w/X records for each i the linear ordering in which the letters of B_i appear as a subsequence within $w = (w_1, w_2, \ldots, w_n)$. Therefore, $c_u/X = c_v/X$ if and only if for each i the letters of B_i appear in the same order in both u and v.

2. Semidefiniteness

As explained after Definition II.1.1 the matrix $\nu_{\mathcal{O}}$ is symmetric and hence the corresponding linear operator is self-adjoint with respect to the usual pairing $\langle -, - \rangle$ on $\mathbb{Z}\mathcal{C}$ that makes the basis vectors c orthonormal. It is also positive semidefinite, as it has the following easily identified "square root".

DEFINITION 2.1. Consider for each intersection subspace X the $\mathbb{Z}$-linear map

$$\begin{array}{ccc} \mathbb{Z}\mathcal{C} & \xrightarrow{\pi_X} & \mathbb{Z}\mathcal{C}(\mathcal{A}/X) \\ c & \longmapsto & c/X \end{array}$$

and having chosen a subset $\mathcal{O} \subseteq \mathcal{L}$, consider the direct sum of maps $\pi_{\mathcal{O}} := \bigoplus_{X \in \mathcal{O}} \pi_X$

$$\mathbb{Z}\mathcal{C} \longrightarrow \bigoplus_{X \in \mathcal{O}} \mathbb{Z}\mathcal{C}(\mathcal{A}/X)$$

PROPOSITION 2.2. *One has the factorization*

$$\nu_{\mathcal{O}} = \pi_{\mathcal{O}}^T \circ \pi_{\mathcal{O}}.$$

In particular, when scalars are extended from $\mathbb{Z}$ to $\mathbb{R}$, one has

$$\ker \nu_{\mathcal{O}} = \ker \pi_{\mathcal{O}}.$$

PROOF. The (c, c')-entry of $\pi_{\mathcal{O}}^T \circ \pi_{\mathcal{O}}$ equals

$$\begin{aligned} &\sum_{X \in \mathcal{O}} \sum_{d \in \mathcal{C}(\mathcal{A}/X)} (\pi_X)_{d,c} (\pi_X)_{d,c'} \\ &= \sum_{X \in \mathcal{O}} \Big| \{ d \in \mathcal{C}(\mathcal{A}/X) : c/X = d = c'/X \} \Big| \\ &= \Big| \{ X \in \mathcal{O} : c/X = c'/X \} \Big| \\ &= \mathrm{noninv}_{\mathcal{O}}(c, c'). \end{aligned}$$

□

3. Equivariant setting

Now assume that one has a finite subgroup W of $\mathrm{GL}(V)$ that preserves the arrangement $\mathcal{A}$ in the sense that for every w in W and every hyperplane H of $\mathcal{A}$, the hyperplane $w(H)$ is also in $\mathcal{A}$. Then W permutes each of the sets $\mathcal{A}, \mathcal{L}, \mathcal{C}$, and hence acts $\mathbb{Z}$-linearly on $\mathbb{Z}\mathcal{C}$.

PROPOSITION 3.1. *If the subset $\mathcal{O} \subseteq \mathcal{L}$ is also preserved by W, then the operator $\nu_{\mathcal{O}}$ on $\mathbb{Z}\mathcal{C}$ is W-equivariant.*

PROOF. This is straightforward from the observation that since W preserves $\mathcal{O}$, one has

$$\mathrm{noninv}_{\mathcal{O}}(c, c') = \mathrm{noninv}_{\mathcal{O}}(w(c), w(c')).$$

□

EXAMPLE 3.2. We resume Example II.1.3 and let $V = \mathbb{R}^n$ and $\mathcal{A}$ the reflection arrangement of the symmetric group $W = \mathfrak{S}_n$. Hence the intersection lattice $\mathcal{L}$ is the lattice of set partitions of $[n]$ ordered by refinement. The group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting coordinates. Thus, $w \in \mathfrak{S}_n$ acts on $\mathcal{L}$ by sending the set partition $[n] = \bigsqcup_i B_i$ to the set partition $[n] = \bigsqcup_i w(B_i)$, where $w(B_i) = \{w(j) \mid j \in B_i\}$. Therefore, the $\mathfrak{S}_n$-orbits on $\mathcal{L}$ are indexed by number partitions $\lambda \vdash n$. The orbit $\mathcal{O}_\lambda$ consist of those intersection subspaces or equivalently set partitions of $[n]$ for which the block sizes ordered in decreasing order are the parts of λ. We call such a set partition a set partition of *type* λ.

For example, for $\lambda = (k, 1^{n-k})$ we obtain as $\mathcal{O}_\lambda$ the $\binom{n}{k}$ set partitions of $[n]$ whose only non-singleton block is a block of size k.

4. $\mathbb{Z}_2$-action and inversions versus noninversions

Let I_V be matrix of the identity endomorphism of V. The scalar matrix $-\mathrm{I}_V$ acting on V preserves any arrangement $\mathcal{A}$, and hence gives rise to an action of $\mathbb{Z}_2 = \{1, \tau\}$ in which τ acts by $-\mathrm{I}_V$. When one has a subgroup W of $\mathrm{GL}(V)$ preserving $\mathcal{A}$, since τ acts by a scalar matrix, this $\mathbb{Z}_2$-action commutes with the action of W, giving rise to a $W \times \mathbb{Z}_2$-action. Of course, if W already contains the element $-\mathrm{I}_V$, this provides no extra information beyond the W-action. But when $-\mathrm{I}_V$ is not an element of W already, it is worthwhile to consider this extra $\mathbb{Z}_2$-action.

We wish to explain how carrying along this $\mathbb{Z}_2$-action naturally eliminates a certain choice we have made. Instead of considering the matrix/operator $\nu_\mathcal{O}$, one might have considered the matrix/operator $\iota_\mathcal{O} = (\mathrm{inv}_\mathcal{O}(c, c'))_{c,c' \in \mathcal{C}}$ having entry $\mathrm{inv}_\mathcal{O}(c, c')$ defined to be the number of subspaces X in $\mathcal{O}$ which are *inversions* for c, c' in the sense that $c/X = -c'/X$. Taking into account the $\mathbb{Z}_2$-action eliminates the need to consider $\iota_\mathcal{O}$ separately:

PROPOSITION 4.1. *The two operators $\nu_\mathcal{O}$ and $\iota_\mathcal{O}$ are sent to each other by the generator τ of the $\mathbb{Z}_2$-action:*

$$\iota_\mathcal{O} = \tau \circ \nu_\mathcal{O} = \nu_\mathcal{O} \circ \tau. \quad \square$$

Thus, if we want to consider the eigenvalues and eigenspaces, it is equivalent to consider either $\nu_\mathcal{O}$ or $\iota_\mathcal{O}$, as long as we also keep track of the $\mathbb{Z}_2$-action on the eigenspaces. In what follows, we prefer to consider the positive semidefinite operator $\nu_\mathcal{O}$ rather than the indefinite operator $\iota_\mathcal{O}$.

EXAMPLE 4.2. We return to the setting of Example II.1.3 and Example II.3.2. For $\lambda = (k, 1^{n-k})$ we had seen that $\mathcal{O}_\lambda$ consists of all set partitions of $[n]$ whose unique non-singleton block is of size k. Thus, $X \in \mathcal{O}_\lambda$ is uniquely defined by specifying a k-subset B of $[n]$. Let $u, v \in \mathfrak{S}_n$ with corresponding chambers c_u and c_v. From Example II.1.3 we know $c_u/X = c_v/X$ if and only if the linear orders defined by u and v coincide on B. Since there are $\binom{n}{k}$ choices for k-subsets B we have

$$\mathrm{inv}_{\mathcal{O}_{(k,1^{n-k})}}(c_u, c_v) = \binom{n}{k} - \mathrm{noninv}_{\mathcal{O}_{(k,1^{n-k})}}(c_u, c_v).$$

In particular, $\mathrm{inv}_{\mathcal{O}_{(2,1^{n-2})}}(c_u, c_v)$ is the number of inversions of $v^{-1}u$.

5. Real reflection groups

We review here some standard facts about real, Euclidean finite reflection groups; a good reference is [**38**].

Here we will adopt the convention that an *(orthogonal) reflection* in $\mathrm{GL}(V)$ for an $\mathbb{R}$-vector space V is an orthogonal involution s whose fixed subspace V^s is some hyperplane H. Necessarily, such an element s has $s^2 = \mathrm{I}_V$ and acts by multiplication by -1 on the line $H^\perp$. A *(real) reflection group* W is a finite subgroup of $\mathrm{GL}(V)$ generated by *reflections*.

To any reflection group W there is naturally associated its *arrangement of reflecting hyperplanes* $\mathcal{A}$, consisting of all hyperplanes H arising as V^s for reflections s in W. In this situation it is known that the set of chambers $\mathcal{C}$ carries a *simply transitive* action of W. Therefore, after making a choice of *fundamental/identity/base chamber* c_1, one can identify the W-action on $\mathbb{Z}\mathcal{C}$ with the left-regular W-action on the group algebra $\mathbb{Z}W$:

$$\begin{array}{ccc} \mathbb{Z}W & \longrightarrow & \mathbb{Z}\mathcal{C} \\ w & \longmapsto & c_w := w(c_1). \end{array} \tag{2}$$

Now assume one is given a W-stable subset $\mathcal{O} \subseteq \mathcal{L}$, and define the statistic

$$\begin{aligned} \mathrm{noninv}_{\mathcal{O}}(w) &:= \mathrm{noninv}_{\mathcal{O}}(c_1, c_w) \\ &= \mathrm{noninv}_{\mathcal{O}}(c_w, c_1) \\ &= \mathrm{noninv}_{\mathcal{O}}(w(c_1), w(c_{w^{-1}})) \\ &= \mathrm{noninv}_{\mathcal{O}}(c_1, c_{w^{-1}}) \\ &= \mathrm{noninv}_{\mathcal{O}}(w^{-1}). \end{aligned}$$

PROPOSITION 5.1. *For any W-stable subset $\mathcal{O} \subseteq \mathcal{L}$, under the identification 2, the operator $\nu_{\mathcal{O}}$ acts on $\mathbb{Z}W$ as right-multiplication by the element*

$$\sum_{w\in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot w.$$

PROOF. Within the group algebra, for any basis element v in W, one has

$$\begin{aligned} v \cdot \left(\sum_{w\in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot w \right) &= \sum_{w\in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot vw \\ &= \sum_{u\in W} \mathrm{noninv}_{\mathcal{O}}(v^{-1}u) \cdot u \\ &= \sum_{u\in W} \mathrm{noninv}_{\mathcal{O}}(c_{v^{-1}u}, c_1) \cdot u \\ &= \sum_{u\in W} \mathrm{noninv}_{\mathcal{O}}(c_u, c_v) \cdot u. \end{aligned}$$ □

By abuse of notation, we will also use the notation $\nu_{\mathcal{O}}$ to denote the element $\sum_{w\in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot w$ of $\mathbb{C}W$.

When W is a real reflection group, the $\mathbb{Z}_2$-action corresponds to the action of the *longest element* w_0 in W, defined uniquely by the property that

$$c_{w_0} = -c_1, \tag{3}$$

where $c_{w_0} = w_0(c_1)$. Note that this forces w_0 to always be an *involution*: $w_0^2 = 1$.

PROPOSITION 5.2. *Under the identification 2, the scalar matrix $-\mathrm{I}_V$ or the generator τ of the $\mathbb{Z}_2$-action on $\mathbb{Z}\mathcal{C}$ acts on $\mathbb{Z}W$ as right-multiplication by w_0.*

PROOF. Applying w on the left of 3 gives

$$c_{ww_0} = ww_0(c_1) = -w(c_1) = -c_w$$

for any w in W. □

It is known that $-\mathrm{I}_V$ is an element of a reflection group W acting on V if and only if W only has *even degrees* $d_1, \ldots, d_n$ for any system of basic invariants $f_1, \ldots, f_n$ that generate the W-invariant polynomials $\mathbb{C}[V]^W = \mathbb{C}[f_1, \ldots, f_n]$. If $-\mathrm{I}_V$ is an element of W, then necessarily $-\mathrm{I}_V = w_0$.

For the *irreducible* real reflection groups, one has

- $-\mathrm{I}_V = w_0$ in types $B_n = C_n$, in type D_n when n is even, in the dihedral types $I_2(m)$ for m even, as well as the exceptional types F_4, E_7, E_8, H_3, H_4.
- $-\mathrm{I}_V \notin W$ in all other cases, that is, in type A_{n-1}, in type D_n for n odd, in dihedral types $I_2(m)$ for m odd, and in type E_6. Thus, in these cases there is an extra $\mathbb{Z}_2$-action to consider.

Example 5.3. Again we return to Example II.1.3. The longest word $w_0 \in \mathfrak{S}_n$ corresponds to the permutation $n\ n-1 \cdots 2\ 1$. Thus, multiplication by w_0 on the right sends a permutation $w(1) \cdots w(n) \in \mathfrak{S}_n$ to the permutation $w(n) \cdots w(1)$.

Given an intersection subspace X, denote by $\mathrm{N}_W(X)$ and $\mathrm{Z}_W(X)$, respectively, its not-necessarily-pointwise stabilizer subgroup and pointwise stabilizer subgroup within W:

$$\begin{aligned} \mathrm{N}_W(X) &= \{w \in W : w(X) = X\}, \\ \mathrm{Z}_W(X) &= \{w \in W : w(x) = x \text{ for all } x \in X\}. \end{aligned}$$

It is well-known (see for example [**1**, Lemma 3.75]) that $\mathrm{Z}_W(X)$ is itself a finite real reflection group, called the *parabolic subgroup* associated to X, which one can view as acting on the quotient space V/X, and having reflection arrangement equal to the localization $\mathcal{A}/X$. Consequently, the chambers $\mathcal{C}(\mathcal{A}/X)$ are in natural bijection with $\mathrm{Z}_W(X)$. This gives the following interpretation to the map $c \longmapsto c/X$ that we have been using.

Proposition 5.4. *Let W be a finite real reflection group W, and X an intersection subspace in $\mathcal{L}$. Then every w in W factors uniquely as $w = z \cdot y$ where z lies in $\mathrm{Z}_W(X)$ and y lies in*

$$^XW := \{y \in W : c_y/X = c_1/X\}.$$

In particular, the map $\pi_X : \mathcal{C} \to \mathcal{C}(\mathcal{A}/X)$ sending $c \longmapsto c/X$ corresponds under (2) *to the map sending $w \longmapsto z$.*

Proof. Given w in W, consider the chamber c_w/X in the localized arrangement $\mathcal{A}/X$. Since this localized arrangement is the reflection arrangement for $\mathrm{Z}_W(X)$, there is a unique element z in $\mathrm{Z}_W(X)$ for which $c_z/X = c_w/X$. In particular, the element $z \in \mathrm{Z}_W(X)$ acts on the chambers of $\mathcal{A}$ and on the chambers of $\mathcal{A}/X$. Conversely, given a factorization $w = zy$ as asserted, it follows that $c_z/X = c_w/X$. Thus, we are done once we check that $y := z^{-1}w$ lies in ^XW. But this follows from

$$\begin{aligned} c_y/X &= c_{z^{-1}w}/X \\ &= (z^{-1}c_w)/(z^{-1}X) \\ &= z^{-1}(c_w/X) \\ &= z^{-1}(c_z/X) \\ &= c_1/z^{-1}X \\ &= c_1/X. \end{aligned}$$

$\square$

In the sequel, for a finite real reflection group W and an intersection subspace X in $\mathcal{L}$, we denote by ${}^XW$ the set $\{y \in W : c_y/X = c_1/X\}$, which by the preceding proposition is a set of right coset representatives of $\mathrm{Z}_W(X)$ in W. We write $W^X = \{w^{-1} : w \in {}^XW\}$ for the corresponding set of left coset representatives. If X intersects the identity chamber c_1 then $\mathrm{Z}_W(X) = W_J = \langle J \rangle$ for some subsets $J \subseteq S$. Here W_J is a (standard) parabolic subgroup for the Coxeter system (W, S) that generates W using the set S of reflections through the walls of c_1. In this case we also write ${}^JW$ for ${}^XW$ and W^J for W^X respectively. The ${}^JW$ and W^J are sets of minimal length right and left coset representatives for W_J.

EXAMPLE 5.5. Returning to Example II.1.3 and Example II.3.2, where $W = \mathfrak{S}_n$ acts on $V = \mathbb{R}^n$ by permuting coordinates and X is the intersection subspace corresponding to the partition $[n] = \bigsqcup_i B_i$, the centralizer $\mathrm{Z}_W(X)$ is the *Young subgroup* $\prod_i \mathfrak{S}_{B_i}$ that permutes each block B_i of coordinates separately. The map $W \longmapsto \mathrm{Z}_W(X)$ that sends $w \mapsto z$ corresponding to $c \longmapsto c/X$ remembers only the ordering of the coordinates within each block B_i.

6. The case where $\mathcal{O}$ is a single W-orbit

When W is a real reflection group, and $\mathcal{O} = X^W := \{w \cdot X : w \in W\}$ is the W-orbit of some intersection subspace X, there are two extra features that will help us to analyze the eigenspaces of $\nu_{\mathcal{O}}$.

6.1. A second square root. First, there is another "square root" for $\nu_{\mathcal{O}}$ when W is the orbit X_0^W of a single subspace X_0. This will connect $\nu_{\mathcal{O}}$ with the BHR random walks in Chapter IV. Given an intersection subspace X, with the associated subgroups $\mathrm{Z}_W(X) \subseteq \mathrm{N}_W(X)$ we have introduced in Proposition II.5.4 and subsequent definitions the parabolic factorizations and coset representatives

$$W = \mathrm{Z}_W(X) \cdot {}^XW$$
$$W = W^X \cdot \mathrm{Z}_W(X).$$

Define

$$n_X := [\mathrm{N}_W(X) : \mathrm{Z}_W(X)]$$
$${}^XR := \sum_{u \in {}^XW} u$$
$$R^X := \sum_{u \in W^X} u.$$

For later use and analogous to our previous convention we write ${}^JR$ and R^J in case X lies in the boundary of the identity chamber c_1 and $\mathrm{Z}_W(X) = W_J$ is a (standard) parabolic subgroup.

PROPOSITION 6.1. *Let W be a real reflection group and $\mathcal{O} = X_0^W \subset \mathcal{L}$ the W-orbit of the intersection subspace X_0. Then*

$$\mathrm{noninv}_{\mathcal{O}}(w) = \frac{1}{n_{X_0}} \left| {}^{X_0}W \cap {}^{X_0}Ww \right|$$

and

$$\nu_{\mathcal{O}} = \frac{1}{n_{X_0}} R^{X_0} \cdot {}^{X_0}R.$$

PROOF. Since $\mathcal{O}$ is the W-orbit of X_0, and $\mathrm{N}_W(X_0)$ the W-stabilizer of X_0, the elements $u \cdot X_0$ as u runs over coset representatives for $W/\mathrm{N}_W(X_0)$ give each X in $\mathcal{O}$ exactly once. Therefore, the elements uX_0 as u runs over the coset representatives W^{X_0} of $W/\mathrm{Z}_W(X_0)$ give each X in $\mathcal{O}$ exactly $n_{X_0} = [\mathrm{N}_W(X_0) : Z_W(X_0)]$ times. Since

$$\mathrm{noninv}_{\mathcal{O}}(w) = \left|\{X \in \mathcal{O} : w \in {}^{X}W\}\right|$$

this implies that

$$n_{X_0} \cdot \mathrm{noninv}_{\mathcal{O}}(w) = \left|\{u \in W^{X_0} : w \in {}^{uX_0}W\}\right|.$$

We wish to rewrite the set appearing on the right side of this equation. Note that u lies in W^{X_0} if and only if u^{-1} lies in ${}^{X_0}W$ if and only if $c_{u^{-1}}/X_0 = c_1/X_0$. Similarly, w lies in ${}^{uX_0}W$ if and only if $c_w/uX_0 = c_1/uX_0$ if and only if $c_{u^{-1}w}/X_0 = c_{u^{-1}}/X_0 = c_1/X_0$ if and only if $u^{-1}w$ lies in ${}^{X_0}W$. Letting $v = u^{-1}w$, one concludes that v lies in both ${}^{X_0}W$ and in ${}^{X_0}Ww$. Thus, so far we have shown that

$$n_{X_0} \cdot \mathrm{noninv}_{\mathcal{O}}(w) = \left|{}^{X_0}W \cap {}^{X_0}Ww\right| = \left|\left\{(u,v) \in W^{X_0} \times {}^{X_0}W : uv = w\right\}\right|.$$

The first equality proves the first assertion. The second assertion follows from the equality between the first and third terms in the preceding equation and the following calculation.

$$R^{X_0} \cdot {}^{X_0}R = \left(\sum_{u \in W^{X_0}} u\right)\left(\sum_{v \in {}^{X_0}W} v\right) = \sum_{w \in W} w \cdot n_{X_0} \cdot \mathrm{noninv}_{\mathcal{O}}(w) \qquad \square$$

6.2. Nested kernels. Second, there is an inclusion of kernels $\ker \nu_{\mathcal{O}} \subseteq \ker \nu_{\mathcal{O}'}$ whenever $\mathcal{O}, \mathcal{O}'$ are W-orbits represented by nested subspaces. To see this, define in the general setting of hyperplane arrangements a map

$$\pi^{\mathcal{O}}_{\mathcal{O}'} : \bigoplus_{X \in \mathcal{O}} \mathcal{C}(\mathcal{A}/X) \to \bigoplus_{X' \in \mathcal{O}'} \mathcal{C}(\mathcal{A}/X')$$

as a direct sum of the natural maps

$$\begin{aligned} \pi^{X}_{X'} : \mathcal{C}(\mathcal{A}/X) &\longrightarrow \mathcal{C}(\mathcal{A}/X') \\ c/X &\longmapsto c/X' \end{aligned}$$

indexed by pairs of subspaces $(X, X') \in \mathcal{O} \times \mathcal{O}'$ for which $X \subseteq X'$. Given $X' \in \mathcal{O}'$, define an integer $c_{\mathcal{O},X'}$ to be the number of $X \in \mathcal{O}$ for which $X \subseteq X'$.

PROPOSITION 6.2. *Let $\mathcal{A}$ be an arrangement with a group of linear symmetries W, and let $\mathcal{O}, \mathcal{O}'$ be two W-orbits within $\mathcal{L}$ represented by two nested subspaces.*

Then the integers $c_{\mathcal{O},X'}$ do not depend upon the choice of X' within $\mathcal{O}'$, and denoting this common integer $c_{\mathcal{O},\mathcal{O}'}$ one has

$$c_{\mathcal{O},\mathcal{O}'} \cdot \pi_{\mathcal{O}'} = \pi^{\mathcal{O}}_{\mathcal{O}'} \circ \pi_{\mathcal{O}}. \tag{4}$$

Consequently,

$$\begin{array}{ccc} \ker \pi_{\mathcal{O}} & \subseteq & \ker \pi_{\mathcal{O}'} \\ \| & & \| \\ \ker \nu_{\mathcal{O}} & & \ker \nu_{\mathcal{O}'} \end{array}$$

PROOF. Because $\pi_{X'} = \pi^X_{X'} \circ \pi_X$, one has generally that

$$\pi^{\mathcal{O}}_{\mathcal{O}'} \circ \pi_{\mathcal{O}} = \sum_{X' \in \mathcal{O}'} c_{\mathcal{O},X'} \; \pi_{X'}.$$

However, whenever $\mathcal{O}, \mathcal{O}'$ are W-orbits, if X', X'' are subspaces in the same W-orbit $\mathcal{O}'$, say with $w \cdot X' = X''$, then the element w gives a bijection between the two sets counted by $c_{\mathcal{O},X'}, c_{\mathcal{O},X''}$. Thus, $c_{\mathcal{O},\mathcal{O}'} := c_{\mathcal{O},X'}$ satisfies

$$\pi^{\mathcal{O}}_{\mathcal{O}'} \circ \pi_{\mathcal{O}} = c_{\mathcal{O},\mathcal{O}'} \sum_{X' \in \mathcal{O}'} \pi_{X'} = c_{\mathcal{O},\mathcal{O}'} \cdot \pi_{\mathcal{O}'}. \qquad \square$$

EXAMPLE 6.3. We again consider the setting of Example II.1.3, Example II.3.2 and the partitions $\lambda = (k, 1^{n-k})$, $1 \le k \le n$. Then for each $1 \le k < k' \le n$ and each subspace $X \in \mathcal{O}_{(k,1^{n-k})}$ there is a subspace $X' \in \mathcal{O}_{(k',1^{n-k'})}$ for which $X' \subseteq X$. Thus, Proposition II.6.2 applies, and we will take advantage of the nesting $\ker \nu_{(k',(1^{n-k'})} \subset \ker \nu_{(k,1^{n-k})}$ in §VI.2.

7. A reduction to isotypic components

The fact that we are considering operators which are right-multiplication on the group algebra $\mathbb{C}W$ by elements of $\mathbb{Z}W$ allows us to take advantage of standard facts from representation theory.

As preparation, let A and B be finite dimensional $\mathbb{C}$-algebras, $\mathfrak{e}$ a primitive idempotent of A and $\mathfrak{f}$ a primitive idempotent of B. Then for any $(A$–$B)$-bimodule C the primitivity of $\mathfrak{e}$ and $\mathfrak{f}$ implies

$$\operatorname{Hom}_A(A\,\mathfrak{e}, C\,\mathfrak{f}) \cong \mathfrak{e}\,C\,\mathfrak{f} \cong \operatorname{Hom}_B(\mathfrak{f}\,B, \mathfrak{e}\,C). \tag{5}$$

Again primitivity implies that $\dim_{\mathbb{C}} \operatorname{Hom}_A(A\,\mathfrak{e}, C\,\mathfrak{f})$ is the multiplicity of the left simple A-module $A\,\mathfrak{e}$ in $C\,\mathfrak{f}$ and $\dim_{\mathbb{C}} \operatorname{Hom}_B(\mathfrak{f}\,B, \mathfrak{e}\,C)$ is the multiplicity of the right simple B-module $\mathfrak{f}\,B$ in $\mathfrak{e}\,C$. This fact is the core of the proof of the following proposition.

PROPOSITION 7.1. *Let W be a real reflection group and let $\mathcal{O} \subseteq \mathcal{L}$ be a W-stable subset. Fix a complex irreducible W-character χ and a representation $\rho_\chi : W \to \mathrm{GL}_{\mathbb{C}}(U^\chi)$ in some complex vectorspace U^χ affording the character χ. Then the multiplicity of χ in*

$$\ker(\nu_{\mathcal{O}} - \lambda \mathrm{I}_{\mathbb{C}W}) \cap \ker(\tau - \epsilon \mathrm{I}_{\mathbb{C}W}) \tag{6}$$

equals the dimension of

$$\ker(\rho_\chi(\nu_{\mathcal{O}}) - \lambda \mathrm{I}_{U^\chi}) \cap \ker(\rho_\chi(w_0) - \epsilon \mathrm{I}_{U^\chi}). \tag{7}$$

In particular, if λ is an eigenvalue of $\nu_{\mathcal{O}}$ and ϵ an eigenvalue of τ, then the multiplicity of χ in the λ-eigenspace for $\nu_{\mathcal{O}}$ intersected with the ϵ-eigenspace for τ is the same as the dimension of the λ-eigenspace for $\rho_\chi(\nu_{\mathcal{O}})$ intersected with the ϵ-eigenspace for $\rho_\chi(w_0)$.

PROOF. We may assume that λ and ϵ are eigenvalues of $\nu_{\mathcal{O}}$ and τ respectively, since otherwise the first part of the claim is trivial.

Let $A = \mathbb{C}W$ and B be the subalgebra of A generated by w_0 and $\nu_{\mathcal{O}}$. Since w_0 and $\nu_{\mathcal{O}}$ commute and are diagonalizable, B is a commutative semisimple algebra. Since λ and ϵ are eigenvalues of $\nu_{\mathcal{O}}$ and w_0 respectively, sending $\nu_{\mathcal{O}}$ to λ and w_0 to ϵ defines a 1-dimensional representation of B with character ϕ. Let $\mathfrak{e}_\chi$ (resp.

$\mathfrak{e}_\phi$) be the primitive idempotent of A (resp. B) corresponding to the irreducible character χ (resp. ϕ). In linear algebra terms $\mathfrak{e}_\phi$ is the product of the projection to the ϵ-eigenspace of w_0 and the λ-eigenspace of ν_0. Hence $\mathbb{C}W\,\mathfrak{e}_\phi$ is the intersection of the λ-eigenspace of $\nu_\mathcal{O}$ and the ϵ-eigenspace of w_0 and thus coincides with (6). Hence, by definition of $\mathfrak{e}_\chi$, we have that $\dim_\mathbb{C} \mathrm{Hom}_A(A\,\mathfrak{e}_\chi, A\,\mathfrak{e}_\phi)$ is the multiplicity of χ in (6). By the same arguments one sees that $\dim_\mathbb{C} \mathrm{Hom}_B(\mathfrak{e}_\phi\, B, \mathfrak{e}_\chi\, A)$ equals the dimension of (7). Now (5) with $C = \mathbb{C}W$ completes the proof. $\square$

The preceding result can also be obtained in a more explicit but less elegant way by partially block diagonalizing $\nu_\mathcal{O}$. The latter technique sometimes goes by the name "Fourier transform".

We note that as a very special case of this, when χ is a *degree one* or *linear* character of W, one can be much more precise.

PROPOSITION 7.2. *For any degree one character χ of W and any W-stable subset $\mathcal{O} \subseteq \mathcal{L}$, multiples of the χ-idempotent*

$$\mathfrak{e}_\chi := \frac{1}{|W|}\sum_{w\in W}\chi(w)\cdot w = \frac{1}{|W|}\sum_{w\in W}\chi(w^{-1})\cdot w$$

in $\mathbb{Q}W$ are eigenvectors for $\nu_\mathcal{O}$, with integer eigenvalue

$$\begin{aligned}\lambda_\mathcal{O}(\chi) :=& \sum_{w\in W}\mathrm{noninv}_\mathcal{O}(w)\chi(w)\\ =& \sum_{X\in\mathcal{O}}\ \sum_{\substack{w\in W:\\ c_w/X=c_1/X}}\chi(w).\end{aligned}$$

In particular, the trivial character $\mathbf{1}$ gives rise to an all positive eigenvector $\mathfrak{e}_\mathbf{1} = \frac{1}{|W|}\sum_{w\in W} w$, having eigenvalue

$$\lambda_\mathcal{O}(\mathbf{1}) = \sum_i\Big([W:\mathrm{N}_W(X_i)]\cdot[W:\mathrm{Z}_W(X_i)]\Big)$$

where $\{X_i\}$ is any set of representatives for the W-orbits within $\mathcal{O}$.

PROOF. First note that since a reflection group W is generated by involutions, any degree one character χ takes values in $\{\pm 1\}$ and satisfies $\chi(w^{-1}) = \chi(w)$. Now check the eigenvalue equation:

$$\begin{aligned}|W|\mathfrak{e}_\chi\cdot\nu_\mathcal{O} &= \left(\sum_{u\in W}\chi(u)\cdot u\right)\left(\sum_{v\in W}\mathrm{noninv}_\mathcal{O}(v)\cdot v\right)\\ &= \sum_{u\in W}\sum_{v\in W}\chi(u)\,\mathrm{noninv}_\mathcal{O}(v)\cdot uv\\ &= \sum_{w\in W} w\left(\sum_{v\in W}\chi(wv^{-1})\,\mathrm{noninv}_\mathcal{O}(v)\right)\\ &= \left(\sum_{w\in W}\chi(w)w\right)\left(\sum_{v\in W}\chi(v^{-1})\,\mathrm{noninv}_\mathcal{O}(v)\right)\\ &= \lambda_\mathcal{O}(\chi)\,(|W|\mathfrak{e}_\chi)\end{aligned}$$

One can also rewrite

$$\begin{aligned}\lambda_{\mathcal{O}}(\chi) &= \sum_{w \in W} \operatorname{noninv}_{\mathcal{O}}(w)\chi(w) \\ &= \sum_{w \in W} \sum_{\substack{X \in \mathcal{O}: \\ c_w/X = c_1/X}} \chi(w) \\ &= \sum_{X \in \mathcal{O}} \sum_{\substack{w \in W: \\ c_w/X = c_1/X}} \chi(w)\end{aligned}$$

Lastly, when $\chi = \mathbf{1}$ one has

$$\begin{aligned}\lambda_{\mathcal{O}}(\chi) &= \sum_{X \in \mathcal{O}} |\{w \in W : c_w/X = c_1/X\}| \\ &= \sum_i \sum_{X'_i \in W \cdot X_i} |\{w \in W : c_w/X'_i = c_1/X'_i\}| \\ &= \sum_i [W : \mathrm{N}_W(X_i)][W : \mathrm{Z}_W(X_i)]\end{aligned}$$

where the last equality uses both the fact that $|W \cdot X_i| = [W : \mathrm{N}_W(X_i)]$ and that Proposition II.5.4 tells us that the elements from ${}^X W = \{w \in W : c_w/X = c_1/X\}$ form a set of coset representatives for $W/\mathrm{Z}_W(X)$. $\square$

EXAMPLE 7.3. We return to the setting of Example II.1.3 with $W = \mathfrak{S}_n$ acting on $V = \mathbb{R}^n$, and $\mathcal{O} = \mathcal{O}_{(k,1^{n-k})}$. There are two degree one characters of W, namely the *trivial* character $\mathbf{1}$, and the *sign* character sgn. Since a representative subspace $x_1 = x_2 = \cdots = x_k$ in $\mathcal{O}$ has $\mathrm{N}_W(X) = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$ and $\mathrm{Z}_W(X) = \mathfrak{S}_k$, for the trivial character $\mathbf{1}$ one finds that

$$\lambda_{\mathcal{O}}(\mathbf{1}) = [W : \mathrm{N}_W(X)][W : \mathrm{Z}_W(X)] = \frac{n!}{k!(n-k)!} \cdot \frac{n!}{k!} = \binom{n}{k}^2 (n-k)!.$$

For the sign character sgn one finds that

$$\begin{aligned}\lambda_{\mathcal{O}}(\operatorname{sgn}) &= \sum_{X \in \mathcal{O}} \sum_{w \in W : c_w/X = c_1/X} \operatorname{sgn}(w) \\ &= \sum_{1 \le i_1 < \cdots < i_k \le n} \sum_{\substack{w \in W: \\ \{i_1,\ldots,i_k\} \text{ appear} \\ \text{left-to-right in } w}} \operatorname{sgn}(w) \\ &= \begin{cases} 1 & \text{if } k = n \\ 1 & \text{if } k = n-1 \text{ and } n \text{ is odd,} \\ 0 & \text{if } k = n-1 \text{ and } n \text{ is even,} \\ 0 & \text{if } 1 \le k \le n-2, \end{cases}\end{aligned}$$

for the following reasons.

When $k = n$ this is because there is only one term in the outer sum, and the inner sum contains only $w = 1$.

When $1 \le k \le n-2$, picking any pair $\{i, j\}$ in the complement $[n] \setminus \{i_1, \ldots, i_k\}$ gives rise to a sign-reversing involution $w \leftrightarrow (i, j) \cdot w$, which shows that the inner sum vanishes.

When $k = n-1$, this calculation appears as [**76**, Proposition 5.3]. Each term in the outer sum is determined by the index i in the complement $[n] \setminus \{i_1, \ldots, i_k\}$, and each w in the inner sum determined by the position j where i appears in w, that is, $j = w^{-1}(i)$. Hence the result is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i-j} = \left(\sum_{i=1}^{n} (-1)^i \right)^2$$

which is 1 for n odd and 0 for n even.

8. Perron-Frobenius and primitivity

Since the matrices representing the $\nu_{\mathcal{O}}$ have non-negative entries, and since the trivial idempotent $\mathfrak{e}_{\mathbf{1}}$ gives an eigenvector with all positive entries, one might wish to apply Perron-Frobenius theory (see e.g. [**37**, Theorem 8.4.4]) to conclude that the eigenspace spanned by $\mathfrak{e}_{\mathbf{1}}$ is simple. This is true in the cases of most interest to us, but we must first deal with a degenerate case that can occur when the reflection group W does not act irreducibly.

Recall that for any finite reflection group W acting on the real vector space V, one can always decompose $W = \prod_{i=1}^t W^{(i)}$ and find an orthogonal decomposition $V = \bigoplus_{i=1}^t V^{(i)}$ such that each $W^{(i)}$ acts as a reflection group *irreducibly* on $V^{(i)}$. In this situation, one has a disjoint decomposition of the arrangement $\mathcal{A}$ of reflecting hyperplanes of the reflections from W as $\mathcal{A} = \bigsqcup_{i=1}^t \mathcal{A}^{(i)}$, where $\mathcal{A}^{(i)}$ is the arrangement of reflecting hyperplanes of the reflections from $W^{(i)}$.

Example 8.1. Let W be of type $A_1 \times A_1$, that is, the reflection group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting on $V = \mathbb{R}^2$ generated by two commuting reflections s_1, s_2 through perpendicular hyperplanes H_1, H_2 (lines, in this case). Thus, $W = W^{(1)} \times W^{(2)}$ where $W^{(i)} = \{\mathbf{1}, s_i\}$. Choose $\mathcal{O} = \{H_1\}$. Then one finds that

w	$\mathrm{noninv}_{\mathcal{O}}(w)$
$\mathbf{1}$	1
s_1	0
s_2	1
$s_1 s_2 = s_2 s_1 = w_0$	0

so that as an element of $\mathbb{Z}W$, one has $\nu_{\mathcal{O}} = \mathbf{1} + s_2$ whose action on $\mathbb{Z}W$ on the right can be expressed in matrix form with respect to the ordered basis $(\mathbf{1}, s_1, s_2, w_0)$ as

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Even though this matrix is non-negative, it is imprimitive in the sense that no power of it will have all strictly positive entries. Thus, one cannot apply the simplest version of the Perron-Frobenius theorem. However, under the identification $\mathbb{Z}W \cong \mathbb{Z}W_1 \otimes_{\mathbb{Z}} \mathbb{Z}W_2$ one has

$$\nu_{\mathcal{O}} = (1 \cdot \mathbf{1} + 0 \cdot s_1) \otimes (1 \cdot \mathbf{1} + 1 \cdot s_2).$$

and correspondingly the above matrix can be rewritten as

$$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \otimes \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}.$$

Note that this second tensor factor is a primitive matrix, to which Perron-Frobenius does apply.

The following proposition can be proven in a completely straightforward fashion.

PROPOSITION 8.2. *Let W be a finite real reflection group and let $\mathcal{A}$ denote its arrangement of reflecting hyperplanes. Let $W = \prod_{i=1}^{t} W^{(i)}$ and $\mathcal{A} = \bigsqcup_{i=1}^{t} \mathcal{A}^{(i)}$ be the decomposition described above.*

Let $\mathcal{O} \subseteq \mathcal{L}$ be W-stable and suppose there is an $1 \leq i \leq t$ such that

- *every X in $\mathcal{O}$ is a subspace of $X^{(i)} := \bigcap_{H \in \mathcal{A}^{(i)}} H$.*

Then every X in $\mathcal{O}$ can be uniquely written as $X = X^{(i)} \cap Y_X$ for some intersection Y_X of hyperplanes from $\bigsqcup_{j \neq i} \mathcal{A}^{(j)}$. Letting $W' := \prod_{j \neq i} W_j$ and identifying $\mathbb{Z}W \cong \mathbb{Z}W^{(i)} \otimes \mathbb{Z}W'$, one has

$$\nu_{\mathcal{O}} = \mathbb{1}_{\mathbb{Z}W^{(i)}} \otimes \nu_{\mathcal{O}'}$$

where $\mathcal{O}' := \{Y_X : X \in \mathcal{O}\}$.

EXAMPLE 8.3. Example II.8.1 illustrates the scenario of Proposition II.8.2 with $V = \mathbb{R}^2 = V^{(1)} \oplus V^{(2)} = \mathbb{R}^1 \oplus \mathbb{R}^1$. Here $i = 1$ with $X = X^{(1)} = H_1$ and $Y = V^{(2)}$ is the second copy of $\mathbb{R}^1$ considered as the empty intersection of hyperplanes from $\mathcal{A}^{(2)}$. In the tensor decomposition of $\nu_{\mathcal{O}}$, the first tensor factor is $\mathbb{1}_{\mathbb{Z}W^{(1)}}$ and the second tensor factor is $\nu_{\mathcal{O}'}$.

Let W be a finite real reflection group that decomposes as $W = \prod_{i=1}^{t} W^{(i)}$ with each $W^{(i)}$ an irreducible reflection group and let $\mathcal{A}^{(i)}$ be the arrangement of reflecting hyperplanes associated with $W^{(i)}$. We say a W-invariant subset $\mathcal{O} \subseteq \mathcal{L}$ is *irreducible* if there is no $1 \leq i \leq t$ such that all $X \in \mathcal{O}$ satisfy $X \subseteq \bigcap_{H \in \mathcal{A}^{(i)}} H$. As a consequence of the proposition, in analyzing the eigenvalues and eigenspaces of $\nu_{\mathcal{O}}$, it suffices to assume that $\mathcal{O}$ is irreducible.

PROPOSITION 8.4. *Let W be a finite real reflection group and $\mathcal{O} \subseteq \mathcal{L}$ an irreducible W-invariant subset of $\mathcal{L}$. Then the non-negative $|W| \times |W|$ matrix $\nu_{\mathcal{O}}$ is primitive in the sense that it has some positive power $\nu_{\mathcal{O}}^m$ whose entries are all strictly positive. In particular, the $\lambda(\mathbf{1})$-eigenspace is simple, spanned by the trivial idempotent $\mathfrak{e}_{\mathbf{1}}$.*

PROOF. Recall that $\nu_{\mathcal{O}} = \sum_{w \in W} \mathrm{noninv}_{\mathcal{O}}(w) \cdot w$ as an element of $\mathbb{Z}W$, and that it has non-negative coefficients. Consequently, it suffices to show: that the set of w in W satisfying $\mathrm{noninv}_{\mathcal{O}}(w) > 0$ is a generating set for W; and that $\mathrm{noninv}_{\mathcal{O}}(\mathbb{1}) > 0$. The first claim implies that for any position in the matrix there will be a power $\nu_{\mathcal{O}}^m$ for which the entry in this position is positive. The second claim shows that the positive entries accumulate and therefore all entries will be positive for a suitable power.

The second claim is obvious since the definition of $\nu_{\mathcal{O}}$ implies $\nu_{\mathcal{O}}(\mathbb{1}) = |\mathcal{O}|$. For the first claim, we will exhibit an explicit generating set for W for which $\mathrm{noninv}_{\mathcal{O}}$ is positive.

As usual we write $W = \prod_{i=1}^{t} W^{(i)}$ as a product of irreducible reflection groups $W^{(i)}$. Recall [**38**, §4.3] that for finite real reflection groups W, the set S of reflections through the hyperplanes which bound the chosen fundamental chamber c_1 gives rise to a *Coxeter presentation* for W, or a *Coxeter system* (W, S). Recall also that here we identify $W^{(i)}$ with its canonical image in $W = \prod_{i=1}^{t} W^{(i)}$ acting on the same space as W. In this situation, for each $i = 1, 2, \ldots, t$, we can choose the fundamental chambers for each group $W^{(i)}$ independently – the fundamental chamber of W being their intersection. We denote by $(W^{(i)}, S^{(i)})$ the corresponding Coxeter systems.

By assumption for each $1 \le i \le t$ there is a $Y^{(i)} \in \mathcal{O}$ such that

$$Y^{(i)} \quad \not\subseteq \quad \bigcap_{H \in \mathcal{A}^{(i)}} H, \tag{8}$$

where $\mathcal{A}^{(i)}$ is the reflection arrangement for $W^{(i)}$. Let $Z^{(i)}$ be the projection of $Y^{(i)}$ to the intersection lattice of $\mathcal{A}^{(i)}$. Using the action of W on $\mathcal{O}$ to alter $Y^{(i)}$, if necessary, we can assume that $Z^{(i)}$ equals the intersection of a subset of walls of the fundamental chamber for $W^{(i)}$. This subset of walls is indexed by a proper subset $J^{(i)}$ of $S^{(i)}$ (that this containment is proper follows from (8)).

Because each $W^{(i)}$ acts irreducibly, the Coxeter system $(W^{(i)}, S^{(i)})$ has connected Coxeter diagram, and one can number its nodes $s_1^{(i)}, s_2^{(i)}, \ldots, s_{|S^{(i)}|}^{(i)}$ in such a way that $s_1^{(i)}$ is not in $J^{(i)}$, and each initial segment of the nodes induces a connected subdiagram. We claim that the union of the sets

$$\left\{ s_1^{(i)}, s_1^{(i)} s_2^{(i)}, \ldots, s_1^{(i)} s_2^{(i)} \cdots s_{|S^{(i)}|}^{(i)} \right\}$$

is a generating set for W, and that $\mathrm{noninv}_{\mathcal{O}}$ is positive for each of these elements. The reason these elements generate W is that $S^{(i)} = \{s_1^{(i)}, s_2^{(i)}, \ldots, s_{|S^{(i)}|}^{(i)}\}$ generates $W^{(i)}$. We want to show that every $w = s_1^{(i)} s_2^{(i)} \cdots s_j^{(i)}$ verifies $\mathrm{noninv}_{\mathcal{O}}(w) > 0$. For that consider the subspace $Y^{(i)}$ of $\mathcal{O}$. We claim that $Y^{(i)}$ forms a noninversion for w. To see this, by Proposition II.5.4 and subsequent comments one needs to check that w is one of the minimal length coset representatives for $W_{J^{(i)}} \backslash W^{(i)}$, that is, it has no reduced expressions that start with an element of $J^{(i)}$ on the left. But by our construction of the word $w = s_1^{(i)} s_2^{(i)} \cdots s_j^{(i)}$, and by Tits' solution to the word problem for W (see [**1**, Theorem 2.33]), this would be impossible because no element of $J^{(i)}$ can be commuted past the $s_1^{(i)}$ on the left.

The fact that the $\lambda(\mathbf{1})$-eigenspace is simple and is spanned by the trivial idempotent $\mathfrak{e}_1$ now follows from the Perron-Frobenius theorem [**37**, Theorem 8.4.4]. □

For future use (in §III.3), we mention another trivial reduction, similar to Proposition II.8.2, that can occur when the finite real reflection group W acting on V does not act irreducibly. Its proof is similarly straightforward.

Proposition 8.5. *Let W be a finite real reflection group and $W = \prod_{i=1}^{t} W^{(i)}$ for irreducible reflection groups $W^{(i)}$. Let $\mathcal{A}^{(i)}$ be the arrangements of reflecting hyperplanes of the reflections from $W^{(i)}$, $1 \le i \le t$. Let $\mathcal{O} \subseteq \mathcal{L}$ be a W-invariant subset of $\mathcal{L}$.*

Assume that there is an $1 \le i \le t$ such that $\mathcal{O}$ contains no subspaces X lying below any hyperplanes from $\mathcal{A}^{(i)}$. Then we can consider $\mathcal{O}$ as a subset of the intersection lattice for the arrangement $\mathcal{A}' := \mathcal{A} \setminus \mathcal{A}^{(i)}$ of the reflection group

$W' := \prod_{j \neq i} W_j$. *We have* $\mathbb{Z}W \cong \mathbb{Z}W^{(i)} \otimes \mathbb{Z}W'$ *and*

$$\nu_{\mathcal{O}} = \mathbb{1}_{\mathbb{Z}W^{(i)}} \otimes \nu_{\mathcal{O}'}$$

where $\mathbb{1}_{\mathbb{Z}W^{(i)}}$ *is represented by the* $|W^{(i)}| \times |W^{(i)}|$ *matrix having all ones as entries.*

Since the eigenvalues and eigenvectors of $\mathbb{1}_{\mathbb{Z}W}$ are easy to write down, by Proposition II.8.5 one is reduced to studying $\nu_{\mathcal{O}'}$ in this situation.

EXAMPLE 8.6. Example II.8.1 also illustrates the scenario of Proposition II.8.5 except now $i = 2$, and one should interpret the first tensor factor as $\nu_{\mathcal{O}'}$ and the second tensor factor as $\mathbb{1}_{\mathbb{Z}W^{(2)}}$.

CHAPTER III

The case where $\mathcal{O}$ contains only hyperplanes

1. Review of twisted Gelfand pairs

We review here some of the theory of (twisted) Gelfand pairs; a good introduction is Stembridge [**72**].

Definition 1.1. Given a finite group G, a subgroup U, and a linear character $\chi : U \to \mathbb{C}^\times$, say that (G, U, χ) forms a *twisted Gelfand pair* (or *triple*) if the induced representation $\mathrm{Ind}_U^G \chi$ is a multiplicity-free $\mathbb{C}G$-module.

One can fruitfully rephrase this is in terms of the algebra structure of $A := \mathbb{C}G$ and the χ-idempotent for U

$$\mathfrak{e} := \frac{1}{|U|} \sum_{u \in U} \chi(u^{-1}) u. \tag{9}$$

It is well-known and easy to see that the left-ideal $A\mathfrak{e}$ carries a left A-module structure isomorphic to $M = \mathrm{Ind}_U^G \chi$. As with any finite dimensional A-module, M can be expressed as $M = \bigoplus_i (S_i)^{\oplus m_i}$ for distinct simple A-modules S_i and uniquely defined multiplicities m_i. One can detect these multiplicities by looking at the commutant algebra $\mathrm{End}_A M$, which is isomorphic to the direct sum of matrix algebras $\oplus_i \mathrm{Mat}_{m_i \times m_i}(\mathbb{C})$. Thus, the commutant algebra is itself a *commutative* algebra if and only if each $m_i = 1$, that is, if and only if M is multiplicity-free as an A-module. Therefore, the condition for (G, U, χ) to be a twisted Gelfand pair is equivalent to $\mathrm{End}_A M$ being commutative.

On the other hand, for any algebra with unit A and idempotent $\mathfrak{e}$, taking $M = A\mathfrak{e}$, the map defined by

$$\begin{aligned} \mathrm{End}_A M = \mathrm{End}_A(A\mathfrak{e}) &\longrightarrow \quad \mathfrak{e}A\mathfrak{e} \\ \varphi &\longmapsto \quad \varphi(\mathfrak{e}) \end{aligned}$$

is easily seen to be an algebra isomorphism. In the case $A = \mathbb{C}G$ and $\mathfrak{e}$ is the idempotent in 9, the algebra $\mathfrak{e}A\mathfrak{e}$ is sometimes called the *(twisted) Hecke algebra*. If one chooses double coset representatives $\{g_1, \ldots, g_t\}$ for $U \backslash G / U$, then it is easy to see that the non-zero elements in the set $\{\mathfrak{e} g_i \mathfrak{e}\}_{i=1,2,\ldots,t}$ form a $\mathbb{C}$-basis for this Hecke algebra $\mathfrak{e}A\mathfrak{e}$. This leads to the following commonly used trick for verifying that one has a twisted Gelfand pair.

Proposition 1.2 (Twisted version of "Gelfand's trick"). *Let G be a finite group, U a subgroup of G and $\chi : U \to \mathbb{C}^\times$ a linear character with $\chi(u^{-1}) = \chi(u)$ for all u in U, that is, χ takes values in $\{\pm 1\}$.*

If every double coset UgU within G for which $\mathfrak{e}g\mathfrak{e} \neq 0$ contains an involution, then (G, U, χ) forms a twisted Gelfand pair.

Proof. As above let $\mathfrak{e} := \frac{1}{|U|}\sum_{u\in U}\chi(u^{-1})u$. Consider the algebra anti-automorphism ψ of $A = \mathbb{C}G$ that sends $g \mapsto g^{-1}$. The assumption that $\chi(u^{-1}) = \chi(u)$ implies $\psi(\mathfrak{e}) = \mathfrak{e}$. Thus, for any involution $g = g^{-1}$ in G, one has that ψ also fixes the element $\mathfrak{e}g\mathfrak{e}$ in $\mathbb{C}G$:

$$\psi(\mathfrak{e}g\mathfrak{e}) = \psi(\mathfrak{e})\psi(g)\psi(\mathfrak{e}) = \mathfrak{e}g^{-1}\mathfrak{e} = \mathfrak{e}g\mathfrak{e}.$$

The assumption that every double coset UgU for which $\mathfrak{e}g\mathfrak{e} \neq 0$ contains an involution therefore implies that ψ fixes every element in a spanning set for the subalgebra $\mathfrak{e}A\mathfrak{e}$ within the group algebra $A = \mathbb{C}G$. Since ψ is an anti-automorphism on all of A, this subalgebra $\mathfrak{e}A\mathfrak{e}$ must be commutative: for any x, y in $\mathfrak{e}A\mathfrak{e}$, one has

$$x \cdot y = \psi(x)\psi(y) = \psi(y \cdot x) = y \cdot x.$$

Thus, $\mathrm{End}_A(A\mathfrak{e}) = \mathfrak{e}A\mathfrak{e}$ is commutative. Hence $A\mathfrak{e}$ is a multiplicity-free left A-module, i.e. (G, U, χ) is a twisted Gelfand pair. □

2. A new twisted Gelfand pair

Recall the statement of Theorem I.4.2 from the introduction.

Theorem I.4.2. *Let $W \leq \mathrm{GL}(V)$ be any finite irreducible real reflection group and H any of its reflecting hyperplanes with associated reflection s.*

Then the linear character χ of the W-centralizer $\mathrm{Z}_W(s)$ given by the determinant on V/H or $H^\perp$ has a multiplicity-free induced W-representation $\mathrm{Ind}_{\mathrm{Z}_W(s)}^W \chi$.

In other words, $(W, \mathrm{Z}_W(s), \chi)$ forms a twisted Gelfand pair.

As preparation for proving this, we begin with some well-known general observations about group actions on cosets, and double cosets. Let $Z := \mathrm{Z}_W(s)$ and $\mathcal{O}$ the orbit of H under the action of W. Then Z is the stabilizer of the element H in the transitive action of W on $\mathcal{O}$. In other words, $\mathcal{O}$ carries the same W-action as the coset action of W left-translating W/Z. One then has inverse bijections between the double cosets $Z\backslash W/Z$ and the W-*orbitals*, that is the W-orbits of the *diagonal* action of W on $\mathcal{O} \times \mathcal{O}$:

$$\begin{array}{rcl} Z\backslash W/Z & \longrightarrow & W\backslash(\mathcal{O}\times\mathcal{O}) \\ ZwZ & \longmapsto & W\cdot(H, w(H)) \end{array}$$

$$\begin{array}{rcl} W\backslash(\mathcal{O}\times\mathcal{O}) & \longrightarrow & Z\backslash W/Z \\ W\cdot(w_1(H), w_2(H)) & \longmapsto & Zw_1^{-1}w_2Z \end{array}$$

Proposition 2.1. *Let (W, S) be a Coxeter system with W finite, and $J \subset S$ such that the Coxeter graph for (W_J, J) is a connected subgraph of the Coxeter graph for (W, S). Then for two reflecting hyperplanes H, H' whose reflections $s_H, s_{H'}$ lie in W_J we have: $s_H, s_{H'}$ lie in the same W-orbit if and only they lie in the same W_J-orbit.*

Proof. Since every reflection in W_J is W_J-conjugate to a simple reflection in J, one may assume without loss of generality that $s_H, s_{H'}$ are simple reflections lying in the subset J. It is well-known (see e.g. [**12**, Chapter 1, Exercise 16, p. 23]) that two simple reflections s, s' in S are W-conjugate if and only if there is a path in the Coxeter graph for (W, S) having all edges with *odd* labels. Since W is finite, the Coxeter graph for (W, S) is a tree. Hence such a path with odd labels exists if and only if it exists within the Coxeter subgraph for (W_J, J), that is, if and only if $s_H, s_{H'}$ are W_J-conjugate. □

Proof of Theorem I.4.2. We will show that the twisted version of Gelfand's trick (Proposition III.1.2) applies. Let $w \in W$ and $H' := w(H)$. Let s_H and $s_{H'}$ be the reflections corresponding to H and H'.
Case 1: H, H' are orthogonal.

In this case we claim that $\mathfrak{e}w\mathfrak{e} = 0$. To see this, note that in this situation, both $s_H, s_{H'}$ lie in Z, with

$$\chi(s_H) = -1$$
$$\chi(s_{H'}) = +1.$$

Thus, factoring the subgroup $Z = \mathrm{Z}_W(s)$ according to cosets $Z/\langle s_{H'}\rangle$ and cosets $\langle s_H\rangle\backslash Z$ gives rise to factorizations

$$\mathfrak{e} = \mathfrak{a}(1 + s_{H'}) = (1 - s_H)\mathfrak{b}$$

for some elements $\mathfrak{a}, \mathfrak{b}$ in $\mathbb{R}W$. One then calculates

$$\begin{aligned}\mathfrak{e}w\mathfrak{e} &= \mathfrak{a}(1 + s_{H'})w(1 - s_H)\mathfrak{b}\\ &= \mathfrak{a}(w - ws_H + s_{H'}w - s_{H'}ws_H)\mathfrak{b}\\ &= \mathfrak{a}\cdot 0\cdot \mathfrak{b}\\ &= 0\end{aligned}$$

where the third line uses the following equalities:

$$\begin{aligned}w(H) &= H', \text{ implying}\\ ws_Hw^{-1} &= s_{H'}\\ ws_H &= s_{H'}w\\ w &= s_{H'}ws_H.\end{aligned}$$

Case 2: H, H' are *not* orthogonal.

A trivial subcase occurs when $H = H'$ and then the double coset $ZwZ = Z$ contains the involution s_H. Hence we are done by Proposition III.1.2.

Otherwise, the parabolic subgroup $\mathrm{Z}_W(H \cap H')$ is dihedral, and W-conjugate to some standard parabolic W_J for some pair $J = \{s, s'\} \subset S$; without loss of generality (by conjugation), $s_H, s_{H'}$ lie in W_J. Since H, H' are not orthogonal, one must have s, s' non-commuting, and hence the Coxeter graph for (W_J, J) is an edge with label $m \geq 3$, forming a connected subgraph of the Coxeter graph of (W, S). Since H, H' were assumed to lie in the same W-orbit, Proposition III.2.1 implies they lie in the same W_J-orbit. However, when $s_H, s_{H'}$ lie within a dihedral group W_J, it is easy to check that if w in W_J sends H to H', then either w or ws_H is a reflection, and hence an involution, sending H to H'. Again the assertion follows from Proposition III.1.2. □

Remark 2.2. The preceding proof is perhaps more subtle than it first appears. When H and H' are *orthogonal* hyperplanes lying in the same W-orbit, so that $H' = w(H)$ for some w in W, it *can* happen that H, H' do *not* lie in the same $\mathrm{Z}_W(H \cap H')$-orbit, and that the double coset ZwZ for $Z = \mathrm{Z}_W(s_H)$ contains *no* involutions.

As an example, this occurs within the Coxeter system (W, S) of type H_3 with Coxeter generators $S = \{s_1, s_2, s_3\}$, satisfying $s_i^2 = 1$ and $(s_1s_2)^5 = (s_1s_3)^2 = (s_2s_3)^3 = e$. The hyperplanes H, H' fixed by s_1, s_3, respectively, are orthogonal. They lie in the same W-orbit, and in fact $w(H) = H'$ for $w = s_2s_1s_2s_3s_1s_2$. However, H, H' do not lie in the same orbit for the rank 2 parabolic $\mathrm{Z}_W(H \cap H') =$

$W_{\{s_1,s_3\}}$, and one finds that the double coset ZwZ for the subgroup $Z = \mathrm{Z}_W(s_1) = \langle s_1, s_3, w_0\rangle$ contains elements of orders 3 and 6, but contains no involutions.

3. Two proofs of Theorem I.4.1

We recall the statement of the theorem.

Theorem I.4.1. *For any finite real reflection group W, and any W-orbit $\mathcal{O}$ of hyperplanes, the matrix $\nu_{\mathcal{O}}$ has all its eigenvalues within the ring of integers of the unique minimal splitting field for W. In particular, when W is crystallographic, these eigenvalues lie in $\mathbb{Z}$.*

We will offer two proofs. In both proofs, one first notes that one can immediately use Proposition II.8.2 to reduce to the case where W acts irreducibly on V. Also note that if $W = \prod_{i=1}^{t} W^{(i)}$ for irreducible reflection groups $W^{(i)}$ and $\mathcal{A}^{(i)}$ the arrangements consisting of the reflecting hyperplanes of the reflections from $W^{(i)}$, then a W-orbit $\mathcal{O}$ of hyperplanes in $\mathcal{A}$ contains only hyperplanes from a single subarrangement $\mathcal{A}^{(i)}$ for some $1 \leq i \leq t$.

Thus, one can assume W acts irreducibly on V, and both proofs will rely on Theorem I.4.2.

3.1. First proof of Theorem I.4.1. The first proof is shorter, but makes forward reference to the equivariant theory of BHR random walks in Chapter IV. This BHR theory will show that when $\nu_{\mathcal{O}}$ acts on $\mathbb{R}W$, its image subspace $U := \ker(\nu_{\mathcal{O}})^{\perp}$ affords the W-representation $\mathbf{1}_W \oplus \mathrm{Ind}_{\mathrm{Z}_W(s)}^{W} \chi$, where $\chi = \det|_{V/H}$. This deduction will be based on Theorem IV.4.2 and Corollary IV.6.1. Note that the image U can have no multiplicity on the trivial representation $\mathbf{1}_W$, since the ambient space $\mathbb{R}W$ contains only one copy of $\mathbf{1}_W$. Hence Theorem I.4.2 tell us that the $\nu_{\mathcal{O}}$-stable subspace U is multiplicity-free as a W-representation. Since $U = \ker(\nu_{\mathcal{O}})^{\perp}$ is a $\mathbb{Q}$-subspace (as $\nu_{\mathcal{O}}$ has $\mathbb{Z}$ entries) an application of Proposition I.3.1 finishes the proof.

3.2. Second proof of Theorem I.4.1. This proof, although longer, does not rely on results to be proven later, and also introduces an important idea, useful both in understanding the eigenspaces of $\nu_{\mathcal{O}}$, and with potential applications to the analysis of linear ordering polytopes (see §III.5). We start by developing this idea here.

For the moment, return to the situation where $\mathcal{A}$ is a central arrangement of hyperplanes in $V = \mathbb{R}^d$ having some finite subgroup W of $\mathrm{GL}(V)$ acting as symmetries, with chambers $\mathcal{C}$, intersection lattice $\mathcal{L}$, and $\mathcal{O}$ any W-stable subset of $\mathcal{L}$. Recall that $\nu_{\mathcal{O}} = \pi_{\mathcal{O}}^T \circ \pi_{\mathcal{O}}$ where

$$\begin{aligned} \pi_{\mathcal{O}} : \mathbb{Z}\mathcal{C} &\longrightarrow \bigoplus_{X\in\mathcal{O}} \mathbb{Z}\mathcal{C}(\mathcal{A}/X) \\ c &\longmapsto (c/X)_{X\in\mathcal{O}} \end{aligned}$$

Note that $\pi_{\mathcal{O}}$ is W-equivariant for the obvious W-actions on the source and targets. It is also equivariant for the commuting $\mathbb{Z}_2$-action that sends $c \mapsto -c$ in the source, and sends $c/X \mapsto -c/X$ in the target.

This gives us the freedom to consider instead of $\nu_{\mathcal{O}} = \pi_{\mathcal{O}}^T \circ \pi_{\mathcal{O}}$, the eigenvectors and eigenvalues of the closely related map

$$\mu_{\mathcal{O}} = \pi_{\mathcal{O}} \circ \pi_{\mathcal{O}}^T : \bigoplus_{H \in \mathcal{O}} \mathbb{Z}^{\mathcal{C}(\mathcal{A}/H)} \longrightarrow \bigoplus_{H \in \mathcal{O}} \mathbb{Z}^{\mathcal{C}(\mathcal{A}/H)}$$

having matrix entries given by

$$(\mu_{\mathcal{O}})_{c_1/X_1, c_2/X_2} = \Big| \{ c \in \mathcal{C} : c/X_1 = c_1/X_1 \text{ and } c/X_2 = c_2/X_2 \} \Big|.$$

PROPOSITION 3.1. *For each non-zero eigenvalue λ in $\mathbb{R}$, the maps $\pi_{\mathcal{O}}$ and $\pi_{\mathcal{O}}^T$ give $W \times \mathbb{Z}_2$-equivariant isomorphisms between the λ-eigenspaces of $\nu_{\mathcal{O}}$ and $\mu_{\mathcal{O}}$.*

PROOF. This is a general linear algebra fact. Assume $A : U \to U'$ and $B : U' \to U$ are $\mathbb{K}$-linear maps of finite-dimensional $\mathbb{K}$-vector spaces U and U' such that all eigenvalues of AB and BA lie in $\mathbb{K}$. We claim that for each potential non-zero eigenvalue λ in $\mathbb{K}$, the maps A and B give isomorphisms between the *generalized λ-eigenspaces* defined to be the subsets of U and U' on which $\lambda - BA$ and $\lambda - AB$ act nilpotently. To see that A, B map between these generalized eigenspaces, note that given a vector v in V with $(\lambda \mathrm{I}_U - BA)^N v = 0$, the fact that

$$(\lambda \mathrm{I}_{U'} - AB)A = A(\lambda \mathrm{I}_U - BA)$$

implies

$$(\lambda \mathrm{I}_{U'} - AB)^N A v = A(\lambda \mathrm{I}_U - BA)^N v = 0.$$

To see that A, B are injective, note that if $Av = 0$ then $(\lambda - BA)v = \lambda v$ and hence

$$0 = (\lambda \mathrm{I}_U - BA)^N v = \lambda^N v$$

would imply that $v = 0$. Since the generalized eigenspaces for non-zero eigenvalues plus the kernel sum up to U, resp. U', it follows from dimension considerations that the maps A and B are indeed isomorphisms between the generalized eigenspaces corresponding to non-zero eigenvalues.

When applying this with $A = \pi_{\mathcal{O}}$ and $B = \pi_{\mathcal{O}}^T$ and $\mathbb{K} = \mathbb{R}$, self-adjointness implies not only that all the eigenvalues λ all lie in $\mathbb{R}$, but also semisimplicity, so that generalized λ-eigenspaces are just λ-eigenspaces. $\square$

Now we specialize to the situation where $\mathcal{A}$ is the reflection arrangement for a finite real reflection group W, and the W-stable subset $\mathcal{O}$ contains only hyperplanes H (but we do not assume yet that $\mathcal{O}$ is a single W-orbit).

In this case, each of the localized subarrangements $\mathcal{A}/H$ has only one hyperplane H, and only two chambers/half-spaces in $\mathcal{C}(\mathcal{A}/H)$, which one can identify with the two unit normals $\pm\alpha$ (or *roots*) to the hyperplane H. Letting $\Phi_{\mathcal{O}}$ denote the union of all such pairs of roots $\pm\alpha$ normal to the hyperplanes H in $\mathcal{O}$, one can identify $\bigoplus_{H \in \mathcal{O}} \mathbb{Z}^{\mathcal{C}(\mathcal{A}/H)}$ with $\mathbb{Z}^{\Phi_{\mathcal{O}}}$, having a basis element E_α for each α in the orbit of roots $\Phi_{\mathcal{O}}$. Let $\Phi_+ \subseteq \Phi_{\mathcal{O}}$ be the set of $\alpha \in \Phi$ for which c_1 and α lie on the same side of the hyperplane H_α orthogonal to α. The elements of Φ_+ are called the *positive roots* inside Φ. Clearly, Φ_+ depends on the choice of c_1. Using this notation and under the above identification, the map $\mathbb{Z}W \overset{\pi_{\mathcal{O}}}{\to} \mathbb{Z}^{\Phi_{\mathcal{O}}}$ has

$$(\pi_{\mathcal{O}})_{w, \mathrm{E}_\alpha} = \begin{cases} 1 & \text{if } w^{-1}(\alpha) \in \Phi_+ \\ 0 & \text{otherwise.} \end{cases}$$

To see this note that by definition $(\pi_{\mathcal{O}})_{w, \mathrm{E}_\alpha} = 1$ if and only if $c_w/H_\alpha = c_1/H_\alpha$ and $\alpha \in \Phi_+$ or $c_w/H_\alpha \neq c_1/H_\alpha$ and $\alpha \notin \Phi_+$. Since in the first case we have

$w^{-1}(\alpha) \in \Phi_+$ if and only if $\alpha \in \Phi_+$ and in the second case $w^{-1}(\alpha) \in \Phi_+$ if and only if $\alpha \notin \Phi_+$ the claim about $(\pi_\mathcal{O})_{w,\mathrm{E}_\alpha}$ follows.

Hence, $\pi_\mathcal{O}$ sends a basis element w in $\mathbb{Z}W$ to the sum of basis elements E_α for which $w^{-1}(\alpha)$ is an element of the positive roots Φ_+. Therefore, the map $\mathbb{Z}^{\Phi_\mathcal{O}} \xrightarrow{\mu_\mathcal{O}} \mathbb{Z}^{\Phi_\mathcal{O}}$ has entry

$$(10) \quad (\mu_\mathcal{O})_{\mathrm{E}_\alpha,\mathrm{E}_\beta} = \#\{w \in W : w^{-1}(\alpha), w^{-1}(\beta) \text{ both lie in } \Phi_+\} = |W| \cdot \frac{\angle\{\alpha,\beta\}}{2\pi}$$

where $\angle\{\alpha,\beta\}$ is the angular measure in radians of the sector which is the intersection of the half-spaces $H_\alpha^+ \cap H_\beta^+$. For the second equality in (10) we use the fact that the elements of the set on the left hand side are the minimal length coset representatives of the group generated by the reflections along H_α and H_β. These reflections generate a dihedral group of order $\frac{2\pi}{\angle\{\alpha,\beta\}}$. Now the equality follows from Lagrange's theorem.

Note that the $\mathbb{Z}_2$-action now sends E_α to $\mathrm{E}_{-\alpha}$. We use this $\mathbb{Z}_2$-action to decompose

$$\mathbb{R}^{\Phi_\mathcal{O}} = \mathbb{R}^{\Phi_\mathcal{O},+} \oplus \mathbb{R}^{\Phi_\mathcal{O},-}$$

in which

$$\mathbb{R}^{\Phi_\mathcal{O},+} \text{ has } \mathbb{R}\text{-basis } \{f_\alpha^+ := \mathrm{E}_\alpha + \mathrm{E}_{-\alpha}\}_{\alpha \in \Phi_\mathcal{O} \cap \Phi_+},$$
$$\mathbb{R}^{\Phi_\mathcal{O},-} \text{ has } \mathbb{R}\text{-basis } \{f_\alpha^- := \mathrm{E}_\alpha - \mathrm{E}_{-\alpha}\}_{\alpha \in \Phi_\mathcal{O} \cap \Phi_+}.$$

For the formulation of the following proposition, recall $\lambda_\mathcal{O}(\chi)$ defined in Proposition II.7.2.

PROPOSITION 3.2. *Acting on $\mathbb{R}^{\Phi_\mathcal{O},+}$, the map $\mu_\mathcal{O}$ has a one-dimensional eigenspace with eigenvalue $\lambda_\mathcal{O}(\mathbf{1}_W)$ carrying the trivial W-representation $\mathbf{1}_W$, and whose orthogonal complement within $\mathbb{R}^{\Phi_\mathcal{O},+}$ lies in the kernel.*

If $\mathcal{O}$ decomposes into W-orbits as $\mathcal{O} = \bigsqcup_{i=1}^t \mathcal{O}_i$ in which $\mathcal{O}_i$ is the orbit of a hyperplane H_i having associated reflection s_i, then $\mathbb{R}^{\Phi_\mathcal{O},-}$ carries the W-representation $\bigoplus_{i=1}^t \mathrm{Ind}_{\mathrm{Z}_W(s_i)}^W \chi_i$, where χ_i is the one-dimensional character $\det_{V/H_i}$.

PROOF. Using the fact that for any w in W, exactly one out of $w^{-1}(\alpha)$ and $w^{-1}(-\alpha)$ will be a positive root, one checks using 10 that

$$\mu_\mathcal{O}(f_\beta^+) = \frac{|W|}{2} \cdot \sum_{\alpha \in \Phi_\mathcal{O} \cap \Phi_+} f_\alpha^+$$

for any β in $\Phi_\mathcal{O} \cap \Phi_+$. This implies that $\mu_\mathcal{O}$ acts on $\mathbb{R}^{\Phi_\mathcal{O},+}$ as an operator of rank one, whose only non-zero eigenspace is the line spanned by $\sum_{\Phi_\mathcal{O} \cap \Phi_+} f_\alpha^+$, affording the trivial W-representation $\mathbf{1}_W$, and with eigenvalue $\lambda_\mathcal{O}(\mathbf{1}_W) = \frac{|W|}{2}|\mathcal{O}|$. Because $\mu_\mathcal{O}$ is self-adjoint, the subspace of $\mathbb{R}^{\Phi_\mathcal{O},+}$ perpendicular to this eigenspace will be preserved, and must lie entirely in the kernel.

Next we turn to the assertion about the W-representation carried by $\mathbb{R}^{\Phi_\mathcal{O},-}$. Let H_i be the representative of the W-orbit $\mathcal{O}_i$, s_i the reflection along H_i and $\alpha_i \in \Phi_+$ the positive root corresponding to H_i. The subgroup $\mathrm{Z}_W(s_i)$ is the stabilizer of the real line spanned by $f_{\alpha_i}^-$. Hence all $z \in \mathrm{Z}_W(s_i)$ have $f_{\alpha_i}^-$ as an eigenvector for eigenvalue ± 1. The eigenvalue for z is $+1$ if α is stabilized by z and -1 if α is sent to $-\alpha$. In either case the eigenvalue coincides with $\det_{V/H_i}(z)$. Thus, the character of $\mathrm{Z}_W(s_i)$ on the line spanned by $f_{\alpha_i}^-$ is given by $\det_{V/H_i}$. Since $\mathrm{Z}_W(s_i)$ is the stabilizer of that line it follows that $\mathrm{Ind}_{\mathrm{Z}_W(s_i)}^W \det_{V/H_i}$ is a character whose degree equals the cardinality of the space spanned by the orbit of f_α^- under the action of W. By a simple calculation one then checks that $\mathrm{Ind}_{Z_W(s_i)}^W \det_{V/H_i}$ and the character on the latter space coincide. Summing up over the W-orbits then yields the asserted formula. ☐

Second proof of Theorem I.4.1. Assuming $\mathcal{O}$ is a transitive W-orbit of some hyperplane H with associated reflection s, Proposition III.3.2 says that the $\mathbb{R}$-subspace $U := \mathbb{R}^{\Phi_{\mathcal{O}},-}$, which is a rational subspace in the sense that $\mathbb{R}^{\Phi_{\mathcal{O}},-} = \mathbb{R} \otimes_{\mathbb{Q}} \mathbb{Q}^{\Phi_{\mathcal{O}},-}$, affords the W-representation $\mathrm{Ind}_{Z_W(s)}^W \chi$. Then Theorem I.4.2 and Proposition I.3.1 imply that the operator $\mu_{\mathcal{O}}$ has all eigenvalues on U lying within the algebraic integers of any splitting field for W. Its remaining eigenvalues on the complementary subspace $\mathbb{R}^{\Phi_{\mathcal{O}},+}$ are either zero or $\lambda_{\mathcal{O}}(\mathbf{1}_W) = \frac{|W|}{2}|\mathcal{O}|$ by Proposition III.3.2. Now an application of Proposition III.3.1 completes the proof. ☐

Remark 3.3. After posting this work on the arXiv, the authors discovered that, independently, P. Renteln [**52**, §4] recently studied the spectrum of the operator $\nu_{\mathcal{O}}$ for a real finite reflection group W, taking $\mathcal{O}$ to be the set of *all* reflecting hyperplanes for W. Note that irreducible finite reflection groups can have at most two W-orbits of hyperplanes, and whenever W has only one orbit of hyperplanes (that is, outside of types $B_n(= C_n)$, F_4 and the dihedral types $I_2(m)$ with m even), Renteln's object of study is the same as our operator $\nu_{\mathcal{O}}$.

In particular, he also uses the technique from our second proof of Theorem I.4.1, introducing the maps $\pi_{\mathcal{O}}$ and $\mu_{\mathcal{O}}$ in his context. We will point out in Remark III.4.5 and Remark III.4.8 below the places where we borrow from and/or extend his work.

4. The eigenvalues and eigenspace representations

We return again to the situation where $\mathcal{O}$ is a single W-orbit of hyperplanes. Having proven Theorem I.4.1 on the integrality of eigenvalues of $\nu_{\mathcal{O}}$ or $\mu_{\mathcal{O}}$, one can still ask for the eigenvalues of $\mu_{\mathcal{O}}$ and the W-irreducible decomposition of its eigenspaces. It turns out that one can be surprisingly explicit here.

Note that Proposition III.3.2 reduces this to the analysis of $\mu_{\mathcal{O}}$ acting on $U := \mathbb{R}^{\Phi_{\mathcal{O}},-}$, which affords the W-representation $\mathrm{Ind}_Z^W \chi$, where $Z = Z_W(s)$ for a reflection s whose hyperplane H represents the orbit $\mathcal{O}$, and $\chi : Z \to \{\pm 1\}$ is the character of Z acting on the line $H^\perp$. We analyze this representation more fully.

We know the W-irreducible decomposition of $\mathrm{Ind}_Z^W \chi$ is multiplicity-free from Theorem I.4.2. Recall this is controlled by the double cosets ZwZ, or W-orbitals $W \cdot (H, H')$ in $\mathcal{O} \times \mathcal{O}$, giving rise to non-zero elements $\mathfrak{e}w\mathfrak{e}$ in the twisted Hecke algebra $\mathfrak{e}\mathbb{R}W\mathfrak{e}$ (see §III.1). We next explain how dihedral angles between hyperplanes play a crucial role here.

Definition 4.1. Given two hyperplanes H, H' within V, define their *dihedral angle* $\angle\{H, H'\}$ to be the unique angle in the interval $[0, \frac{\pi}{2}]$ separating them.

Proposition 4.2. *Let W be a finite real reflection group, and H,H',H'' hyperplanes in the same W-orbit $\mathcal{O}$, but with neither H' nor H'' orthogonal to H. Then $(H, H'), (H, H'')$ lie in the same W-orbital on $\mathcal{O} \times \mathcal{O}$ if and only if $\angle\{H, H'\} = \angle\{H, H''\}$.*

Proof. The forward implication is clear. For the reverse, assume $\angle\{H, H'\} = \angle\{H, H''\}$, and consider three cases based on the codimension of $X := H \cap H' \cap H''$.

Case 1: X has codimension 1.

This case is trivial, since then $H = H' = H''$.

Case 2: X has codimension 2.

This case is also straightforward. One checks inside the dihedral reflection subgroup $\mathrm{Z}_W(X)$ containing $s_H, s_{H'}, s_{H''}$ that whenever $\angle\{H, H'\} = \angle\{H, H''\}$, either one is in the trivial case $H' = H''$, or else s_H sends (H, H') to (H, H'').

Case 3: X has codimension 3. Then by conjugation, one may assume that the rank 3 reflection subgroup $\mathrm{Z}_W(X)$ containing $s_H, s_{H'}, s_{H''}$ is a standard parabolic subgroup W_J for some triple $J = \{s_1, s_2, s_3\} \subset S$ among the Coxeter generators S of W. In fact, (W_J, J) must be a connected subgraph of the Coxeter graph of (W, S), else W_J contains no three reflections $s_H, s_{H'}, s_{H''}$ with $H \cap H' \cap H''$ of codimension 3 having $\angle\{H, H'\} = \angle\{H, H''\} \neq \frac{\pi}{2}$. Thus, Proposition III.2.1 implies that H, H', H'' lie in the same $\mathrm{Z}_W(X)$-orbit, since they lie in the same W-orbit. Finiteness of W further forces $\mathrm{Z}_W(X)$ to be one of the rank three irreducible types $A_3(\cong D_3)$ or $B_3(\cong C_3)$ or H_3. Now it is not hard to check by brute force in any of these three types that a triple H, H', H'' in the same $\mathrm{Z}_W(X)$-orbit having $\angle\{H, H'\} = \angle\{H, H''\} \neq \frac{\pi}{2}$ will have (H, H') and (H, H'') in the same $\mathrm{Z}_W(X)$-orbital. This then implies that (H, H') and (H, H'') lie in the same W-orbital. $\square$

The following example shows that the non-orthogonality assumption in Proposition III.4.2 is perhaps more subtle than it first appears. Indeed, if $\angle\{H, H'\} = \angle\{H, H''\} = \frac{\pi}{2}$, it is possible that $(H, H'), (H, H'')$ lie in different W-orbits of $\mathcal{O} \times \mathcal{O}$.

Example 4.3. Let W be of type D_n for $n \geq 4$, and

$$\begin{aligned} H &= \{x_1 = x_2\} \\ H' &= \{x_1 = -x_2\} \\ H'' &= \{x_3 = x_4\}. \end{aligned}$$

Then it is easily checked that $(H, H'), (H, H'')$ lie in different W-orbits of $\mathcal{O} \times \mathcal{O}$. The problem here is that $X = H \cap H' \cap H''$ has $\mathrm{Z}_W(X)$ of the reducible type $A_1 \times A_1 \times A_1$, so that Proposition III.2.1 does not apply.

Proposition III.4.2 has very strong consequences in the *crystallographic* case, that is, where W is a finite *Weyl* group. For this we distinguish two cases for a given reflecting hyperplane H for a finite reflection group W and its W-orbit $\mathcal{O}$:

$(\frac{\pi}{3})$ There is a hyperplane $H' \in \mathcal{O}$ for which for which $\angle\{H, H'\} = \frac{\pi}{3}$.

$(\cancel{\frac{\pi}{3}})$ There is no hyperplane $H' \in \mathcal{O}$ for which for which $\angle\{H, H'\} = \frac{\pi}{3}$.

Note that $(\cancel{\frac{\pi}{3}})$ occurs only in the situation when W is of type $B_n(\cong C_n)$, and the reflection s_H along H is the special "non-simply-laced" node, corresponding to a sign change in a coordinate of $V = \mathbb{R}^n$.

Corollary 4.4. *Let W be a finite irreducible Weyl group acting on $V \cong \mathbb{R}^n$ in its reflection representation, $\mathcal{O}$ the W-orbit of a reflecting hyperplane H with reflection s, and $Z = \mathrm{Z}_W(s)$. Let $\chi : Z \to \{\pm 1\}$ be the character of Z on $H^\perp$.*

Then:

(i) *In situation $(\frac{\pi}{3})$ we have*

$$\mathrm{Ind}_Z^W \chi = V \oplus V'$$

for a unique W-irreducible V' of dimension $|\mathcal{O}| - n$.

(ii) *In situation $(\not\frac{\pi}{3})$ we have*

$$\mathrm{Ind}_Z^W \chi = V.$$

Moreover, in (i)*, one can realize the W-irreducible V' as the subspace of $\mathbb{R}^{\Phi_{\mathcal{O}},-}$ that is $\mathbb{R}$-linearly spanned by the vectors*

$$\psi_{\alpha,\beta,\gamma} := \mathrm{E}_\alpha + \mathrm{E}_\beta + \mathrm{E}_\gamma - (\mathrm{E}_{-\alpha} + \mathrm{E}_{-\beta} + \mathrm{E}_{-\gamma})$$

as $\{\alpha, \beta, \gamma\}$ run through all triples of roots in the W-orbit $\mathcal{O}$ having $\alpha + \beta + \gamma = 0$ and having normal hyperplanes $H_\alpha, H_\beta, H_\gamma$ with pairwise dihedral angles of $\frac{\pi}{3}$.

Proof. By Proposition III.4.2 and Case 1 of the proof of Theorem I.4.2, the number of W-irreducible constituents in $\mathrm{Ind}_Z^W \chi$ is the number of dihedral angles $\angle\{H, H'\}$ other than $\frac{\pi}{2}$ which occur among pairs $\{H, H'\}$ in the W-orbit $\mathcal{O}$. By conjugation, one may assume $\mathrm{Z}_W(H \cap H')$ is a standard parabolic subgroup W_J, of some dihedral type $I_2(m)$ with $m \geq 3$. Since W is a Weyl group, this limits m to be $3, 4, 6$, and then one can check that $\{H, H'\}$ lying in the same W-orbit $\mathcal{O}$ forces either $H = H'$ or $\angle\{H, H'\} = \frac{\pi}{3}$. It follows that $\mathrm{Ind}_Z^W \chi$ has exactly two irreducible constituents in situations $(\frac{\pi}{3})$, and exactly one irreducible constituent in situation $(\not\frac{\pi}{3})$.

To prove the remaining assertions, let $Y \subset \mathbb{R}^{\Phi_{\mathcal{O}},-} \subset \mathbb{R}^{\Phi_{\mathcal{O}}}$ be the subspace spanned by the vectors $\psi_{\alpha,\beta,\gamma}$ described above. Consider the $\mathbb{R}$-linear map $\mathbb{R}^{\Phi_{\mathcal{O}}} \xrightarrow{g} V$ that sends $\mathrm{E}_\alpha \longmapsto \alpha$. It is easy to see that g is W-equivariant, and also $\mathbb{Z}_2$-equivariant for the $\mathbb{Z}_2$-action on $\mathbb{R}^{\Phi_{\mathcal{O}}}$ that swaps $\mathrm{E}_\alpha \leftrightarrow \mathrm{E}_{-\alpha}$ and the $\mathbb{Z}_2$-action on V by the scalar -1. The calculations

$$\begin{array}{rcll} \mathrm{E}_\alpha - \mathrm{E}_{-\alpha} & \overset{g}{\longmapsto} & \alpha - (-\alpha) & = 2\alpha \\ \mathrm{E}_\alpha + \mathrm{E}_{-\alpha} & \overset{g}{\longmapsto} & \alpha + (-\alpha) & = 0 \\ \psi_{\alpha,\beta,\gamma} & \overset{g}{\longmapsto} & 2(\alpha + \beta + \gamma) & = 0 \end{array}$$

then show that

- the kernel $\ker(g)$ contains $\mathbb{R}^{\Phi_{\mathcal{O}},+}$, and hence g induces a map $\mathbb{R}^{\Phi_{\mathcal{O}},-} \xrightarrow{\bar{g}} V$,
- the map g, and hence also $\bar{g}$, surjects onto V, since V is irreducible, and
- the subspace Y lies in the kernel of g, so also $Y \subset \ker\left(\mathbb{R}^{\Phi_{\mathcal{O}},-} \overset{\bar{g}}{\twoheadrightarrow} V\right)$.

The surjection $\bar{g}$ shows that $\mathrm{Ind}_Z^W \chi$ always contains V as one of its constituents. Hence, there are no other constituents in situation $(\not\frac{\pi}{3})$. However, in situation $(\frac{\pi}{3})$, the subspace Y is nonzero, and hence must form the other irreducible constituent of $\mathrm{Ind}_Z^W \chi$. $\square$

Remark 4.5. Here we have borrowed from Renteln's paper [**52**, §4.8.1] the explicit realization of V' by the vectors $\psi_{\alpha,\beta,\gamma}$, and its proof via the map g, although we substitute our argument via irreducibility for his dimension-counting argument.

Example 4.6. In type A_{n-1}, when $W = \mathfrak{S}_n$ and $\mathcal{O}$ is the unique W-orbit of hyperplanes, one can check that

$$\begin{aligned} \mathrm{Ind}_Z^W \chi &= \mathrm{Ind}_{\mathfrak{S}_2 \times \mathfrak{S}_{n-2}}^{\mathfrak{S}_n} \mathrm{sgn} \otimes \mathbf{1} \\ &= \chi^{(n-1,1)} + \chi^{(n-2,1,1)} \\ &= V \oplus \wedge^2 V \end{aligned}$$

using standard calculations with the $\mathfrak{S}_n$-irreducible characters χ^λ indexed by integer partitions λ of n. Thus, the irreducible $V' \cong \wedge^2 V \cong \chi^{(n-2,1,1)}$ in this case.

Based on the W-irreducible description for $\mathbb{R}^{\Phi_{\mathcal{O}},-} \cong \mathrm{Ind}_Z^W \chi$ given in Corollary III.4.4, one can now be more precise about the eigenspaces of $\nu_{\mathcal{O}}$ or $\mu_{\mathcal{O}}$.

THEOREM 4.7. *Let W be a finite irreducible Weyl group acting on $V \cong \mathbb{R}^n$ in its reflection representation, $\mathcal{O}$ the W-orbit of a reflecting hyperplane H with reflection s, and $Z = \mathrm{Z}_W(s)$. Let $\chi : Z \to \{\pm 1\}$ be the character of Z on $H^\perp$.*

Then either of $\nu_{\mathcal{O}}$ or $\mu_{\mathcal{O}}$ have non-zero eigenvalues and accompanying W-irreducible eigenspaces described as follows:

(i) *There is a 1-dimensional eigenspace carrying the trivial W-representation with eigenvalue $\lambda_{\mathcal{O}} = \frac{|\mathcal{O}||W|}{2}$.*

(ii) *In the case of situation $(\frac{\pi}{3})$ there is an $(|\mathcal{O}| - n)$-dimensional eigenspace carrying the W-representation V' with eigenvalue $\frac{|W|}{6}$.*

(iii) *In either situation $(\frac{\pi}{3})$ and $(\frac{\not\pi}{3})$, there is an n-dimensional eigenspace carrying the W-representation V with eigenvalue*

$$\begin{cases} \frac{(2|\mathcal{O}|+n)|W|}{6n} & \text{in situation } (\frac{\pi}{3}), \\ 2^{n-1} n! & \text{in situation } (\frac{\not\pi}{3}). \end{cases}$$

Furthermore, in the subcase of situation $(\frac{\pi}{3})$ where W is simply-laced (type A_n, D_n, or E_6, E_7, E_8), one can rewrite this eigenvalue as $\frac{(h+1)|W|}{6}$, where h is the Coxeter number.

PROOF. Proposition III.3.2 already shows assertion (i), and the fact that $\mathbb{R}^{\Phi_{\mathcal{O}},-}$ gives the remaining non-kernel eigenspaces of $\mu_{\mathcal{O}}$. Calculating traces, one sees from 10 that the diagonal entry $(\mu_{\mathcal{O}})_{\alpha,\alpha} = \frac{|W|}{2}$ for each root α in $\Phi_{\mathcal{O}}$, so that $\mu_{\mathcal{O}}$ has trace $\frac{|W||\Phi_{\mathcal{O}}|}{2} = |W||\mathcal{O}|$ when acting on $\mathbb{R}^{\Phi_{\mathcal{O}}}$. Since the eigenvalues of $\mu_{\mathcal{O}}$ on $\mathbb{R}^{\Phi_{\mathcal{O}},+}$ are all zero except for the eigenvalue $\lambda_{\mathcal{O}} = \frac{|W||\mathcal{O}|}{2}$ with multiplicity one, one concludes that $\mu_{\mathcal{O}}$ has trace $|W||\mathcal{O}| - \frac{|W||\mathcal{O}|}{2} = \frac{|W||\mathcal{O}|}{2}$ when restricted to $\mathbb{R}^{\Phi_{\mathcal{O}},-}$.

Thus, in situation $(\frac{\not\pi}{3})$, where $\mathbb{R}^{\Phi_{\mathcal{O}},-} \cong V \cong \mathbb{R}^n$, it acts with eigenvalue $\frac{|W||\mathcal{O}|}{2n} = 2^{n-1} n!$.

In situation $(\frac{\pi}{3})$, Schur's Lemma implies that the W-irreducible constituent V' of $\mathbb{R}^{\Phi_{\mathcal{O}},-}$ will lie in a single eigenspace for $\mu_{\mathcal{O}}$. Since this copy of V' is realized as the span of the elements $\{\psi_{\alpha,\beta,\gamma}\}$, one can, for example, determine this eigenvalue by using 10 to compute that the coefficient of E_α in $\mu_{\mathcal{O}}(\psi_{\alpha,\beta,\gamma})$ is

$$\frac{|W|}{2\pi}\left(\pi + \frac{\pi}{3} + \frac{\pi}{3} - 0 - \frac{2\pi}{3} - \frac{2\pi}{3}\right) = \frac{|W|}{6}.$$

Thus, V' is an eigenspace for $\mu_{\mathcal{O}}$ with eigenvalue $\frac{|W|}{6}$, having dimension $|\mathcal{O}| - n$. Since the only other constituent V of $\mathbb{R}^{\Phi_{\mathcal{O}},-}$ has dimension n, it must lie in a single eigenspace, whose eigenvalue λ satisfies $\lambda \cdot n = \frac{|W||\mathcal{O}|}{2} - \frac{|W|}{6}(|\mathcal{O}| - n) = \frac{(2|\mathcal{O}|+n)|W|}{6}$, and hence $\lambda = \frac{(2|\mathcal{O}|+n)|W|}{6n}$.

For the last assertion, in the simply-laced case, one has that $\mathcal{O}$ is the set of all hyperplanes, whose cardinality is well-known [**38**, §3.18] to be $\frac{nh}{2}$. The formula for the eigenvalue follows. ☐

type	factored characteristic polynomial	
$A_{n-1} = \mathfrak{S}_n$	$\left(x - \frac{(n+1)!}{6}\right)^{n-1}$	$\left(x - \frac{n!}{6}\right)^{\binom{n-1}{2}}$
$B_n, s =$ sign change	$(x - 2^{n-1}n!)^n$	
$B_n, s =$ transposition	$\left(x - \frac{2^{n-1}\cdot n!\cdot(2n-1)}{3}\right)^n$	$\left(x - \frac{2^{n-1}n!}{3}\right)^{n(n-2)}$
D_n	$\left(x - \frac{2^{n-2}\cdot n!\cdot(2n-1)}{3}\right)^n$	$\left(x - \frac{2^{n-2}n!}{3}\right)^{n(n-2)}$
E_6	$(x - 112320)^6$	$(x - 8640)^{30}$
E_7	$(x - 9192960)^7$	$(x - 483840)^{56}$
E_8	$(x - 3599769600)^8$	$(x - 116121600)^{112}$
F_4	$(x - 1344)^4$	$(x - 192)^8$
H_3	$(x^2 - 248x + 3856)^3$ $= (x - 124 \pm 48\sqrt{5})^3$	$(x - 24)^4 \cdot (x - 12)^5$ $(x - 24)^4 \cdot (x - 12)^5$
H_4	$(x^2 - 79680x + 94233600)^4$ $= (x - 39840 \pm 17280\sqrt{5})^4$	$(x - 3840)^{16} \cdot (x - 1440)^5$ $(x - 3840)^{16} \cdot (x - 1440)^5$

FIGURE 1. Factored characteristic polynomials for $\nu_{\mathcal{O}}$ or $\mu_{\mathcal{O}}$ on their eigenspaces affording $\mathrm{Ind}_{Z_W(s)}^W \chi$, where $\chi = \det|_{V/H}$ if $s = s_H$.

REMARK 4.8. The above assertion about the structure of the eigenspaces of $\mu_{\mathcal{O}}$ in the simply-laced subcase of situation $(\frac{\pi}{3})$ was a conjecture in the previous version of our paper, and turned out to be Renteln's [**52**, Theorem 39]. We have adapted his method of proof to give the more general statement above.

We have implemented in `Mathematica` [**79**] the calculation of this matrix for $\mu_{\mathcal{O}}$ acting on $\mathbb{R}^{\Phi_{\mathcal{O}},-}$, and produced the characteristic polynomials shown in Figure 1. Theorem III.4.7 predicts the answers for all rows of the figure corresponding to Weyl groups, but makes no prediction for the non-crystallographic groups H_3, H_4. Note that we have omitted any data on the dihedral types $I_2(m)$, as here the matrices for $\mu_{\mathcal{O}}$ are easily-analyzed circulant matrices, discussed thoroughly in [**52**, §4.1 and §4.6].

REMARK 4.9. Theorem I.4.1 can fail without the hypothesis that $\mathcal{O}$ is a single W-orbit of hyperplanes. For example, when $W = B_2 = I_2(4)$ and $\mathcal{O}$ is the set of all four hyperplanes, one finds that

$$\det(t\mathrm{I}_{\mathbb{R}^8} - \nu_{\mathcal{O}}) = t^3(t - 16)(t^2 - 8t + 8)^2,$$

which contains quadratic factors irreducible over $\mathbb{Q}$, the unique minimal splitting field of W in characteristic 0. The issue here is that $\mathcal{O}$ contains two different W-orbits of hyperplanes, so that Theorem I.4.2 does not apply. It turns out that the irreducible quadratic factors $(t^2 - 8t + 8)^2$ are the characteristic polynomial for $\nu_{\mathcal{O}}$ acting on two eigenspaces that both afford the reflection representation V for W.

5. Relation to linear ordering polytopes

We pause here to discuss a topic from discrete geometry and polytopes that motivated some of these explorations. We refer to Ziegler's book [**80**] for basic facts and unexplained terminology from polytope theory.

Given the hyperplane arrangement $\mathcal{A}$, with some possible subset of linear symmetries W, and (W-stable) subset $\mathcal{O}$ of $\mathcal{L}$, note that the map from Definition II.2.1

$$\begin{array}{rcl} \pi_{\mathcal{O}} : \mathbb{Z}\mathcal{C} & \longrightarrow & \oplus_{X \in \mathcal{O}} \mathbb{Z}\mathcal{C}(\mathcal{A}/X) \\ c & \longmapsto & \bigoplus_{X \in \mathcal{O}} c/X \end{array}$$

is W-equivariant for the natural W-permutation actions in the source and target. Clearly, $\pi_{\mathcal{O}}$ extends to a mapping from $\mathbb{R}\mathcal{C}$ to $\oplus_{X \in \mathcal{O}} \mathbb{R}\mathcal{C}(\mathcal{A}/X)$, which allows a definition of a new class of polytopes. Recall for the definition that the image of a convex polytope under a linear map is again a convex polytope.

Definition 5.1. Let $\mathcal{A}$ be an arrangement of hyperplanes and $\mathcal{C}$ its set of chambers. Denote by $\Delta_{|\mathcal{C}|-1}$ the standard $(|\mathcal{C}|-1)$-dimensional simplex $\Delta_{|\mathcal{C}|-1}$ which is the convex hull of the standard basis vectors within $\mathbb{R}\mathcal{C}$. The convex polytope $\mathrm{Lin}_{\mathcal{O}}$ is defined to be

$$\mathrm{Lin}_{\mathcal{O}} = \pi_{\mathcal{O}}(\Delta_{|\mathcal{C}|-1}),$$

the image of the polytope $\Delta_{|\mathcal{C}|-1}$ under the linear map $\pi_{\mathcal{O}}$.

Since the map $\pi_{\mathcal{O}}$ has all entries in $\{0, 1\}$ when expressed with respect to the standard basis, $\mathrm{Lin}_{\mathcal{O}}$ is a $0/1$-*polytope*, and its vertex set will simply be the distinct images (after eliminating duplicates) $\pi_{\mathcal{O}}(c)$ of the chambers c in $\mathcal{C}$. Letting $\mathcal{A}(\mathcal{O})$ denote the subset of hyperplanes H in $\mathcal{A}$ that contain at least one subspace X in $\mathcal{O}$, it is easy to see that two chambers in $\mathcal{C}$ have distinct images under $\pi_{\mathcal{O}}$ if and only if they do not lie in the same chamber of the arrangement $\mathcal{A}(\mathcal{O})$. Thus, $\mathrm{Lin}_{\mathcal{O}}$ has vertex set in bijection with the chambers $\mathcal{C}(\mathcal{A}(\mathcal{O}))$.

Proposition 5.2. *The polytope* $\mathrm{Lin}_{\mathcal{O}}$ *has dimension* $r - 1$ *where*

$$r := \operatorname{rank} \pi_{\mathcal{O}} = \operatorname{rank} \nu_{\mathcal{O}} = \operatorname{rank} \mu_{\mathcal{O}}.$$

In particular, when $\mathcal{A}$ *is a reflection arrangement and* $\mathcal{O}$ *is a* W*-stable subset of hyperplanes* H*, the dimension of* $\mathrm{Lin}_{\mathcal{O}}$ *is the cardinality* $|\mathcal{O}|$*.*

Proof. Consider the vector $v_{\mathbf{1}} := \sum_{c \in \mathcal{C}} c$ inside $\mathbb{R}\mathcal{C}$ that has all coordinates equal to 1, and note that its image $\pi_{\mathcal{O}}(v_{\mathbf{1}})$ within $\oplus_{X \in \mathcal{O}} \mathbb{R}\mathcal{C}(\mathcal{A}/X)$ is non-zero. On the other hand, the perpendicular space $v_{\mathbf{1}}^{\perp}$, which is spanned by the elements $c - c'$ for $c, c' \in \mathcal{C}$, is sent by $\pi_{\mathcal{O}}$ into the codimension one subspace of $\oplus_{X \in \mathcal{O}} \mathbb{R}\mathcal{C}(\mathcal{A}/X)$ where the sum of the coordinates is zero. This is easily checked on the above spanning set for $v_{\mathbf{1}}^{\perp}$.

This shows that $\pi_{\mathcal{O}}$ restricts to a linear map out of $v_{\mathbf{1}}^{\perp}$ that has rank $r - 1$, where r is the rank of $\pi_{\mathcal{O}}$. Since the simplex $\Delta_{|\mathcal{C}|-1}$ contains an open neighborhood within the affine translate of $v_{\mathbf{1}}^{\perp}$ where the sum of coordinates is 1, the image of the simplex under $\pi_{\mathcal{O}}$ will also have dimension $r - 1$.

When $\mathcal{A}$ is a reflection arrangement and $\mathcal{O}$ is a W-stable subset of hyperplanes H, either the BHR theory (see Corollary IV.6.1 and Example IV.6.2) or Proposition III.3.2 shows that the space perpendicular to the kernel of $\pi_{\mathcal{O}}$ carries the W-representation

$$\mathbf{1}_W \oplus \left(\bigoplus_{i=1}^{t} \mathrm{Ind}_{Z_W(s_i)}^{W} \chi_i \right).$$

Since the dimension of the representation $\mathrm{Ind}_{\mathrm{Z}_W(s_i)}^W \chi_i$ is $[W : \mathrm{Z}_W(s_i)] = |\mathcal{O}_i|$, this shows that the rank of $\pi_{\mathcal{O}}$ is $1 + \sum_{i=1}^t |\mathcal{O}_i| = 1 + |\mathcal{O}|$. $\square$

Example 5.3. Let $W = \mathfrak{S}_n$ and $\mathcal{A}$ its reflection arrangement. Consider the

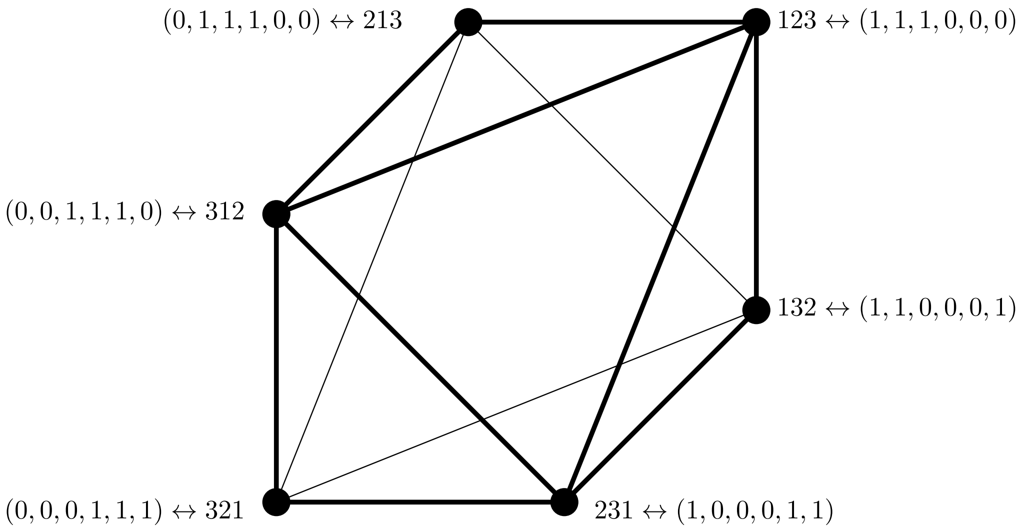

Figure 2. Linear ordering polytope for $\mathfrak{S}_3$

case when $\mathcal{O}$ is the set of all hyperplanes $\mathcal{A}$. The polytope $\mathrm{Lin}_{\mathcal{O}}$ lives in a space isomorphic to $\mathbb{R}^{n(n-1)}$ whose coordinates are indexed by ordered pairs (i, j) with $1 \leq i \neq j \leq n$. The vertices of $\mathrm{Lin}_{\mathcal{O}}$ are labelled by the $n!$ elements of $\mathfrak{S}_n$ or, equivalently, the different linear orders $\preceq$ on $[n]$. If we consider the vertex labelled by $w \in \mathfrak{S}_n$, then its coordinate indexed by (i, j) is 1 if $w(i) < w(j)$ and 0 otherwise. If we choose the labeling by linear orders, then the vertex labelled by $\preceq$ has a 1 in coordinate (i, j) whenever $i \preceq j$, and 0 otherwise. Figure 2 shows the linear ordering polytope for $\mathfrak{S}_3$ with coordinates indexed by $(1,2)$, $(1,3)$, $(2,3)$, $(2,1)$, $(3,1)$, $(3,2)$.

Note that $\mathrm{Lin}_{\mathcal{O}}$ lies in an affine subspace where the sum of the (i, j) and (j, i) coordinates is 1. Therefore, $\mathrm{Lin}_{\mathcal{O}}$ is affinely isomorphic to its projection onto the space $\mathbb{R}^{\binom{n}{2}}$ via the map p preserving the coordinates (i, j) with $i < j$, and forgetting the rest of the coordinates.

This projection of $\mathrm{Lin}_{\mathcal{O}}$ onto $\mathbb{R}^{\binom{n}{2}}$ is called the *linear ordering polytope*, and has a rich history, having appeared in several guises (see [**27**]), with great importance in combinatorial optimization; see e.g. [**33**], [**26**]. Its possible first appearance was in mathematical psychology, where the question—phrased in our terms—was the following. Consider $\Delta_{n!-1}$ as the set of all probability distributions on $\mathfrak{S}_n$ or equivalently on the set of linear orders on $[n]$.

Question 5.4. Describe the set of vectors $(u_{ij})_{1 \leq i < j \leq n}$ in $\mathbb{R}^{\binom{n}{2}}$ for which

$$u_{ij} = \sum_{\substack{\pi \in \mathfrak{S}_n \\ \pi(i) < \pi(j)}} \mathcal{P}(\pi)$$

as $\mathcal{P}$ ranges over all probability distributions $\mathcal{P} \in \Delta_{n!-1}$.

Note that in the terminology used above the set described in Question III.5.4 is given as $p \circ \pi_{\mathcal{O}}(\Delta_{n!-1})$ and hence is the linear ordering polytope. The description asked for in mathematical psychology is the same crucial question asked in optimization: find a list of facet inequalities. Since it is known (see [**33**]) that optimization of a general linear cost function over the linear ordering polytope is NP-hard, providing a polynomial size description of its facets would prove P=NP. However, this suggests the following problem.

Problem 5.5. Let $W = \mathfrak{S}_n$ and $\mathcal{O} = \mathcal{A}$ the set of all reflecting hyperplanes so that $\Phi_{\mathcal{O}} = \Phi$ is the set of all roots. Can one make use of the explicit $\mathbb{R}W$-module orthogonal decomposition of $\mathbb{R}^{\Phi} = V \oplus \wedge^2 V$, coming from Corollary III.4.4 worked out in this special case in Example III.4.6, as a good coordinate system in which to study the polytope $\mathrm{Lin}_{\mathcal{O}}$, which is isomorphic to the linear ordering polytope?[1]

Example 5.6. Let W be the hyperoctahedral group of all signed permutations, that is, the Weyl group of type B_n, and let $\mathcal{O}$ be the set of all reflecting hyperplanes. Then $\mathcal{O}$ is a union of two W-orbits, namely the coordinate hyperplanes $x_i = 0$, and the hyperplanes of the form $x_i \pm x_j = 0$ for $1 \leq i < j \leq n$. Then the polytope $\mathrm{Lin}_{\mathcal{O}}$ is affinely isomorphic to one considered by Fiorini and Fishburn [**25**], having the linear ordering polytope as one of its faces.

[1] A partial answer to this problem was given in [**40**]. It is shown that this coordinate system gives rise to two mutually orthogonal $\mathfrak{S}_n \times \mathbb{Z}_2$-equivariant projections of $\mathrm{Lin}_{\mathcal{O}}$ into the n^{th} permutahedron and the $(n-1)^{\mathrm{st}}$ linear ordering polytope. The paper also lists other guises in which the same coordinate system has arisen.

CHAPTER IV

Equivariant theory of BHR random walks

It will turn out to be useful to exploit a relation between the operators $\nu_{\mathcal{O}}$ which we have been considering and certain operators studied by Bidigare, Hanlon and Rockmore [**11**]. We begin by defining these operators, and then exhibit the special case which is relevant for us when considering $\nu_{\mathcal{O}}$ for a reflection arrangement $\mathcal{A}$ and $\mathcal{O}$ a single W-orbit.

After this we review the (non-equivariant) aspects of the theory, followed by the equivariant versions that we will need, which are in some cases stronger than what we find in the literature, that is, [**11**, **15**, **16**, **55**]. However, we generally borrow some of the proofs from the literature directly, or in other cases, simply beef-up the techniques. One new feature here is the consideration of the extra $\mathbb{Z}_2$-action that comes from the antipodal action on chambers and faces of a central arrangement.

1. The face semigroup

Given a real, central arrangement of hyperplanes $\mathcal{A}$ in a d-dimensional real vector space V, we have already discussed the dissection of the complement $V \setminus \bigcup_{H\in\mathcal{A}} H$ into the chambers $\mathcal{C}$. More generally, $\mathcal{A}$ dissects V into relatively open polyhedral cones which we will call the *faces* $\mathcal{F}$, that are the equivalence classes for the relation $\equiv$ having $v \equiv v'$ whenever v and v' lie within exactly the same subset of the closed half-spaces defined by all the hyperplanes H in $\mathcal{A}$.

There is a natural semigroup structure on $\mathcal{F}$ defined as follows. Given two faces x, y, define a new face $x \circ y$ (*x pulled by y*) to be the unique face that one enters first (possibly x itself) when following a straight line from a point in the relative interior of the cone x toward a point in the relative interior of the cone y. More formally, the face $x \circ y$ is uniquely defined by the properties that for each hyperplane H of $\mathcal{A}$ the points of $x \circ y$ lie

- on the same side of H as x if $x \not\subset H$,
- on the same side of H as y if $x \subset H$, but $y \not\subseteq H$, and
- inside H if $x, y \subset H$.

It is not hard to see that if c is a chamber then $x \circ c$ is always a chamber, and hence $\mathbb{K}\mathcal{C}$ becomes a left-ideal within the semigroup algebra $\mathbb{K}\mathcal{F}$ of the semigroup $\mathcal{F}$ with coefficients in $\mathbb{K}$.

For our subsequent considerations we need the following simple lemma which connects the multiplication in $\mathbb{K}\mathcal{F}$ with the operators $\nu_{\mathcal{O}}$.

LEMMA 1.1. *Let $\mathcal{A}$ be central arrangement of hyperplanes in $\mathbb{R}^d$. Let $F \in \mathcal{F}$ and X be the linear subspace spanned by F. Then for $C, C' \in \mathcal{C}$ the we have $F\circ C = F\circ C'$ if and only if $C/X = C'/X$.*

PROOF. Since $C, C' \in \mathcal{C}$ there are no hyperplanes from $\mathcal{A}$ containing C or C'. Therefore, $F \circ C = F \circ C'$ if and only if for all hyperplanes H containing F we have that C and C' lie on the same side of H. The hyperplanes containing F are exactly the hyperplanes containing X. Therefore, their images H/X are the hyperplanes of the arrangement $\mathcal{A}/X$. But then the assertion follows, since $C/X = C'/X$ if and only if C/X and C'/X lie on the same side of the hyperplanes from $\mathcal{A}/X$. $\square$

DEFINITION 1.2. We will define a BHR *random walk* or BHR *operator* to be any of the family of $\mathbb{K}$-linear operators on the left-ideal $\mathbb{K}\mathcal{C}$ within $\mathbb{K}\mathcal{F}$ that comes from multiplication on the left by an element

$$\sum_{x \in \mathcal{F}} p_x x \tag{11}$$

for some p_x in $\mathbb{K}$.

Note that we are not assuming that the p_x are real, nor even non-negative, nor that they sum to 1 as in the case of a probability distribution on the faces $\mathcal{F}$; for the moment, they lie in an arbitrary field $\mathbb{K}$.

2. The case relevant for $\nu_{\mathcal{O}}$

When W is a finite subgroup of $\mathrm{GL}(V)$ acting as symmetries of $\mathcal{A}$, it permutes the faces in $\mathcal{F}$, and it is easily seen that the W-action respects the semigroup structure, that is, $w(x) \circ w(y) = w(x \circ y)$. Thus, W acts as a group of algebra automorphisms on $\mathbb{K}\mathcal{F}$. Let $(\mathbb{K}\mathcal{F})^W$ be the algebra of W-invariants of this action. Then for $w \in W$, $x \in \mathbb{K}\mathcal{F}$ and $y \in (\mathbb{K}\mathcal{F})^W$ we have that $w(x) \circ y = w(x \circ y)$. Thus, $\mathbb{K}\mathcal{F}$ becomes a $(W\text{–}(\mathbb{K}\mathcal{F})^W)$ bimodule, as does the left-ideal $\mathbb{K}\mathcal{C}$ within $\mathbb{K}\mathcal{F}$.

For the remainder of this subsection, assume that $\mathcal{A}$ is the reflection arrangement for a finite real reflection group W acting on V. As discussed in §II.5, having picked a fundamental base chamber c_1, the simply transitive W-action on the chambers $\mathcal{C}$ leads to a W-equivariant identification $\mathbb{K}W \to \mathbb{K}\mathcal{C}$ that sends $w \mapsto c_w := w(c_1)$.

Let S denote the set of Coxeter generators for W that come from the reflections through the walls of c_1. It is well-known that every face x in $\mathcal{F}$ lies in the W-orbit of a unique subface $x(J)$ of c_1, stabilized by the parabolic subgroup $W_J := \langle J \rangle$ for some unique subset $J \subseteq S$.

Consequently, the W-invariant subalgebra $(\mathbb{K}\mathcal{F})^W$ of $\mathbb{K}\mathcal{F}$ will have $\mathbb{K}$-basis given by the $2^{|S|}$ elements

$$\left\{ \sum_{y \in x(J)^W} y \right\}_{J \subseteq S}$$

where as usual $x(J)^W$ denotes the W-orbit of the face $x(J)$.

The following observation, due originally to Bidigare [**10**, §3.8.3] (see also [**15**, Theorem 8]), is crucial. For the formulation, we use the notation ${}^J W$, W^J and ${}^J R$, R^J from Proposition II.5.4, subsequent comments and §II.6.1.

PROPOSITION 2.1. *Under the W-equivariant isomorphism $\mathbb{K}W \to \mathbb{K}\mathcal{C}$, multiplication on the right of $\mathbb{K}W$ by the element $R^J := \sum_{u \in W^J} u$ of $\mathbb{K}W$ corresponds to the action on $\mathbb{K}\mathcal{C}$ coming from multiplication on the left by the element $\sum_{y \in x(J)^W} y$ of $(\mathbb{K}\mathcal{F})^W$.*

PROOF. We wish to show that for each w in W,

$$\left(\sum_{y \in x(J)^W} y \right) \circ c_w = \sum_{u \in W^J} c_{wu}.$$

Acting by w^{-1} on the left, and using the W-equivariance, this is equivalent to showing

$$w^{-1} \left(\sum_{y \in x(J)^W} y \right) \circ c_1 = \sum_{u \in W^J} c_u.$$

Since $\sum_{y \in x(J)^W} y$ lies in $\mathbb{K}\mathcal{F}^W$, this means showing

$$\sum_{y \in x(J)^W} y \circ c_1 = \sum_{u \in W^J} c_u.$$

On the other hand, as W_J is the W-stabilizer subgroup for the face $x(J)$, and W^J are coset representatives for W/W_J, one has $x(J)^W = \{u \cdot x(J) : u \in W^J\}$. Thus, it suffices to show that for $u \in W^J$ one has $u \cdot x(J) \circ c_1 = c_u$. This follows because

$$\begin{aligned} u \in W^J &\Leftrightarrow u^{-1} \in {}^J W \\ &\Leftrightarrow c_{u^{-1}}/X = c_1/X \\ &\Leftrightarrow x(J) \circ c_{u^{-1}} = c_1 \\ &\Leftrightarrow u \cdot x(J) \circ c_1 = c_u \end{aligned}$$

where X is the subspace fixed pointwise by W_J, that is the linear subspace spanned by $x(J)$. Here the third equivalence follows from Lemma IV.1.1 and the last equivalence comes from applying the left-action of u. $\square$

This has the following consequence. Denote by b_J the linear operator on $\mathbb{K}\mathcal{C}$ given by multiplication on the left by $\sum_{y \in x(J)^W} y$. Let b_J^T denote its adjoint operator with respect to the standard inner product on $\mathbb{K}\mathcal{C}$ in which the elements of $\mathcal{C}$ form an orthonormal basis.

COROLLARY 2.2. *Let W be a finite real reflection group and $\mathcal{O} \subseteq \mathcal{L}$ a single W-orbit of intersection subspaces. Choose a representative subspace X_0 for the W-orbit $\mathcal{O}$ that contains a face $x(J)$ of the fundamental chamber c_1, for some $J \subseteq S$.*

Then the action of $\nu_{\mathcal{O}}$ multiplying $\mathbb{K}W$ on the right corresponds under the W-equivariant isomorphism $\mathbb{K}W \to \mathbb{K}\mathcal{C}$ to the operator $\frac{1}{n_{X_0}} b_J^T b_J$. In particular, $\nu_{\mathcal{O}}$ and b_J share the same kernel.

PROOF. Proposition II.6.1 asserts that as an element of $\mathbb{K}W$ one has $\nu_{\mathcal{O}} = \frac{1}{n_{X_0}} R^{X_0} \cdot {}^{X_0}R$. Here we choose minimal length coset representatives W^{X_0} and ${}^{X_0}W$ in the definition of R^{X_0} (see §II.6.1). Note that multiplication on the right by R^{X_0} and ${}^{X_0}R$ are adjoint with respect to the standard inner product on $\mathbb{K}W$ (since multiplying on the right by w and by w^{-1} are adjoint). Thus, one must show that multiplication on the right by $R^{X_0} = R^J$ in kW corresponds to multiplication on the left by b_J. This is exactly Proposition IV.2.1. $\square$

3. Some non-equivariant *BHR* theory

It is an interesting and non-trivial fact that, when working over $\mathbb{K} = \mathbb{R}$ and assuming that the coefficients p_x are non-negative, the BHR operators as in 11 act semisimply. We recapitulate in this section a beautiful argument for this due to Brown [**15**]. We remark that there are now simpler proofs of this result in the literature. The simplest proof can be found in [**71**]. Other simple proofs begin with a construction of the eigenvectors for the BHR operators [**21**, **57**].

Brown begins with an interesting way to capture the *minimal polynomial* of an element a in a finite-dimensional $\mathbb{K}$-*algebra* A, via generating functions. Recall that this minimal polynomial is the unique monic polynomial $m_a(T)$ in the univariate polynomial ring $\mathbb{K}[T]$ that generates the principal ideal which is the kernel of the map defined by

$$\begin{aligned} \mathbb{K}[T] &\longrightarrow A \\ T &\longmapsto a. \end{aligned}$$

For the sake of factoring $m_a(T)$, extend coefficients to the algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$, that is, replace $\mathbb{K}$ with $\overline{\mathbb{K}}$, and replace A with $\overline{\mathbb{K}} \otimes_{\mathbb{K}} A$. Then one can uniquely express

$$m_a(T) = \prod_i (T - \lambda_i)^{m_i}$$

for some distinct λ_i in $\mathbb{K}$ and positive integers m_i.

It turns out that the roots λ_i and multiplicities m_i can be read off from a certain generating function

$$f_a(z) := \sum_{\ell \geq 0} a^\ell z^\ell = \frac{1}{1 - a \cdot z}.$$

We claim that $f_a(z)$ makes sense an element of $A \otimes_{\mathbb{K}} \mathbb{K}[\![z]\!]$: if we choose for A some $\mathbb{K}$-basis $\{a_j\}_{j=1,2,\ldots,t}$, then expressing each power a^ℓ uniquely as $a^\ell = \sum_{j=1}^t c_{\ell,j} a_j$ one has

$$f_a(z) = \sum_{j=1}^t a_j \otimes f_{a,j}(z)$$

where $f_{a,j}(z) = \sum_\ell c_{\ell,j} z^\ell$ lies in $\mathbb{K}[\![z]\!]$.

PROPOSITION 3.1. *In the above setting, each coefficient $f_{a,i}(z)$ in $\mathbb{K}[\![z]\!]$ is a rational function in z, that is, it lies in $\mathbb{K}(z)$. Furthermore, one can recover the roots λ_i and multiplicities m_i in the minimal polynomial of a from the location and orders of poles in the partial fraction expansion of the related function*

$$\begin{aligned} g_a(z) &:= \frac{1}{z} f_a\left(\frac{1}{z}\right) = \frac{1}{z - a} \\ &= \sum_i \left(\frac{b_i^0}{z - \lambda_i} + \frac{b_i^1}{(z - \lambda_i)^2} + \cdots + \frac{b_i^{m_i - 1}}{(z - \lambda_i)^{m_i}} \right) \end{aligned}$$

where the b_i are some elements of A satisfying $b_i^{m_i} = 0$ but $b_i^{m_i - 1} \neq 0$.

PROOF. The Chinese Remainder Theorem says that the subalgebra R of A generated by a is isomorphic as an algebra to the product $\prod_i \mathbb{K}[T]/(T - \lambda_i)^{m_i}$. From this one can immediately reduce to the case where the minimal polynomial has only a single factor $(T-\lambda)^m$. Then one can express $a = \lambda + b$ where $b^m = 0$ but $b^{m-1} \neq 0$, and compute directly that

$$\begin{aligned} g_a(z) &= \frac{1}{z-a} = \frac{1}{(z-\lambda)-b} = \frac{1}{(z-\lambda)} \cdot \frac{1}{1-\frac{b}{z-\lambda}} \\ &= \frac{1}{(z-\lambda)} \sum_{\ell \geq 0} \frac{b^\ell}{(z-\lambda)^\ell} \\ &= \frac{b^0}{z-\lambda} + \frac{b^1}{(z-\lambda)^2} + \cdots + \frac{b^{m-1}}{(z-\lambda)^m}. \end{aligned} \qquad \square$$

Brown then applies this criterion to elements a of the face semigroup algebra $A = \mathbb{R}\mathcal{F}$, and more generally for semigroup algebras $\mathbb{K}\mathcal{F}$ of semigroups $\mathcal{F}$ that satisfy the *left-regular band* axioms:

$$x^2 = x \qquad \text{and} \qquad xyx = xy.$$

He shows that for any left-regular band $\mathcal{F}$, one recovers a semilattice[1] $\mathcal{L}$, playing the role of the intersection lattice for the arrangement when $\mathcal{F}$ is the face semigroup, in the following fashion. Consider the quasiorder (reflexive, transitive, but not antisymmetric) on $\mathcal{F}$ defined by $x \preceq y$ if $xy = x$, and then let $\mathcal{L}$ be the associated poset structure on the equivalence classes. One obtains in this way a (meet) semilattice $\mathcal{L}$, endowed naturally with a surjection of posets $\operatorname{supp} : \mathcal{F} \twoheadrightarrow \mathcal{L}$, sending $F \in \mathcal{F}$ to the subspace spanned by F. This surjection fulfills the following:

$$\begin{aligned} \operatorname{supp}(xy) &= \operatorname{supp}(x) \wedge \operatorname{supp}(y), \\ xy = x &\text{ if } \operatorname{supp}(y) \supseteq \operatorname{supp}(x). \end{aligned}$$

If the semigroup $\mathcal{F}$ has an identity element $\mathbb{1}$, which we assume from now on, then $\operatorname{supp}(\mathbb{1}) = \hat{0}$ is a minimum element of $\mathcal{L}$.

This leads to the following considerations for factorizations of elements of $\mathcal{F}$, which will help expand the generating function $f_a(z)$.

Definition 3.2. Given a word $\mathbf{x} := (x_1, \ldots, x_\ell)$ in $\mathcal{F}^\ell$, let $\ell(\mathbf{x}) := \ell$ denote its length, and let $\prod \mathbf{x} := x_1 \cdots x_\ell$ denote its product as an element of the semigroup $\mathcal{F}$. Define for $i = 1, 2, \ldots, \ell$ the elements $X_i(\mathbf{x}) := \operatorname{supp}(x_1 x_2 \cdots x_i)$ in $\mathcal{L}$, with convention $X_0(\mathbf{x}) = \hat{0}$, so that

$$\hat{0} = X_0(\mathbf{x}) \leq X_1(\mathbf{x}) \leq \cdots \leq X_\ell(\mathbf{x})$$

is a multichain in the semilattice $\mathcal{L}$. Say that $\mathbf{x}$ is *reduced* if this multichain is actually a chain, that is, the $\{X_i(\mathbf{x})\}_{i=0}^{\ell}$ are distinct.

Given the word $\mathbf{x}$, uniquely define a reduced subword $\tilde{\mathbf{x}}$ of $\mathbf{x}$ by repeatedly removing any letter x_i for which $\operatorname{supp}(x_i) \leq \operatorname{supp}(x_1 x_2 \cdots x_{i-1})$. Note that $\prod \tilde{\mathbf{x}} = \prod \mathbf{x}$ in $\mathcal{F}$.

From this we can now calculate the generating function $f_a(z)$ that determines the minimal polynomial of any element $a = \sum_{x \in \mathcal{F}} p_x x$ in the semigroup algebra $\mathbb{K}\mathcal{F}$. Having fixed a, define

$$\lambda_X := \sum_{\substack{x \in \mathcal{F}: \\ \operatorname{supp}(x) \subseteq X}} p_x$$

[1] Brown orders the semilattice $\mathcal{L}$ using the opposite order that we have chosen here. Explicitly, he orders intersection subspaces of a hyperplane arrangement by inclusion rather than reverse-inclusion.

for X in $\mathcal{L}$, and define $\mathbf{p}_{\mathbf{x}} := p_{x_1} \cdots p_{x_\ell}$ for words $\mathbf{x} = (x_1, \ldots, x_\ell)$.

PROPOSITION 3.3. *Given a left-regular band $\mathcal{F}$ with identity, and $a = \sum_{x \in \mathcal{F}} p_x x$ in $\mathbb{K}\mathcal{F}$, as above, one has the rational expansion*

$$g_a(z) = \sum_{\text{reduced words } \mathbf{y}} \left(\prod \mathbf{y}\right) \cdot \frac{\mathbf{p}_{\mathbf{y}}}{(z - \lambda_{X_0(\mathbf{y})})(z - \lambda_{X_1(\mathbf{y})}) \cdots (z - \lambda_{X_{\ell(\mathbf{y})}(\mathbf{y})})}. \tag{12}$$

PROOF.

$$\begin{aligned} f_a(z) = \sum_{\ell \geq 0} a^\ell z^\ell &= \sum_{\text{words } \mathbf{x}} z^{\ell(\mathbf{x})} \left(\prod \mathbf{x}\right) \mathbf{p}_{\mathbf{x}} \\ &= \sum_{\text{reduced words } \mathbf{y}} \sum_{\substack{\text{words } \mathbf{x}: \\ \tilde{\mathbf{x}} = \mathbf{y}}} z^{\ell(\mathbf{x})} \left(\prod \mathbf{x}\right) \mathbf{p}_{\mathbf{x}} \end{aligned}$$

For a given reduced word $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_\ell)$, the set of all words $\mathbf{x}$ having $\tilde{\mathbf{x}} = \mathbf{y}$ is obtained by inserting between y_i and y_{i+1} an arbitrary collection of elements of $\mathcal{F}$ having support contained in $X_i(\mathbf{y})$; this means elements of support $\hat{0} = X_0(\mathbf{y})$ can be inserted before y_1, and elements of support $X_\ell(\mathbf{y})$ after $\mathbf{y}_\ell$. From this one concludes that

$$\begin{aligned} f_a(z) &= \sum_{\text{reduced words } \mathbf{y}} \left(\prod \mathbf{y}\right) \mathbf{p}_{\mathbf{y}} \frac{1}{1 - z \cdot \lambda_{X_0(\mathbf{y})}} \frac{1}{1 - z \cdot \lambda_{X_1(\mathbf{y})}} \cdots \frac{1}{1 - z \cdot \lambda_{X_{\ell(\mathbf{y})}(\mathbf{y})}} \\ &= \sum_{\text{reduced words } \mathbf{y}} \left(\prod \mathbf{y}\right) \cdot \frac{\mathbf{p}_{\mathbf{y}}}{(1 - z \cdot \lambda_{X_0(\mathbf{y})})(1 - z \cdot \lambda_{X_1(\mathbf{y})}) \cdots (1 - z \cdot \lambda_{X_{\ell(\mathbf{y})}(\mathbf{y})})}. \end{aligned}$$

The formula claimed for $g_a(z) := \frac{1}{z} f\left(\frac{1}{z}\right)$ then follows. ☐

COROLLARY 3.4. *Assume $a = \sum_{x \in \mathcal{F}} p_x x$ lies in $\mathbb{R}\mathcal{F}$ for a left-regular band $\mathcal{F}$, and that the p_x are non-negative. Then the minimal polynomial of a has only simple roots, contained in the set $\{\lambda_X\}_{X \in \mathcal{L}}$.*

In particular, a generates a semisimple subalgebra of A, and a acts semisimply on any finite-dimensional A-module U, with eigenvalue support contained in $\{\lambda_X\}_{X \in \mathcal{L}}$.

PROOF. Under the above hypotheses, the only terms in the sum (12) for $g_a(z)$ that contribute with $\mathbf{p}_{\mathbf{y}} \neq 0$ will be indexed by reduced words $\mathbf{y} = (y_1, \ldots, y_\ell)$ for which

$$\lambda_{X_0(\mathbf{y})} < \lambda_{X_1(\mathbf{y})} < \cdots < \lambda_{X_\ell(\mathbf{y})}$$

since y_i is an element of $X_i(\mathbf{y}) \setminus X_{i-1}(\mathbf{y})$ with $p_{y_i} > 0$ for each $i = 1, 2, \ldots, \ell(\mathbf{y})$. Hence the corresponding product term in the summation

$$\frac{1}{(z - \lambda_{X_0(\mathbf{y})})(z - \lambda_{X_1(\mathbf{y})}) \cdots (z - \lambda_{X_{\ell(\mathbf{y})}(\mathbf{y})})}$$

for $g_z(z)$ has only simple poles at each of these $\lambda_{X_i(\mathbf{y})}$. Thus, $g_a(z)$ itself has only simple poles, all of which are contained in the set $\{\lambda_X\}_{X \in \mathcal{L}}$. Now apply Proposition IV.3.1 to conclude the asserted form for the minimal polynomial of a. The remaining assertions are immediate from this. ☐

4. Equivariant structure of eigenspaces

We now return to the setting of a central, essential hyperplane arrangement $\mathcal{A}$ in $V = \mathbb{R}^d$, having $\mathcal{F}$ as its face semigroup (with identity, since $\mathcal{A}$ is central). Recall that the BHR operator may be thought of as the action by left multiplication of an element $a = \sum_{x \in \mathcal{F}} p_x x$ inside $\mathbb{R}\mathcal{F}$ on the left-ideal $\mathbb{R}\mathcal{C}$ spanned $\mathbb{R}$-linearly by the chambers of $\mathcal{A}$.

Bidigare, Hanlon, and Rockmore computed the eigenvalue multiplicities. In their re-proof of this result, Brown and Diaconis [**16**] introduced an important exact sequence[2] of $\mathbb{K}\mathcal{F}$-modules, allowing them to compute the eigenvalue multiplicities for any BHR operator inductively, using the recurrence for the Möbius function of the intersection lattice $\mathcal{L}$.

In this section, we will recall their exact sequence, and then use it in the *equivariant setting*, where W is some finite subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{R})$ that preserves the arrangement $\mathcal{A}$, to identify the $\mathbb{R}W$-module structure on the BHR-eigenspaces.

To this end, recall that in §II.1 we defined for each subspace X in $\mathcal{L}$ the *localized arrangement*

$$\mathcal{A}/X := \{H/X : H \in \mathcal{A}, H \supset X\}$$

inside the quotient space V/X, having intersection lattice $\mathcal{L}(\mathcal{A}/X) \cong [V, X]_{\mathcal{L}}$. Accompanying this is the *restriction arrangement* of hyperplanes

$$\mathcal{A}|_X := \{H \cap X : H \in \mathcal{A}, H \not\supset X\}$$

inside the subspace X, having intersection lattice $\mathcal{L}(\mathcal{A}|_X) \cong [X, \{0\}]_{\mathcal{L}}$. We will use $\mathcal{C}_X$ to denote the subset of faces in $\mathcal{F}$ that represent chambers of $\mathcal{A}|_X$.

The exact sequence used by Brown and Diaconis then takes the form

$$0 \longrightarrow \mathbb{K}\mathcal{F}_d \xrightarrow{\partial_d} \cdots \longrightarrow \mathbb{K}\mathcal{F}_i \xrightarrow{\partial_i} \cdots \longrightarrow \mathbb{K}\mathcal{F}_1 \xrightarrow{\partial_1} \mathbb{K}\mathcal{F}_0 \xrightarrow{\partial_0} \mathbb{K} \longrightarrow 0 \tag{13}$$

in which $\mathcal{F}_i$ is the set of faces x in $\mathcal{F}$ for which $\mathrm{supp}(x)$ has codimension i. Thus,

$$\mathbb{K}\mathcal{F}_i = \bigoplus_{\substack{X \in \mathcal{L}: \\ \dim V/X = i}} \mathbb{K}\mathcal{C}_X$$

so, for example, $\mathbb{K}\mathcal{F}_0 = \mathbb{K}\mathcal{C}$ and $\mathbb{K}\mathcal{F}_1 = \bigoplus_{H \in \mathcal{A}} \mathbb{K}\mathcal{C}_H$. The boundary map ∂_0 sends each chamber c of $\mathcal{A}$ to the same element $\mathbb{1}$ in $\mathbb{K}$. The boundary map ∂_i for $i \geq 1$ sends a face x to the sum

$$\sum_y [x : y] y$$

where y ranges over all faces containing x as a codimension one subface, and where $[x : y]$ are certain *incidence coefficients* taking values ± 1 defined in the following way. First choose an arbitrary orientation on each subspace X in $\mathcal{L}$, and then decree $[x : y]$ to be the sign with respect to the orientation in $\mathrm{supp}(y)$ of any basis for y that is obtained by appending to a positively oriented basis for $\mathrm{supp}(x)$ any vector that points from x into y. Exactness of (13) follows because it is essentially the complex of cellular chains for the regular CW-decomposition into faces of the *zonotope* having $\mathcal{A}$ as its normal fan.

[2]Later observed in [**56**] to be a projective resolution of $\mathbb{K}$ as a $\mathbb{K}\mathcal{F}$-module.

Each $\mathbb{K}\mathcal{C}_X$ carries the structure of a (left-)$\mathbb{K}\mathcal{F}$-module by deforming the product in $\mathbb{K}\mathcal{F}$ as follows: for $x \in \mathcal{F}$ and $y \in \mathcal{C}_X$, set

$$x \lozenge y := \begin{cases} xy, & \text{if } x \subseteq X \text{ (so that } xy \in \mathcal{C}_X), \\ 0, & \text{otherwise.} \end{cases}$$

One can check that this makes the exact sequence (13) a complex of (left-)$\mathbb{K}\mathcal{F}$-modules. Consequently for any choice of $a = \sum_{x\in\mathcal{F}} p_x x$ in $\mathbb{K}\mathcal{F}$, it becomes an exact sequence of $\mathbb{K}[T]$-modules by letting T act as the element a.

An important feature to note about this structure is that for each subspace X in $\mathcal{L}$ having $\dim V/X = i$, the subspace $\mathbb{K}\mathcal{C}_X$ inside $\mathbb{K}\mathcal{F}_i$ is again a $\mathbb{K}\mathcal{F}$-module and $\mathbb{K}[T]$-module of the same type as $\mathbb{K}\mathcal{C}$. Combined with the semisimplicity of the $\mathbb{K}[T]$-structure when $\mathbb{K} = \mathbb{R}$ and $p_x \geq 0$, this will allow for arguments about the T-eigenspaces by induction on $\dim V$. For example, Brown and Diaconis use such an argument, along with the defining recurrence for the Möbius function of $\mathcal{L}$, to show in this setting that the BHR eigenvalue λ_X occurrs in $\mathbb{K}\mathcal{C}$ with multiplicity $|\mu(V,X)|$, where μ denotes the Möbius function of $\mathcal{L}$.

There are two preliminary observations we need before proving the W-equivariant version of this assertion. First, note that in order to place the desired $\mathbb{K}W$-module structure on $\mathbb{K}\mathcal{C}$, and to have (13) be a complex of $\mathbb{K}W$-modules, each summand $\mathbb{K}\mathcal{C}_X$ inside the term $\mathbb{K}\mathcal{F}_i$ with $i = \dim_{\mathbb{R}}(V/X)$ has to be twisted by $\det_{V/X}$. This means that, as a $\mathbb{K}W$-module, $\mathbb{K}\mathcal{F}_i$ has the following description:

$$\mathbb{K}\mathcal{F}_i = \bigoplus_{\substack{X^W \in \mathcal{L}/W: \\ \dim_{\mathbb{R}}(V/X)=i}} \operatorname{Ind}_{W_X}^{W} \left(\mathbb{R}\mathcal{C}_X \otimes \det{}_{V/X}\right). \tag{14}$$

Note that the sign twist $\det_{V/X}$ arises from the fact that for the definition of (13) one has to orient the subspaces in $\mathcal{L}$. Indeed, it is a simple calculation to show that the twist $\det_{V/X}$ makes the differentials from (13) equivariant.

Secondly, we will need a W-equivariant version of the Möbius function recurrence. It can be deduced from [**73**, Lemma 1.1] and [**74**, Proposition 2.2]. However, since the proof of Proposition 2.2 in [**74**] only invites the reader to verify that the non-equivariant proof generalizes, we give an explicit proof here for completeness. The proof proceeds via the equivariant generalization of a standard sign-reversing-involution proof for P. Hall's Möbius function formula.

As preliminary notation, when a group W acts on a set M, let M/W denote the set of W-orbits, with the W-orbit containing some element m of M denoted by m^W. Let $\operatorname{Stab}_W(m) := \{w \in W : w(m) = m\}$ denote the W-stabilizer of m, so that one can identify the permutation W-action on m^W with the action of W on the left cosets $W/\operatorname{Stab}_W(m)$ by left multiplication. Note that if $\mathcal{A}$ is an arrangement that is invariant under the action of some linear group W, then W acts as a group of permutations on $\mathcal{L}$ and for any $X \in \mathcal{L}$ we have $\operatorname{Stab}_W(X) = \mathrm{N}_W(X)$. Let $\Gamma(\mathbb{K}W)$ denote the Grothendieck group of virtual $\mathbb{K}W$-modules (see [**7**, §5.1]). Finally, for a poset P and $X \leq Y$ in P we denote by (X,Y) the *open interval* $\{Z \in P : X < Z < Y\}$. By $\widetilde{\mathrm{H}}^i(P;\mathbb{K})$, respectively $\widetilde{\mathrm{H}}^i((X,Y);\mathbb{K})$, we denote the i^{th} *reduced cohomology group* of the *order complex* of P, respectively (X,Y). Recall that the order complex of a poset is the *simplicial complex* of all chains in the poset.

If P has a unique smallest element then we call this element the *bottom* of P and we denote it by $\hat{0}$. Analogously, if P has a unique largest element then we call this element the *top* of P and we denote it by $\hat{1}$. Also recall that a poset P with bottom and top element is called *Cohen-Macaulay* over $\mathbb{K}$ if it is ranked and $\widetilde{\mathrm{H}}^i((X,Y);\mathbb{K}) = 0$ for all $i \neq \mathrm{rank}(Y) - \mathrm{rank}(X) - 2$ and all $\hat{0} \leq X < Y \leq \hat{1}$. For more details on homological and geometric aspects of posets and their order complexes we refer the reader to the survey article by Wachs [**77**].

PROPOSITION 4.1. *Let P be a finite poset with bottom and top elements $\hat{0}$ and $\hat{1}$, respectively, and W a finite group acting as a group of poset automorphisms on P. Then in $\Gamma(\mathbb{K}W)$ one has*

$$\sum_{\substack{X^W \in P/W \\ i \geq -1}} (-1)^i \operatorname{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^i((X,\hat{1});\mathbb{K}) = 0.$$

and

$$\sum_{\substack{X^W \in P/W \\ i \geq -1}} (-1)^i \operatorname{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^i((\hat{0},X);\mathbb{K}) = 0.$$

In particular, if P is a poset which is Cohen-Macaulay over $\mathbb{K}$, with rank function $\mathrm{rank}_P(-)$*, then one has*

$$\sum_{X^W \in P/W} (-1)^{t(X)} \operatorname{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^{t(X)}((X,\hat{1});\mathbb{K}) = 0.$$

and

$$\sum_{X^W \in P/W} (-1)^{b(X)} \operatorname{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^{b(X)}((\hat{0},X);\mathbb{K}) = 0.$$

where $t(X) := \mathrm{rank}_P(\hat{1}) - \mathrm{rank}_P(X) - 2$ *and* $b(X) := \mathrm{rank}_P(X) - \mathrm{rank}_P(\hat{0}) - 2$.

PROOF. Since the assertions about the cohomology groups of intervals $(\hat{0},X)$ follow from the ones about intervals $(X,\hat{1})$ by considering the poset with the opposite order relation, it suffices to verify the assertions about intervals $(X,\hat{1})$ to the top. By the Hopf trace formula (see [**77**, Theorem 2.3.9]), it is equivalent to show that

$$\sum_{\substack{X^W \in P/W \\ i \geq -1}} (-1)^i \operatorname{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{C}}^i((X,\hat{1});\mathbb{K}) = 0,$$

where $\widetilde{\mathrm{C}}^i((X,\hat{1});\mathbb{K})$ is the i^{th} reduced cochain group of the order complex of $(X,\hat{1})$ with coefficients in $\mathbb{K}$. Because W acts by poset automorphisms on P, the stabilizer $\mathrm{Stab}_W(X)$ acts as a group of poset automorphisms on $(X,\hat{1})$ and therefore $\mathrm{Stab}_W(X)$ acts on the cochain group $\widetilde{\mathrm{C}}^i((X,\hat{1});\mathbb{K})$ as a permutation representation: the usual $\mathbb{K}$-bases dual to oriented simplicial chains $[X_1,\ldots,X_{i+1}]$, listed in their P-order $X_1 < \cdots < X_{i+1}$, will be permuted without any $\pm$ sign.

Consequently, if we let M be the set of all pairs (X,C) where X is an element of P and $\mathrm{C} = \{X_1,\ldots,X_{i+1}\}$ satisfies $X < X_1 < \ldots < X_{i+1} < \hat{1}$ in P, then it is enough to show

$$\sum_{(X,\mathrm{C})^W \in M/W} (-1)^{|\mathrm{C}|} \operatorname{Ind}^W_{\mathrm{Stab}_W(X,\mathrm{C})} \mathbf{1} = 0.$$

To show this, note that every $X \neq \hat{0}$ in P has $\mathrm{Stab}_W(X,\mathrm{C}) = \mathrm{Stab}_W(\hat{0},\{X\}\cup\mathrm{C})$. Hence the two terms $\operatorname{Ind}^W_{\mathrm{Stab}_W(X,\mathrm{C})} \mathbf{1}$ and $\operatorname{Ind}^W_{\mathrm{Stab}_W(\hat{0},\{X\}\cup\mathrm{C})} \mathbf{1}$ cancel in the sum. $\square$

We can now state and prove our W-equvariant description of the BHR eigenspaces when $\mathbb{K} = \mathbb{R}$ and the coefficients p_x in $a = \sum_{x \in \mathcal{F}} p_x x$ are chosen not only non-negative, but also W-invariant:

$$p_{gx} = p_x \text{ for all } g \in W, x \in \mathcal{F}.$$

Of course, when W is the trivial group, one recovers the usual theory. Our statement will involve $\widetilde{\mathrm{H}}^*((V,X);\mathbb{R})$. As usual $\widetilde{\mathrm{H}}^*((V,X);\mathbb{R})$ denotes $\oplus_i \widetilde{\mathrm{H}}^i((V,X);\mathbb{R})$. But since $\mathcal{L}$ is a geometric lattice any interval (V,X) will have its cohomology trivial except for the top dimension $\mathrm{rank}(X)-2$ (see [**77**, Section 3.2.2]). Thus, one can also see $\widetilde{\mathrm{H}}^*((V,X);\mathbb{R})$ as a short notation for $\widetilde{\mathrm{H}}^{\mathrm{rank}(X)-2}((V,X);\mathbb{R})$. Since $\mathrm{Stab}_W(X)$ acts on the order complex of (V,X) it then follows that $\widetilde{\mathrm{H}}^*((V,X);\mathbb{R})$ is indeed a $\mathrm{Stab}_W(X)$-module.

One further bit of notation: for an $\mathbb{R}[T]$-module U, and an eigenvalue λ, let U_λ denote the λ-eigenspace of T on U, that is, $U_\lambda := \ker(T - \lambda \mathrm{I}_U)$.

THEOREM 4.2. *For any choice of coefficients $\{p_x\}_{x \in \mathcal{F}}$ which are non-negative and W-invariant, the $\mathbb{R}[T][W]$-module structure on $\mathbb{R}\mathcal{C}$ is semisimple. The T-eigenvalues are contained in the set $\{\lambda_X\}_{X \in \mathcal{L}}$, and the λ-eigenspace has the following description as an element of the Grothendieck group $\Gamma(\mathbb{R}W)$:*

$$(\mathbb{R}\mathcal{C})_\lambda = \sum_{\substack{X^W \in \mathcal{L}/W: \\ \lambda_X = \lambda}} \mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \left(\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X} \right).$$

PROOF. Corollary IV.3.4 tells us that $\mathbb{R}\mathcal{C}$ is a semisimple $\mathbb{R}[T][W]$-module and that its T-eigenvalues are contained in the set $\{\lambda_X\}_{X \in \mathcal{L}}$. We claim that it suffices to show the assertion of the theorem only for those choices of p_x which make $\lambda_V > \lambda_X$ for $X \subsetneq V$. First we explain why this is a valid reduction. Note that such choices of p_x form a dense subset of all the relevant choices of p_x in the theorem. Also note that the theorem can be viewed as asserting for each W-irreducible χ, that the operator $T_\chi(p_x)$ acting on the χ-isotypic component $\mathbb{R}\mathcal{C}^\chi$ of $\mathbb{R}\mathcal{C}$ has a certain factorization for its characteristic polynomial

$$\det(t\mathrm{I}_{\mathbb{R}\mathcal{C}^\chi} - T_\chi(p_x)) = \prod_{X \in \mathcal{L}} (t - \lambda_X(p_x))^{m_X}$$

with m_X independent of $\{p_x\}$, but with the operators $T_\chi(p_x)$ and the eigenvalues $\lambda_X(p_x)$ depending polynomially on the $\{p_x\}$. If this identity holds on a dense set of $\{p_x\}$, it holds for all of them.

So assume $\lambda_V > \lambda_X$ for $X \subsetneq V$, and we will prove the assertion of the theorem by induction on $d := \dim_{\mathbb{R}}(V)$. The base case $d = 0$ is easily verified.

For the inductive step, we first consider the exact sequence of $\mathbb{R}W$-modules obtained by restricting the terms $\mathcal{F}_i$ in (13) to their eigenspaces

$$(\mathcal{F}_i)_\lambda = \bigoplus_{\substack{X \in \mathcal{L}: \\ \dim V/X = i}} (\mathbb{R}\mathcal{C}_X)_\lambda.$$

Thus, in the Grothendieck group $\Gamma(\mathbb{R}W)$ we obtain:

$$(\mathbb{R}\mathcal{C})_\lambda = -\sum_{i \geq 1} (-1)^i (\mathbb{R}\mathcal{F}_i)_\lambda.$$

By induction, and because $\lambda_V > \lambda_X$ for $X \subsetneq V$, only the last two terms $\mathbb{R}\mathcal{F}_0 = \mathbb{R}\mathcal{C}$ and $\mathbb{R}$ have a non-zero λ_V-eigenspace. Hence the two λ_V-eigenspaces are isomorphic, which proves the assertion for $\lambda = \lambda_V$. For $\lambda < \lambda_V$ we refine (14) to an eigenspace decomposition.

$$\begin{aligned}(\mathbb{R}\mathcal{F}_i)_\lambda &= \sum_{\substack{Y^W \in \mathcal{L}/W: \\ \dim_{\mathbb{R}}(V/Y)=i}} \mathrm{Ind}^W_{\mathrm{Stab}_W(Y)} \left(\mathbb{R}\mathcal{C}_Y \otimes \det{}_{V/Y}\right)_\lambda \\ &= \sum_{\substack{Y^W \in \mathcal{L}/W: \\ \dim_{\mathbb{R}}(V/Y)=i}} \mathrm{Ind}^W_{\mathrm{Stab}_W(Y)} \left((\mathbb{R}\mathcal{C}_Y)_\lambda \otimes \det{}_{V/Y}\right).\end{aligned}$$

By assumption on λ, we can apply the induction hypothesis to each $(\mathcal{C}_Y)_\lambda$, thus obtaining the following decomposition of $(\mathbb{R}\mathcal{F}_i)_\lambda$.

$$-\sum_{\substack{Y^W \in \mathcal{L}/W: \\ \dim_{\mathbb{R}}(V/Y)=i}} (-1)^i \mathrm{Ind}^W_{\mathrm{Stab}_W(Y)} \left(\sum_{\substack{X^{\mathrm{Stab}_W(Y)} \in \\ (\mathcal{L}/Y)/\,\mathrm{Stab}_W(Y): \\ \lambda_X=\lambda}} \mathrm{Ind}^{\mathrm{Stab}_W(Y)}_{\mathrm{Stab}_{\mathrm{Stab}_W(Y)}(X)} \begin{pmatrix} \widetilde{\mathrm{H}}^*((Y,X);\mathbb{R}) \\ \otimes \\ \det{}_{Y/X} \end{pmatrix} \otimes \det{}_{V/Y} \right)$$

We can simplify the two sums to a single sum over pairs $(Y^W, X^{\mathrm{Stab}_W(Y)})$ in which Y^W is a W-orbit in $\mathcal{L}$ not equal to $\{V\}$, and $X^{\mathrm{Stab}_W(Y)}$ is a $\mathrm{Stab}_W(Y)$-orbit on the set $\{X \in \mathcal{L} : X \subseteq Y, \lambda_X = \lambda\}$. Note that a set of representatives (Y, X) for such pairs is the same as for the pairs $(X^W, Y^{\mathrm{Stab}_W(X)})$ in which X^W is a W-orbit not equal to $\{V\}$ on the set $\{X \in \mathcal{L} : \lambda_X = \lambda\}$ and $Y^{\mathrm{Stab}_W(X)}$ is a $\mathrm{Stab}_W(X)$-orbit not equal to $\{V\}$ on the set $\{Y \in \mathcal{L} : X \subseteq Y\}$. Consequently, using the fact that $\det{}_{Y/X} \det{}_{V/X} = \det{}_{V/Y}$ and transitivity of induction, one obtains

$$(15)\ (\mathbb{R}\mathcal{C})_\lambda = -\sum_{(X^W, Y^{\mathrm{Stab}_W(X)})} (-1)^{\dim_{\mathbb{R}} V/Y} \mathrm{Ind}^{\mathrm{Stab}_W(X)}_{\mathrm{Stab}_{\mathrm{Stab}_W(X)}(Y)} \widetilde{\mathrm{H}}^*((Y,X);\mathbb{R}) \otimes \det{}_{V/X}.$$

The lattice $\mathcal{L}$ is a geometric lattice and therefore the full lattice and all its intervals are Cohen-Macaulay [**77**, Lecture 4]. Its rank function is given by $\mathrm{rank}(X) = \dim V/X$. Therefore, by Proposition IV.4.1 we have that in the Grothendieck group $(-1)^{\dim V/X - 2}\widetilde{\mathrm{H}}^*((V,X);\mathbb{R})$ equals

$$(16) - \sum_{Y^{\mathrm{Stab}_W(X)} \in \mathcal{L}/\,\mathrm{Stab}_W(X)} (-1)^{\dim Y/X - 2} \mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^{\dim Y/X - 2}((Y,X);\mathbb{R}).$$

We multiply (16) with $(-1)^{-\dim Y/X + 2}$ and use $\dim V/X - \dim Y/X = \dim V/Y$ to obtain

$$\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) = -\sum_{Y^{\mathrm{Stab}_W(X)} \in \mathcal{L}/\,\mathrm{Stab}_W(X)} (-1)^{\dim V/Y} \mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^{\dim Y/X - 2}((Y,X);\mathbb{R}).$$

Observe that the right hand side of this equation contains exactly the terms from the righthand side of (15) for which (X, Y) is in the appropriate range. Thus, combining the two equations shows

$$(\mathbb{R}\mathcal{C})_\lambda = \sum_{\substack{X^W \in \mathcal{L}/W: \\ \lambda_X = \lambda}} \mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \left(\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X}\right). \qquad \square$$

Note for future use the following consequence of Theorem IV.4.2 which simply ignores the $\mathbb{R}[T]$-structure.

COROLLARY 4.3. *For any finite subgroup $W \subset \mathrm{GL}(V)$ that preserves $\mathcal{A}$, one has in the Grothendieck group $\Gamma(\mathbb{R}W)$*

$$\mathbb{R}\mathcal{C} = \sum_{X^W \in \mathcal{L}/W} \mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \left(\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X} \right).$$

Corollary IV.4.3 can also be seen as a special case of the equivariant version [**74**, Theorem 2.5 (ii)] of the Goresky-MacPherson formula for the cohomology of the complement of a *subspace arrangement*. For a hyperplane arrangement $\mathcal{A}$ invariant under W, the complement is the union of the (open) chambers in $\mathcal{C}$. Its non-reduced cohomology with coefficients in $\mathbb{R}$ is $\mathbb{R}^{|\mathcal{C}|}$ carrying the representation induced by the action of W on $\mathcal{C}$. This reduces [**74**, Theorem 2.5 (ii)] to Corollary IV.4.3 once one observes that the representation of non-reduced cohomology differs from reduced cohomology by a copy of the trivial representation and the fact that the representation of W on the unique non-vanishing homology of $X^\perp$ intersected with a W-invariant sphere is $\det{}_{V/X}$.

5. $(W \times \mathbb{Z}_2)$-equivariant eigenvalue filtration

Because we have been working with a *central* hyperplane arrangement $\mathcal{A}$, the map on the set $\mathcal{F}$ of faces that sends $x \mapsto -x$ gives a $\mathbb{Z}_2$-action on $\mathcal{F}$, on $\mathbb{K}\mathcal{F}$, and on the complex (13). Furthermore, it commutes with the action of any group of symmetries $W \subset GL(V)$ of $\mathcal{A}$.

If we only assume that $p_{gx} = p_x$ for $g \in W$ and $x \in X$, but make no assumption that $p_{-x} = p_x$, then in general this $\mathbb{Z}_2$-action does not commute with the T-action coming from the element $a = \sum_{x \in \mathcal{F}} p_x x$. However, the $\mathbb{Z}_2$-action will preserve a certain natural filtration that comes from the T-eigenspaces, as we now show. Given a semisimple $\mathbb{K}[T]$-module U having only real T-eigenvalues, and a real number λ, let

$$U_{\leq \lambda} := \bigoplus_{\mu \leq \lambda} U_\mu.$$

Note that since there are only finitely many different eigenvalues μ, these subspaces $U_{\leq\lambda}$ as λ increases through all real numbers form a finite filtration of U. For the formulation of the following result we denote by $\chi^+ := \mathbf{1}_{\mathbb{Z}_2}$ the trivial character of $\mathbb{Z}_2$ and by χ^- the unique non-trivial character of $\mathbb{Z}_2$.

THEOREM 5.1. *Let $\{p_x\}_{x\in\mathcal{F}}$ be real numbers such that $p_{wx} = p_x$ for all $w \in W$ and $x \in \mathcal{F}$. Then the $\mathbb{Z}_2$-action on $\mathbb{R}\mathcal{C}$ preserves the filtration of $\mathbb{R}\mathcal{C}$ by $\{(\mathbb{R}\mathcal{C})_{\leq\lambda}\}$. Furthermore, in the Grothendieck group $\Gamma(\mathbb{R}[W \times \mathbb{Z}_2])$, one has*

$$(\mathbb{R}\mathcal{C})_{\leq\lambda} = \sum_{\substack{X^W \in \mathcal{L}/X: \\ \lambda_X \leq \lambda}} \left(\mathrm{Ind}^W_{\mathrm{Stab}_W(X)} \widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X} \right) \otimes \left(\chi^-\right)^{\otimes \dim_{\mathbb{R}} V/X}.$$

PROOF. We first show $(\mathbb{R}\mathcal{C})_{\leq\lambda}$ is $\mathbb{Z}_2$-stable. Since all eigenvalues of T on $\mathbb{R}\mathcal{C}$ lie in $\{\lambda_X\}_{X\in\mathcal{L}}$ by Corollary IV.3.4, when $\lambda \geq \lambda_V$ one has $(\mathbb{R}\mathcal{C})_{\leq\lambda} = \mathbb{R}\mathcal{C}$ and the stability is trivial. When $\lambda < \lambda_V$, one can induct on the dimension d of the ambient space (with the base case $d = 0$ trivial as before) using the exact sequence (13) restricted to the spaces $(\mathbb{R}\mathcal{F}_i)_{\leq\lambda}$. From this restricted exact sequence one concludes that

$$(\mathbb{R}\mathcal{C})_{\leq\lambda} = (\mathrm{im}\,\partial_1)_{\leq\lambda} = \partial_1 \left(\bigoplus_{H\in\mathcal{A}} (\mathbb{R}\mathcal{C}_H)_{\leq\lambda} \right).$$

Since $(\mathbb{R}\mathcal{C}_H)_{\leq\lambda}$ is $\mathbb{Z}_2$-stable by induction, and since the $\mathbb{Z}_2$-action does commute with the ∂_i, this shows that $(\mathbb{R}\mathcal{C})_{\leq\lambda}$ is $\mathbb{Z}_2$-stable.

Regarding the description of the $\mathbb{R}[W \times \mathbb{Z}_2]$-module structure of $(\mathbb{R}\mathcal{C})_{\leq\lambda}$, one also checks this in two cases. When $\lambda \geq \lambda_V$ so that $(\mathbb{R}\mathcal{C})_{\leq\lambda} = \mathbb{R}\mathcal{C}$, it follows by applying Theorem IV.4.2 to the finite subgroup

$$\hat{W} := W \times \mathbb{Z}_2 = W \times \langle -\mathrm{I}_V \rangle \subset GL(V)$$

and noting that

- $\mathbb{Z}_2$ acts trivially on the lattice $\mathcal{L}$,
- so in particular, W and $\hat{W}$ have the same orbits on $\mathcal{L}$, and
- the $\mathbb{Z}_2$-characters $(\chi^-)^{\otimes \dim_{\mathbb{R}} V/X}$ and $\det_{V/X}$ are the same when the generator of $\mathbb{Z}_2$ acts by $-\mathrm{I}_V$ on V.

When $\lambda < \lambda_V$, one proceeds by induction on d using the exact sequence (13) restricted to the $(\mathbb{R}\mathcal{F}_i)_{\leq\lambda}$, proceeding exactly as in the proof of Theorem IV.4.2. $\square$

EXAMPLE 5.2. It is worth examining the $d = 1$ case of the preceding results in detail. Here the central arrangement $\mathcal{A}$ inside the real line $V = \mathbb{R}^1$ has two chambers c, c', separated by the unique hyperplane $H = \{0\}$. So the face semigroup $\mathcal{F}$ is $\{H, c, c'\}$, and H is the identity element of $\mathcal{F}$. The sequence (13) is

$$\begin{array}{ccccccc} 0 \to & \mathbb{R}\mathcal{F}_1 & \xrightarrow{\partial_1} & \mathbb{R}\mathcal{F}_0 & \xrightarrow{\partial_0} & \mathbb{R} & \to 0 \\ & \| & & \| & & \| & \\ & \mathbb{R}\{H\} & & \mathbb{R}\{c, c'\} & & \mathbb{R}\{\varnothing\} & \end{array}$$

with $\partial_1(H) = c - c'$ and $\partial_0(c) = \partial_0(c') = \varnothing$. Let T act by the element

$$a = p_0 H + pc + p'c'$$

in $\mathbb{R}\mathcal{F}$. Then the $\mathbb{R}[T]$-module structure on $\mathbb{R}\mathcal{F}_1$ has T scaling H by p_0, and on $\mathbb{R}\mathcal{F}_{-1}$ has T scaling $\varnothing$ by $p_0 + p + p'$, while on $\mathbb{R}\mathcal{F}_0$, the element T acts in the ordered basis (c, c') by

$$\begin{bmatrix} p_0 + p & p \\ p' & p_0 + p'. \end{bmatrix}$$

Changing to an ordered basis of T-eigenvectors $(c - c',\ pc + p'c')$, will diagonalize the action of T on $\mathbb{R}\mathcal{C}$:

$$\begin{bmatrix} p_0 & 0 \\ 0 & p_0 + p + p' \end{bmatrix}.$$

Note that

$$(\mathbb{R}\mathcal{C})_{\lambda_V} = (\mathbb{R}\mathcal{C})_{p_0+p+p'} = \mathbb{R}\{pc + p'c'\}$$

is not $\mathbb{Z}_2$-stable unless $p = p'$. However

$$(\mathbb{R}\mathcal{C})_{\lambda_{\{0\}}} = (\mathbb{R}\mathcal{C})_{p_0} = \mathbb{R}\{c - c'\}$$

is always $\mathbb{Z}_2$-stable.

REMARK 5.3. Theorem IV.5.1 suggests a conjectural stronger statement in the case of a reflection arrangement $\mathcal{A}$ corresponding to reflection group W, tying in with the work of Hanlon and Hersh [**35**] in type A. We discuss this briefly here.

For a reflection arrangement, one can identify the $W \times \mathbb{Z}_2$-action on $\mathbb{K}\mathcal{C}$ discussed above with the $W \times \mathbb{Z}_2$-action on $\mathbb{K}W$ where W acts via left-translation and the generator of $\mathbb{Z}_2$ acts via right-translation by w_0. Note that here the face semigroup $\mathbb{K}\mathcal{F}$ also acts on the left on the ideal $\mathbb{K}\mathcal{C}$ inside $\mathbb{K}\mathcal{F}$.

Since w_0 is the unique element of W having descent set all of S, it not only lies in the group algebra $\mathbb{K}W$, but also inside the *descent algebra*, which is spanned by the sums over w in W having a *fixed descent set*. Furthermore in type A_{n-1}, it lies in a subalgebra called the *Eulerian subalgebra*, spanned by the sums over w in W having[3] a *fixed number of descents*. There is a complete system of orthogonal idempotents for this Eulerian subalgebra known as the *Eulerian idempotents* $\{\mathfrak{e}_n^{(j)}\}_{j=1,2,\dots,n}$ defined by the generating function

$$\sum_{j=1}^{n} \mathfrak{e}_n^{(j)} t^j = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (t - \operatorname{des}(\sigma))^{\uparrow n} \sigma, \tag{17}$$

where $\operatorname{des}(\sigma)$ is the number of descents of the permutation $\sigma \in \mathfrak{S}_n$ and $t^{\uparrow n}$ denotes the *increasing factorial* $t^{\uparrow n} = t(t+1)(t+2)\cdots(t+n-1)$. These idempotents decompose spaces U on which the Eulerian subalgebra acts into their *Hodge decomposition* $U = \oplus_j U\mathfrak{e}_n^{(j)}$, and have the property that

$$\begin{aligned} 1 &= \sum_{j=1}^{n} \mathfrak{e}_n^{(j)}, \\ (-1)^n w_0 &= \sum_{j=1}^{n} (-1)^j \mathfrak{e}_n^{(j)}. \end{aligned}$$

(These identifies can be proved by taking $t = 1$ and $t = -1$, respectively, in 17.) Consequently, the two projectors onto the χ^+ and χ^--isotypic components for the group $\mathbb{Z}_2 = \{1, w_0\}$ can be expressed as

$$\begin{aligned} \frac{1}{2}(1 + (-1)^n w_0) &= \sum_{j \text{ even}} \mathfrak{e}_n^{(j)} \\ \frac{1}{2}(1 - (-1)^n w_0) &= \sum_{j \text{ odd}} \mathfrak{e}_n^{(j)}. \end{aligned}$$

In light of this, the following result generalizes Theorem IV.4.2 as well as the results of [**35**, Section 2].

Theorem 5.4 (Saliola). *Let $\mathcal{A}$ be a hyperplane arrangement, $\mathcal{L}$ its intersection lattice (ordered by reverse-inclusion, as usual), and $\mathcal{F}$ its semigroup of faces.*

(i) *There is a natural filtration of $\mathbb{K}\mathcal{C}$ by $\mathbb{K}\mathcal{F}$-modules indexed by the order ideals of $\mathcal{L}$. Explicitly, there is an inclusion-reversing map $\mathcal{I} \mapsto U_{\mathcal{I}}$ where $\mathcal{I}$ is any order ideal of $\mathcal{L}$ and*

$$U_{\mathcal{I}} := \{a \in \mathbb{K}\mathcal{C} : xa = 0 \text{ for every } x \in \mathcal{F} \text{ with } \operatorname{supp}(x) \in \mathcal{I}\}.$$

(ii) *For any choice of $a \in \mathbb{R}\mathcal{F}$ giving a $\mathbb{R}[T]$-module structure on $\mathbb{R}\mathcal{C}$, this poset-filtration refines the T-eigenvalue filtration $(\mathbb{R}\mathcal{C})_{\leq\lambda}$ in the following fashion: one has $(\mathbb{R}\mathcal{C})_{\leq\lambda} = U_{\mathcal{I}}$ for the order ideal $\mathcal{I} = \{X \in \mathcal{L} : \lambda_X \leq \lambda\}$.*

[3]Note that Hanlon and Hersh [**35**] and other authors put a coefficient of $\det(w)$ in front of each w in the sum. Thus, for our purposes we need to twist by the automorphism $w \mapsto \det(w) \cdot w$ in order to compare our BHR operators with the signed random-to-top shuffle operator they are using.

(iii) *Now assume $\mathcal{A}$ is the type A_{n-1} reflection arrangement and the p_x are chosen W-invariant, so that the W-invariant subalgebra of $\mathbb{R}\mathcal{F}$ acts on $\mathbb{R}\mathcal{C} \cong \mathbb{R}W$, and can be identified with the descent algebra acting on the right within $\mathbb{R}W$. Then for any W-stable order ideal $\mathcal{I}$ of $\mathcal{L}$, the j^{th} Hodge decomposition component $U_{\mathcal{I}}\mathfrak{e}_n^{(j)}$ of the poset-filtration space $U_{\mathcal{I}}$ carries $\mathbb{R}W$-module structure isomorphic to*

$$U_{\mathcal{I}}\,\mathfrak{e}_n^{(j)} \cong \sum \mathrm{Ind}_{\mathrm{Stab}_W(X)}^{W} \left(\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X} \right),$$

where the sum ranges over all W-orbits X^W in $\mathcal{L}/W$ with $X \in \mathcal{I}$ and $\dim_{\mathbb{R}}(X) = j$.

A proof of this theorem would lead us too far afield; it will be published separately [**58**].

6. Consequences for the kernels

For $\mathcal{A}$ a real hyperplane arrangement and W a finite group of linear symmetries, introduce a notation for the following $\mathbb{R}W$-modules that recur in the W-equivariant BHR theory:

$$\mathrm{WH}_{\mathcal{O}_X} = \mathrm{Ind}_{\mathrm{N}_W(X)}^{W} \left(\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X} \right) \tag{18}$$

where $\mathcal{O}_X := X^W$ is the W-orbit on $\mathcal{L}$ represented by some subspace X. The module $\mathrm{WH}_{\mathcal{O}_X}$ is almost a submodule of the *Whitney cohomology* $\mathrm{WH}^*(P;\mathbb{R})$ with real coefficients of a poset P with unique minimal element $\hat{0}$. The latter was introduced by Baclawski [**3**] by truncating the usual differential of the simplicial cochain complex. It follows that $\mathrm{WH}^*(P;\mathbb{R}) := \bigoplus_{p\in P} \widetilde{\mathrm{H}}^*((\hat{0},p);\mathbb{R})$. From the definition it is obvious that if a finite group W acts as a group of poset automorphisms on P then $\mathrm{WH}^*(P;\mathbb{R})$ becomes a W-module with submodule $\bigoplus_{p\in\mathcal{O}} \widetilde{\mathrm{H}}^*((\hat{0},p);\mathbb{R}) \cong \mathrm{Ind}_{\mathrm{Stab}_W(q)}^{W} \widetilde{\mathrm{H}}^*((\hat{0},q);\mathbb{R})$ for any W orbit $\mathcal{O}$ of P and $q \in \mathcal{O}$. Clearly, if W is a finite subgroup of $\mathrm{GL}(V)$ acting on $\mathcal{A}$ then W acts on $\mathcal{L}$ and except for the twist with $\det{}_{V/X}$ our module $\mathrm{WH}_{\mathcal{O}_X}$ coincides with a submodule of the Whitney cohomology of $\mathcal{L}$. We have chosen this twist since if facilitates the formulation of our applications of Whitney cohomology.

Also, define a partial order on the W-orbits $\mathcal{O}$ in $\mathcal{L}/W$ by setting $\mathcal{O} \leq \mathcal{O}'$ if there exist representatives X, X' in $\mathcal{O}, \mathcal{O}'$ with $X \leq X'$ in $\mathcal{L}$, that is $X' \subseteq X$.

Corollary 6.1. *For $\mathcal{O} \subseteq \mathcal{L}$ a single W-orbit, one has*

$$\ker(\nu_{\mathcal{O}}) \cong \bigoplus_{\substack{\mathcal{O}'\in\mathcal{L}/W:\\ \mathcal{O}'\not\leq\mathcal{O}}} \mathrm{WH}_{\mathcal{O}'} \otimes (\chi^-)^{\otimes \dim_{\mathbb{R}}(V/X)}$$

$$\mathrm{im}(\nu_{\mathcal{O}}) \cong \bigoplus_{\substack{\mathcal{O}'\in\mathcal{L}/W:\\ \mathcal{O}'\leq\mathcal{O}}} \mathrm{WH}_{\mathcal{O}'} \otimes (\chi^-)^{\otimes \dim_{\mathbb{R}}(V/X)}$$

as $\mathbb{R}[W \times \mathbb{Z}_2]$-modules.

Proof. By the semisimplicity of the self-adjoint operator $\nu_{\mathcal{O}}$ acting on $\mathbb{R}\mathcal{C}$, together with Corollary IV.4.3, it suffices to prove the assertion about $\ker(\nu_{\mathcal{O}})$.

Corollary IV.2.2 tells us that $\ker(\nu_{\mathcal{O}}) = \ker(b_J)$ where b_J is a BHR-operator that has $p_x > 0$ if and only if x is in the W-orbit of some particular face $x(J)$

whose support subspace X_0 lies in the W-orbit $\mathcal{O}$. Since $\ker(b_J) = (\mathbb{K}\mathcal{C})_{\leq 0}$ for this BHR-operator b_J, one deduces from Theorem IV.5.1 that its kernel carries the $\mathbb{R}[W \times \mathbb{Z}_2]$-module structure which is the sum of $\mathrm{WH}_{\mathcal{O}'} \otimes (\chi^-)^{\otimes \dim_{\mathbb{R}}(V/X)}$ over those W-orbits $\mathcal{O}'$ for which each representative subspace X has $\lambda_X = \sum_{x \subset X} p_x = 0$. This occurs if and only if each subspace X in $\mathcal{O}'$ contains no face in the W-orbit of $x(J)$, which occurs if and only if X contains no subspace in the W-orbit $\mathcal{O}$ of X_0, which occurs if and only if $\mathcal{O}' \not\leq \mathcal{O}$. □

EXAMPLE 6.2. When $\mathcal{A}$ is the reflection arrangement for a finite real reflection group W, and $\mathcal{O}$ is the W-orbit of an intersection subspace having low codimension, Corollary IV.6.1 says that $\ker \nu_{\mathcal{O}}$ will be large, and $\operatorname{im} \nu_{\mathcal{O}}$ small.

In particular, if $\mathcal{O}$ is the W-orbit of some hyperplane H corresponding to a reflection s, then $\operatorname{im} \nu_{\mathcal{O}}$ is the following sum over two W-orbits: $\mathcal{O}$ itself and the singleton orbit $\{V\}$.

$$\operatorname{im} \nu_{\mathcal{O}} \cong \left(\mathrm{Ind}_{Z_W(s)}^{W} \det{}_{V/H} \otimes \chi^-\right) \oplus \left(\mathbf{1}_W \otimes \chi^+\right).$$

EXAMPLE 6.3. For future use in Chapter V and Chapter VI, we wish to discuss two further examples in which $\mathcal{A}$ is the reflection arrangement of type A_{n-1} with $W = \mathfrak{S}_n$. Recall from Example II.1.3 that an intersection subspace X here corresponds to the set partition $[n] = \bigsqcup_i B_i$ whose blocks B_i tell us which coordinates x_j are equal on the subspace X. The W-orbit $\mathcal{O}_X$ is then determined by the number partition λ of n whose parts λ_i are the weakly decreasing reorderings of the block sizes $|B_i|$. Let X_λ be any representative of this W-orbit indexed by λ. Note that $\dim_{\mathbb{R}}(V/X_\lambda) = n - \ell(\lambda)$ where $\ell(\lambda)$ is the number of parts of λ.

If $\mathcal{O}, \mathcal{O}'$ are the orbits of X_λ, X_μ, one finds that $\mathcal{O} \leq \mathcal{O}'$ if and only μ refines λ, that is, if one can combine some of the parts of μ to obtain λ.

Therefore, Corollary IV.6.1 implies that if $\mathcal{O}$ is the orbit of X_λ then $\ker \nu_{\mathcal{O}}$ and $\operatorname{im} \nu_{\mathcal{O}}$, respectively, are the sums of $\mathrm{WH}_{\mathcal{O}_\mu} \otimes (\chi^-)^{\otimes n-\ell(\mu)}$ over all μ which do or do not, respectively, refine λ.

An interesting instance is when $\lambda = (n-k, 1^k)$, and the set of μ which do not refine λ are those μ that have at most $k-1$ parts of size 1. When $k = 1$, this is the set of all μ having no parts of size 1.

Another interesting instance is when $\lambda = (2^k, 1^{n-2k})$, and the set of μ which do refine λ are those of the form $\mu = (2^j, 1^{n-2j})$ for $j \leq k$.

7. Reformulation of $\mathrm{WH}_{\mathcal{O}_X}$

When $\mathcal{A}$ is the reflection arrangement for a finite real reflection group W, the representation $\mathrm{WH}_{\mathcal{O}_X}$ in (18) has some well-known extra features and reformulations, which we discuss below. When $W = \mathfrak{S}_n$, there are even more reformulations, also discussed below, some of which will be used in Chapter V and Chapter VI.

7.1. Reformulations for any reflection group. Our first reformulation originates in topology. Let $\mathcal{A}$ be an arrangement of (real) hyperplanes in $V = \mathbb{R}^d$. The chambers $\mathcal{C}$ of $\mathcal{A}$ are the connected components of the complement. Since each of them is easily seen to be contractible the complement is not an interesting topological space. One gains interesting topology when one extends scalars to $\mathbb{C}$ and considers the arrangement $\mathcal{A} \otimes \mathbb{C}$ of complex hyperplanes $H \otimes \mathbb{C}$, $H \in \mathcal{A}$, in

$\mathbb{C}^d$ defined by the same linear forms as H. We call $\mathcal{A}\otimes\mathbb{C}$ the *complexification* of $\mathcal{A}$ in $V_{\mathbb{C}} = \mathbb{C}^d$. The *complexified complement*

$$M_{\mathcal{A}} := \mathbb{C}^d \setminus \bigcup_{H\in\mathcal{A}} H\otimes\mathbb{C}$$

is a rich and complicated mathematical object (see for example [**48**]). It has cohomology algebra $\mathrm{OS}(\mathcal{A}) := \mathrm{H}^*(M_{\mathcal{A}};\mathbb{R})$ described by the *Orlik-Solomon presentation* [**46**, Theorem 5.2], which we now recall.

Choose for each hyperplane $H\in\mathcal{A}$ a linear form $\ell_H : \mathbb{C}^d \to \mathbb{C}$ with kernel $H\otimes\mathbb{C}$. Then there is an $\mathbb{R}$-algebra surjection from the exterior algebra $\bigwedge\mathcal{A}$ over $\mathbb{R}$ on generators $\{\mathrm{E}_H\}_{H\in\mathcal{A}}$ onto the cohomology algebra $\mathrm{H}^*(M_{\mathcal{A}};\mathbb{R})$

$$\begin{aligned}\bigwedge(\mathcal{A}) &\longrightarrow \mathrm{H}^*(M_{\mathcal{A}};\mathbb{R})\\ \mathrm{E}_H &\longmapsto \frac{d\ell_H}{\ell_H}\end{aligned}$$

whose kernel is generated by the elements

$$\sum_{s=0}^{t}(-1)^s \mathrm{E}_{H_1}\wedge\cdots\wedge\widehat{\mathrm{E}_{H_s}}\wedge\cdots\wedge\mathrm{E}_{H_t}$$

as $\{H_1,\ldots,H_t\}$ runs through all (minimal) subsets of hyperplanes in $\mathcal{A}$ that are dependent (in the sense that $\bigcap_{i=1}^t H_i$ has codimension strictly less than t). The algebra $\mathrm{OS}(\mathcal{A})$ is called the *Orlik-Solomon algebra* of $\mathcal{A}$. Note that the result by Orlik and Solomon holds even for integer coefficients. We use coefficients in the real numbers since we will consider the Orlik-Solomon algebra as a module.

The above presentation of the Orlik-Solomon algebra leads to a direct sum decomposition of $\mathrm{OS}(\mathcal{A})$ that comes from the subspaces $\mathrm{OS}(\mathcal{A})_X$ which are the images of the decomposable wedges $\mathrm{E}_{H_1}\wedge\cdots\wedge\mathrm{E}_{H_t}$ having $\bigcap_{i=1}^t H_i = X$ for some fixed X in $\mathcal{L}$:

$$\mathrm{OS}(\mathcal{A}) = \bigoplus_{X\in\mathcal{L}} \mathrm{OS}(\mathcal{A})_X.$$

PROPOSITION 7.1 (Theorem 5.2 [**46**] and Lemma 2.5 [**43**]). *For any arrangement $\mathcal{A}$ in $V\cong\mathbb{R}^d$ and subspace X in $\mathcal{L}$, there is a natural isomorphism*

$$\widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \cong \mathrm{OS}(\mathcal{A})_X.$$

Consequently, when a finite subgroup W of $\mathrm{GL}(V)$ acts on $\mathcal{A}$, one has

$$\mathrm{Ind}_{\mathrm{N}_W(X)}^W \widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \cong \mathrm{Ind}_{\mathrm{N}_W(X)}^W \mathrm{OS}(\mathcal{A})_X.$$

In particular,

$$\mathrm{WH}_{\mathcal{O}_X} \cong \mathrm{Ind}_{\mathrm{N}_W(X)}^W \mathrm{OS}(\mathcal{A})_X \otimes \det{}_{V/X}.$$

In the case of reflection arrangements, the dimension of $\mathrm{WH}_{\mathcal{O}_X}$ has a well-known reformulation.

PROPOSITION 7.2 (Lemma 4.7 [**47**]). *For a finite real reflection group W acting on the arrangement $\mathcal{L}$ in $V=\mathbb{R}^d$, and for any intersection subspace X in $\mathcal{L}$, one has*

$$\mu(V,X) = (-1)^{\dim V/X}\Big|\{w\in W : \mathrm{Fix}_w(V) = X\}\Big|,$$

where $\mathrm{Fix}_w(V)$ *is the set of elements in* V *fixed by the action of* w*. Consequently,*

$$\dim \mathrm{WH}_{\mathcal{O}_X} = \Big| \{ w \in W : \mathrm{Fix}_w(V) \in \mathcal{O}_X \} \Big|.$$

This reformulation suggested an interesting conjecture of Lehrer and Solomon, which they verified in [**43**] for $W = \mathfrak{S}_n$ of type A_{n-1} (see also Proposition IV.7.4) and also for dihedral groups $W = I_2(m)$. Note that the subset $\{w \in W : \mathrm{Fix}_w(V) \in \mathcal{O}_X\}$ of W is stable under conjugation, and hence a union of W-conjugacy classes.

CONJECTURE 7.3 (Conjecture 1.6 [**43**]). *There is an isomorphism of* W*-modules*

$$\mathrm{WH}_{\mathcal{O}_X} \cong \bigoplus_v \mathrm{Ind}_{\mathrm{Z}_W(v)}^W \xi_v$$

where v *runs over a system of representatives of the* W*-conjugacy classes which comprise* $\{w \in W : \mathrm{Fix}_w(V) \in \mathcal{O}_X\}$*, and* $\xi_v : \mathrm{Z}_W(v) \to \mathbb{C}^\times$ *is a degree one character of* $\mathrm{Z}_W(v)$*.*

7.2. Reformulations in type A**.** When $W = \mathfrak{S}_n$, one can both

- be much more explicit in the Lehrer-Solomon reformulation, and
- tie this in with other interesting reformulations, involving Lyndon words, free Lie algebras, etc.

As explained in Example II.1.3, an intersection subspace X for $W = \mathfrak{S}_n$ will correspond to a set partition $[n] = \bigsqcup_i B_i$ of $[n]$, and its W-orbit $\mathcal{O}_X$ is determined by the number partition λ of n whose parts give the block sizes $|B_i|$. Let X_λ be any representative of this W-orbit indexed by λ, and say that λ contains the part of size j with multiplicity m_j for each j. A typical element $v_\lambda \in W$ having $V^{v_\lambda} = X_\lambda$ will be a product of disjoint cycles of sizes λ_i supported on the blocks B_i. One then has that

$$\begin{aligned} \mathrm{N}_W(X_\lambda) &\cong \prod_i \mathfrak{S}_{m_i}[\mathfrak{S}_i] \\ \mathrm{Z}_W(v_\lambda) &\cong \prod_i \mathfrak{S}_{m_i}[\mathbb{Z}_i]. \end{aligned}$$

where $\mathfrak{S}_m[G]$ denotes the *wreath product* of G with the symmetric group $\mathfrak{S}_m$, and $\mathbb{Z}_i$ denotes the cyclic group of order i.

In [**43**] Lehrer and Solomon describe a degree one character ξ_λ of $\mathrm{Z}_W(v_\lambda)$ that sends each of the disjoint cycles of size j in v_λ to the same primitive j^{th} root of unity. This character fits the motivating type A case of their Conjecture IV.7.3 which they prove in their paper.

PROPOSITION 7.4 (Theorem 4.5 [**43**]). *Let* $W = \mathfrak{S}_n$ *and* λ *a partition of* n*. For an element* $v_\lambda \in W$ *such that* $V^{v_\lambda} = X_\lambda$ *we have*

$$\mathrm{WH}_{\mathcal{O}_{X_\lambda}} \cong \mathrm{Ind}_{\mathrm{Z}_W(v_\lambda)}^W \xi_\lambda.$$

One has a reformulation of the previous proposition in the language of symmetric functions; see [**44**] and [**64**, Chapter 7 (see in particular Exercise 7.89)] for the basic facts used here.

Recall Frobenius's *characteristic map* ch giving an isomorphism between virtual $\mathfrak{S}_n$-characters and symmetric functions of degree n. Under this isomorphism, one has

$$\mathrm{ch}\left(\mathrm{Ind}_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}} \chi_1 \otimes \chi_2 \right) = \mathrm{ch}(\chi_1) \cdot \mathrm{ch}(\chi_2)$$

where the product on the right is in the ring of symmetric functions. Also, given representations U, V of $\mathfrak{S}_m, \mathfrak{S}_n$, one can construct a representation $U[V]$ of $\mathfrak{S}_m[\mathfrak{S}_n]$ on $V^{\otimes m} \otimes U$ by having $(\mathfrak{S}_n)^m$ act on $V^{\otimes m}$ in the usual way, while $\mathfrak{S}_m$ permutes the tensor factors of $V^{\otimes m}$ and simultaneously acts on U; this construction is well-known (see for example [**44**, I.8 Remark 2, p. 136]) to have

$$\mathrm{ch}\, U[V] = \mathrm{ch}\, U[\mathrm{ch}\, V]$$

where $f[g]$ denotes the *plethysm* operation on symmetric functions f, g. We denote by $h_m := \mathrm{ch}(\mathbf{1}_{\mathfrak{S}_m})$ the m^{th} *homogeneous symmetric function*.

For $W = \mathfrak{S}_n$, let ω_n denote the W-representation $\mathrm{WH}_{\mathcal{O}_{X_{(n)}}}$, corresponding to the subspace $X_{(n)}$ which is the intersection of all the hyperplanes. Thus, by definition of $\mathrm{WH}_{\mathcal{O}_X}$ the representation ω_n is the $\mathfrak{S}_n$-representation on the top homology of the order complex of the proper part of the lattice of all set partitions of n, twisted by the character $\det_V = \mathrm{sgn}$. Based on work of Hanlon [**34**] it was shown by Stanley [**63**, Theorem 7.3] that $\omega_n = \mathrm{Ind}_{\mathbb{Z}_n}^{\mathfrak{S}_n} \exp(\frac{2\pi i}{n})$ is induced from a degree one character of the cyclic group $\mathbb{Z}_n$ generated in $\mathfrak{S}_n$ by a fixed n-cycle with character value $\exp(\frac{2\pi i}{n})$. From this the case $W = \mathfrak{S}_n$ and $\mathcal{O}_X = \mathcal{O}_{X_{(n)}}$ of Proposition IV.7.4 follows. This fact implies that Proposition 4.4 (iii) from [**43**] proved along the way to Proposition IV.7.4 actually can be stated as follows.

PROPOSITION 7.5. *For $W = \mathfrak{S}_n$ and λ any partition of n,*

$$\mathrm{WH}_{\mathcal{O}_{X_\lambda}} \cong \mathrm{Ind}_{\mathfrak{S}_{m_1}[\mathfrak{S}_1] \times \mathfrak{S}_{m_2}[\mathfrak{S}_2] \times \cdots}^{\mathfrak{S}_n} \left(\mathbf{1}_{\mathfrak{S}_{m_1}}[\omega_1] \otimes \mathbf{1}_{\mathfrak{S}_{m_2}}[\omega_2] \otimes \cdots \right),$$

or in other words,

$$\mathrm{ch}\, \mathrm{WH}_{\mathcal{O}_{X_\lambda}} = \prod_i h_{m_i}[\mathrm{ch}(\omega_i)]. \tag{19}$$

A proof of this result based solely on the description of $\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$ in terms of simplicial homology was first presented by Sundaram [**73**, Theorem 1.7]. Note that the result by Sundaram is stated for $\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$ tensored with the sign representation. Tensoring once more by the sign and some standard transformations of plethysms show the equivalence of her formula with the above formula.

Proposition IV.7.5 can be further reformulated in terms of the combinatorics of *Lyndon words* and *quasisymmetric functions*. Let $A = \{a_1 < a_2 < a_3 < \cdots\}$ be a linearly ordered alphabet. Recall that a word $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_n$ with letters $\mathbf{x}_i \in A$, $1 \le i \le n$, is called a *Lyndon word* if it is lexicographically strictly smaller than any of its cyclic rearrangements (see [**53**, §5] for more details). It is also well known [**53**, Theorem 5.1] that every word $\mathbf{x}$ has a unique *Lyndon factorization* $\mathbf{x} = \mathbf{x}^{(1)}\mathbf{x}^{(2)} \cdots \mathbf{x}^{(g)}$, meaning that the $\mathbf{x}^{(i)}$, $1 \le i \le g$, are Lyndon words satisfying

$$\mathbf{x}^{(1)} \ge_{lex} \mathbf{x}^{(2)} \ge_{lex} \cdots \ge_{lex} \mathbf{x}^{(g)}.$$

The *Lyndon type* of $\mathbf{x}$ is the weakly decreasing rearrangement of the lengths of the $\mathbf{x}^{(i)}$. We use this to define a power series in the ring of formal power series $\mathbb{C}[\![t_a \mid a \in A]\!]$ by

$$\mathfrak{L}_\lambda(\mathbf{t}) := \sum_{\substack{\text{words } \mathbf{x} \\ \text{of Lyndon type } \lambda}} t_{\mathbf{x}}$$

where $t_{\mathbf{x}} := t_{\mathbf{x}_1} t_{\mathbf{x}_2} \cdots t_{\mathbf{x}_n}$ for $\mathbf{x} = \mathbf{x}_1 \cdots \mathbf{x}_n$. In [**31**, Theorem 3.6] it is stated that $\mathfrak{L}_\lambda(\mathbf{t})$ coincides with the symmetric function on the right hand side of (19) from Proposition IV.7.5. This then yields the following description of $\mathrm{ch}\, \mathrm{WH}_{\mathcal{O}_{X_\lambda}}$.

PROPOSITION 7.6. *For $W = \mathfrak{S}_n$ and λ any partition of n,*

$$\operatorname{ch} \mathrm{WH}_{\mathcal{O}_{X_\lambda}} = \mathfrak{L}_\lambda(\mathbf{t}).$$

This reformulation is proved by Gessel and Reutenauer in [**31**] in parallel with a reformulation in terms of quasisymmetric functions. It is derived from a bijection attributed to Gessel (see e.g. [**53**, p. 175], [**23**], [**76**]) and described in [**31**]. The bijection allows one to expand $\mathfrak{L}_\lambda(\mathbf{t})$ in terms of the *fundamental quasisymmetric functions* associated with subsets $D \subseteq [n-1]$:

$$F_D := \sum_{\substack{1 \le i_1 \le \cdots \le i_n : \\ i_j < i_{j+1} \text{ if } j \in D}} t_{i_1} \cdots t_{i_n}.$$

Given a permutation $w = (w_1, \ldots, w_n)$ in $\mathfrak{S}_n$, define its *descent set* by

$$\operatorname{Des}(w) := \{ j \in [n-1] : w_j > w_{j+1} \}.$$

Theorem 3.6 in [**31**] states that $\sum_{w \text{ of cycle type } \lambda} F_{\operatorname{Des}(w)}$ equals $\mathfrak{L}_\lambda$. Thus, we arrive at the desired reformulation.

PROPOSITION 7.7. *For $W = \mathfrak{S}_n$ and λ any partition λ of n,*

$$\operatorname{ch} \mathrm{WH}_{\mathcal{O}_{X_\lambda}} = \sum_{w \text{ of cycle type } \lambda} F_{\operatorname{Des}(w)}.$$

In parallel to the study of $\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$ motivated by the action of the symmetric group on the Orlik-Solomon algebra, the module $\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$ appeared in a different guise already in the 1940's, in the context of the *free Lie algebra* (see [**75**], [**14**]). Before we can explain this connection we introduce the free Lie algebra and related notation (see [**53**] for more background information). The tensor algebra

$$T(V) := \bigoplus_{d \ge 0} V^{\otimes d}$$

is an associative algebra, and hence also a Lie algebra for the usual bracket operation $[x, y] = xy - yx$. The Lie subalgebra $\mathrm{Lie}(V)$ of $T(V)$ generated by its degree one part $T^1(V) = V$ is called the *free Lie algebra*. It inherits a grading

$$\mathrm{Lie}(V) = \oplus_{d \ge 0} \mathrm{Lie}(v)_d$$

where $\mathrm{Lie}(v)_d = \mathrm{Lie}(V) \cap V^{\otimes d}$. Because (see [**53**, Theorem 0.5]) $T(V)$ turns out to be the universal enveloping algebra for $\mathrm{Lie}(V)$, the Poincaré-Birkhoff-Witt theorem (see [**53**, Theorem 0.2]) gives a $\mathfrak{S}_n$-equivariant vector space isomorphism

$$T^n(V) = V^{\otimes n} \cong \sum_{\lambda \vdash n} \mathrm{Lie}_\lambda(V)$$

where for a partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots)$ one defines

$$\mathrm{Lie}_\lambda(V) := \mathrm{Sym}^{m_1}(\mathrm{Lie}(V)_1) \otimes \mathrm{Sym}^{m_2}(\mathrm{Lie}(V)_2) \otimes \cdots .$$

Here $\mathrm{Sym}^m(U)$ denotes the m^{th} graded component of the *symmetric algebra* over a vector space U. For a partition λ of n each $\mathrm{Lie}_\lambda(V)$ is itself an $\mathfrak{S}_n$-module. Assume $\dim V = n$, fix a basis of $T^1(V) = V$ and denote by E_n the space spanned in $T^n(V) = V^{\otimes n}$ by the tensors of the $n!$ permutations of the basis elements. For λ a partition of n the multilinear part of $\mathrm{Lie}_\lambda(V)$ is its intersection with E_n. Since both E_n and $\mathrm{Lie}_\lambda(V)$ are $\mathfrak{S}_n$-modules it follows that the multilinear part is also an $\mathfrak{S}_n$-module. For $\lambda = (n)$ it was shown by Klyachko [**41**] that $\mathrm{Lie}_{(n)}(V) \cap E_n$ is isomorphic as an $\mathfrak{S}_n$-module to $\mathrm{Ind}_{\mathbb{Z}_n}^{\mathfrak{S}_n} \exp(\frac{2\pi i}{n})$ and hence by the above mentioned result of Stanley [**63**, Theorem 7.3], it is isomorphic to $\mathrm{WH}_{\mathcal{O}_{(n)}}$. More generally a result by F. Bergeron, N. Bergeron and A. Garsia [**8**] (see also [**53**, Theorem 8.23]) shows that the characteristic of the $\mathfrak{S}_n$-representation on the multilinear part of $\mathrm{Lie}_\lambda(V)$ is the symmetric function on the right hand side of (19) from Proposition IV.7.5.

PROPOSITION 7.8. *Let $V \cong \mathbb{R}^n$ and λ any partition of n. Then there is an isomorphism of $\mathfrak{S}_n$-modules*

$$\mathrm{WH}_{\mathcal{O}_{X_\lambda}} \cong E_n \cap \mathrm{Lie}_\lambda(V).$$

Indeed, using this reformulation, Theorem 8.24 of [**53**], which was first proved in [**8**], gives an alternative proof of Proposition IV.7.4.

The action of the group $\mathrm{GL}(V)$ on $T^1(V)$ induces a diagonal action of $\mathrm{GL}(V)$ on each $T^d(V)$. By standard facts about the representation theory of $\mathrm{GL}(V)$ and from Proposition IV.7.8 and Proposition IV.7.6, the character of $\mathrm{Lie}_\lambda(V)$ as a $\mathrm{GL}(V)$-module (that is, the trace of a diagonal element of $\mathrm{GL}(V)$ having eigenvalues $x_1, \ldots, x_n$) is $\mathfrak{L}_\lambda(\mathbf{x})$. In other words, one has the following.

PROPOSITION 7.9. *For $W = \mathfrak{S}_n$ and λ any partition of n, the symmetric function $\mathrm{ch}\,\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$ is the $\mathrm{GL}(V)$-character of $\mathrm{Lie}_\lambda(V)$.*

Thus, the following problems are equivalent, and attributed by Schocker [**59**] to Thrall [**75**].

PROBLEM 7.10. Find any of

- the $\mathfrak{S}_n$-irreducible decomposition of $\mathrm{ch}\,\mathrm{WH}_{\mathcal{O}_{X_\lambda}}$,
- the $\mathrm{GL}(V)$-irreducible decomposition of $\mathrm{Lie}_\lambda(V)$, or
- the Schur function expansion of $\mathfrak{L}_\lambda(\mathbf{x})$.

Only partial results are known in this regard. For example for $\lambda = (n)$ it was shown by Kraskiewićz and Weyman in a preprint from 1987, published as [**42**], that

$$\mathfrak{L}_{(n)}(\mathbf{x}) = \sum_{\substack{T \text{ in } SYT_n: \\ \mathrm{maj}(T) \equiv 1 \mod n}} s_{\lambda(T)}.$$

This result can also be seen as a reformulation of a result by Springer from [**61**].

CHAPTER V

The family $\nu_{(2^k,1^{n-2k})}$

Our goal here is to prove Theorem I.4.3, whose statement we recall.

Theorem I.4.3. *The operators from the family* $\{\nu_{(2^k,1^{n-2k})}\}_{k=1,2\dots,\lfloor \frac{n}{2}\rfloor}$ *pairwise commute, and have only integer eigenvalues.*

Recall that here one is considering the reflection arrangement for the reflection group $W = \mathfrak{S}_n$, and the W-orbit on $\mathcal{L}$, denoted here $\mathcal{O}_{(2^k,1^{n-2k})}$, consisting of all intersection subspaces of the form

$$\{x_{i_1} = x_{i_2}\} \cap \{x_{i_3} = x_{i_4}\} \cap \cdots \cap \{x_{i_{2k-1}} = x_{i_{2k}}\}.$$

Then $\nu_{(2^k,1^{n-2k})} = \nu_{\mathcal{O}_{(2^k,1^{n-2k})}}$. Define also $\pi_{(2^k,1^{n-2k})} = \pi_{\mathcal{O}_{(2^k,1^{n-2k})}}$.

1. A Gelfand model for $\mathfrak{S}_n$

Theorem I.4.3 will follow from applying our eigenvalue integrality principle Proposition I.3.1 to the following representation-theoretic fact, which will identify the non-zero eigenspaces of the operators $\{\nu_{(2^k,1^{n-2k})}\}$ with a certain *Gelfand model* for $W = \mathfrak{S}_n$, whose construction is related to the construction of the Gelfand model of $\mathfrak{S}_n$ in [**2**]. Recall that a Gelfand model for $W = \mathfrak{S}_n$ is an $\mathfrak{S}_n$-module that carries exactly one copy of each $\mathfrak{S}_n$-irreducible χ^λ; see [**2**] for further references. Given a number partition λ, let $\mathrm{oddcols}(\lambda)$ denote the number of columns of odd length in the Ferrers diagram for λ, or the number of parts of odd length in the conjugate partition λ'.

Proposition 1.1. *For $W = \mathfrak{S}_n$ and non-negative integers a, b with $2a + b = n$ one has*

$$\mathrm{WH}_{\mathcal{O}_{(2^a,1^b)}} = \bigoplus_{\substack{\lambda:\\ \mathrm{oddcols}(\lambda)=b}} \chi^\lambda.$$

Consequently, we obtain a Gelfand model for $\mathfrak{S}_n$ by combining the modules:

$$\bigoplus_{a=0}^{\lfloor \frac{n}{2}\rfloor} \mathrm{WH}_{\mathcal{O}_{(2^a,1^{n-2a})}} = \bigoplus_{\lambda \vdash n} \chi^\lambda.$$

PROOF. By Proposition IV.7.5, one has

$$\begin{aligned}\sum_{a,b\geq 0} \mathrm{ch}\left(\mathrm{WH}_{\mathcal{O}_{(2^a,1^b)}}\right) t^b &= \sum_{a,b\geq 0} h_a[\mathrm{ch}\,\omega_2]\; h_b[\mathrm{ch}\,\omega_1]\; t^b \\ &= \sum_{a,b\geq 0} h_a[e_2(\mathbf{x})]\; h_b(\mathbf{x})\; t^b \\ &= \prod_{i<j}(1-x_ix_j)^{-1}\prod_i(1-tx_i)^{-1} \\ &= \sum_\lambda t^{\mathrm{oddcols}(\lambda)} s_\lambda(\mathbf{x})\end{aligned}$$

where the last equality uses a well-known identity (see [**64**, Exercise 7.28(b)], [**44**, Chap. I, §5, Ex. 7]). $\square$

2. Proof of Theorem I.4.3

We will actually prove following version of Theorem I.4.3, which is more precise and tells us more about the eigenvalues of the operators $\nu_{(2^k,1^{n-2k})}$.

THEOREM 2.1. *There exists an orthogonal direct sum decomposition of* $\mathbb{R}[\mathfrak{S}_n\times\mathbb{Z}_2]$*-modules*

$$\mathbb{R}\mathfrak{S}_n = K \oplus \left(\bigoplus_{\lambda\vdash n} U^\lambda\right) \tag{20}$$

with the following properties.

(i) *The subspace K is annihilated by each of the operators*

$$\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,\lfloor\frac{n}{2}\rfloor}.$$

(ii) *For $k = 0,1,2,\ldots,\lfloor\frac{n}{2}\rfloor$ the subspace U^λ lies inside the eigenspace for $\nu_{(2^k,1^{n-2k})}$ having eigenvalue*

$$\gamma_{(2^k,1^{n-2k}),\lambda} = \sum_{w\in\mathfrak{S}_n} \mathrm{noninv}_{(2^k,1^{n-2k})}(w)\cdot\chi^\lambda(w). \tag{21}$$

(iii) *The subspace U^λ affords the irreducible $\mathbb{R}[\mathfrak{S}_n\times\mathbb{Z}_2]$-module*

$$\chi^\lambda\otimes(\chi^-)^{\otimes\frac{n-\mathrm{oddcols}(\lambda)}{2}}.$$

PROOF OF Theorem V.2.1. We set $k_{\max} := \lfloor\frac{n}{2}\rfloor$, and define

$$K := \ker\pi_{(2^{k_{\max}},1^{n-2k_{\max}})} = \ker\nu_{(2^{k_{\max}},1^{n-2k_{\max}})}.$$

Since one can find a nested chain of representative subspaces for the W-orbits $\mathcal{O}_{(2^k,1^{n-2k})}$ as k varies, Proposition II.6.2 implies the following inclusions of kernels:

$$\begin{array}{cccccccc} K = & \ker\pi_{(2^{k_{\max}},1^{n-2k_{\max}})} & \subset\cdots\subset & \ker\pi_{(2^2,1^{n-4})} & \subset & \ker\pi_{(2^1,1^{n-2})} & \subset & \ker\pi_{(1^n)} \\ & \| & & \| & & \| & & \| \\ & \ker\nu_{(2^{k_{\max}},1^{n-2k_{\max}})} & & \ker\nu_{(2^2,1^{n-4})} & & \ker\nu_{(2^1,1^{n-2})} & & \ker\nu_{(1^n)} \end{array} \tag{22}$$

In particular, K is annihilated, and hence preserved, by every one of the self-adjoint operators $\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,k_{\max}}$.

Hence they also preserve the perpendicular space $U := K^{\perp}$, a $\mathbb{Q}$-rational subspace of $\mathbb{R}\mathfrak{S}_n$. Note that Corollary IV.2.2 implies that the $\mathfrak{S}_n$-representation afforded by U is the same as that afforded by the non-zero eigenspaces of a certain BHR operator b_J. Meanwhile, Example IV.6.3 shows that this $\mathfrak{S}_n$-representation is the sum of $\bigoplus_{a=0}^{k_{\max}} \mathrm{WH}_{\mathcal{O}_{(2^a,1^{n-2a})}}$. Note that this sum is isomorphic to the multiplicity-free Gelfand model described in Proposition V.1.1.

This multiplicity-freeness has two consequences. First, it shows that by combining the $\mathfrak{S}_n$-isotypic decomposition $U = \bigoplus_\lambda U^\lambda$ together with the complementary space K, one obtains a direct sum decomposition as in (20) that simultaneously diagonalizes all of the operators $\{\nu_{(2^k,1^{n-2k})}\}_{k=0,1,2,\ldots,k_{\max}}$.

Secondly, the eigenvalue integrality principle, Proposition I.3.1, implies that each operator $\nu_{(2^k,1^{n-2k})}$ acts on U with integer eigenvalues. Since $\nu_{(2^k,1^{n-2k})}$ also annihilates the subspace $K = U^{\perp}$ complementary to U, it has only integer eigenvalues on all of $\mathbb{R}\mathfrak{S}_n$.

However, we know more about the eigenvalue[1] $\gamma_{(2^k,1^{n-2k}),\lambda}$ with which $\nu_{(2^k,1^{n-2k})}$ acts on U^λ. Picking any realization $\mathfrak{S}_n \overset{\rho_\lambda}{\to} GL_{\mathbb{C}}(V)$ of the irreducible $\mathfrak{S}_n$-representation with character χ^λ, Proposition II.7.1 tells us that $\rho_\lambda(\nu_{(2^k,1^{n-2k})})$ has $\gamma_{(2^k,1^{n-2k}),\lambda}$ as its only potential non-zero eigenvalue, and hence

$$
\begin{aligned}
\gamma_{(2^k,1^{n-2k}),\lambda} &= \operatorname{Trace} \rho_\lambda(\nu_{(2^k,1^{n-2k})}) \\
&= \operatorname{Trace} \left(\sum_{w \in \mathfrak{S}_n} \operatorname{noninv}_{(2^k,1^{n-2k})}(w) \cdot \rho_\lambda(w) \right) \\
&= \sum_{w \in \mathfrak{S}_n} \operatorname{noninv}_{(2^k,1^{n-2k})}(w) \cdot \operatorname{Trace} \rho_\lambda(w) \\
&= \sum_{w \in \mathfrak{S}_n} \operatorname{noninv}_{(2^k,1^{n-2k})}(w) \cdot \chi^\lambda(w).
\end{aligned}
$$

Lastly, to see how $\mathbb{Z}_2$ acts on U^λ, note that Proposition V.1.1 implies that U^λ lies in

$$\operatorname{im}(\nu_{(2^a,1^{n-2a})}) \cap \operatorname{im}(\nu_{(2^{a-1},1^{n-2a+2})})^{\perp} \tag{23}$$

where $a := \frac{n - \operatorname{oddcols}(\lambda)}{2}$. Since $\nu_{(2^a,1^{n-2a})}, \pi_{(2^a,1^{n-2a})}$ share the same kernels, one has an isomorphism of $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-modules

$$\operatorname{im}(\nu_{(2^a,1^{n-2a})}) \cong \operatorname{im}(\pi_{(2^a,1^{n-2a})}).$$

Consequently the space (23) carries $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-module structure isomorphic to that of

$$\operatorname{im}(\pi_{(2^a,1^{n-2a})}) / \operatorname{im}(\pi_{(2^{a-1},1^{n-2a+2})})$$

which is $\mathrm{WH}_{\mathcal{O}_{(2^a,1^{n-2a})}} \otimes (\chi^-)^{\otimes a}$ by Example IV.6.3. Thus, $\mathbb{Z}_2$ acts by $(\chi^-)^{\otimes a}$ on U^λ. $\square$

Remark 2.2. One does not have that the associated BHR-operators b_J pairwise commute, in contrast with the situation for the original family $\{\nu_{(k,1^{n-k})}\}_{k=1,2\ldots,n}$.

[1] The first author thanks C.E. Csar for discussions leading to this expression for $\gamma_{(2^k,1^{n-2k}),\lambda}$.

REMARK 2.3. The formula for the eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$ given in (21) is somewhat explicit, but still leaves something to be desired. For example, the character values $\chi^\lambda(w)$ for w in $\mathfrak{S}_n$ are integers, but they can be negative. Thus (21) does not manifestly show the fact that $\gamma_{(2^k,1^{n-2k}),\lambda}$ is *non-negative*, nor does it show the fact that $\gamma_{(2^k,1^{n-2k}),\lambda}$ vanishes unless $\operatorname{oddcols}(\lambda) \geq n-2k$. This suggests the following problem.

PROBLEM 2.4. For each partition λ of n, and each k with $\operatorname{oddcols}(\lambda) \geq n-2k$, find a more explicit formula for the non-zero eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$ of $\nu_{(2^k,1^{n-2k})}$ acting on its (non-kernel) eigenspace U^λ affording χ^λ.

We have computed some of these eigenvalues using `Sage` [**68**], and we present this data for $3 \leq n \leq 6$ in the tables below. The data is presented as follows:

- each row of the table corresponds to the subspace U^λ affording χ^λ;
- the entry in the column indexed by $\nu_{(2^k,1^{n-2k})}$ is the eigenvalue $\gamma_{(2^k,1^{n-2k}),\lambda}$;
- the entry in the column indexed by w_0 is the eigenvalue for the $\mathbb{Z}_2$-action on U^λ.

To enhance the presentation of the data, every zero eigenvalue has been replaced by a dot.

	$\nu_{(1^3)}$	$\nu_{(2^1,1^1)}$	$\mathbb{Z}_2$
χ^{3}	6	9	1
χ^{21}	·	4	−1
χ^{111}	·	1	−1

	$\nu_{(1^4)}$	$\nu_{(2^1,1^2)}$	$\nu_{(2^2)}$	$\mathbb{Z}_2$
χ^{4}	24	72	18	1
χ^{31}	·	20	10	−1
χ^{211}	·	4	2	−1
χ^{22}	·	·	8	1
χ^{1111}	·	·	2	1

	$\nu_{(1^5)}$	$\nu_{(2^1,1^3)}$	$\nu_{(2^2,1^1)}$	$\mathbb{Z}_2$
χ^{5}	120	600	450	1
χ^{41}	·	120	180	−1
χ^{311}	·	20	30	−1
χ^{32}	·	·	68	1
χ^{221}	·	·	12	1
χ^{2111}	·	·	12	1
χ^{11111}	·	·	2	1

	$\nu_{(1^6)}$	$\nu_{(2^1,1^4)}$	$\nu_{(2^2,1^2)}$	$\nu_{(2^3)}$	$\mathbb{Z}_2$
χ^{6}	720	5400	8100	1350	1
χ^{51}	·	840	2520	630	−1
χ^{411}	·	120	360	90	−1
χ^{42}	·	·	616	308	1
χ^{321}	·	·	96	48	1
χ^{3111}	·	·	84	42	1
χ^{222}	·	·	24	12	1
χ^{21111}	·	·	12	6	1
χ^{33}	·	·	·	204	−1
χ^{2211}	·	·	·	36	−1
χ^{111111}	·	·	·	6	−1

Eigenvalues of $\nu_{(2^k,1^{n-2k})}$ acting on the non-kernel eigenspace afforded by χ^{λ}.

	$\nu_{(1^7)}$	$\nu_{(2^1,1^5)}$	$\nu_{(2^2,1^3)}$	$\nu_{(2^3,1^1)}$	$\mathbb{Z}_2$
χ^{7}	5040	52920	132300	66150	1
χ^{61}	·	6720	33600	25200	-1
χ^{511}	·	840	4200	3150	-1
χ^{52}	·	·	6048	9072	1
χ^{421}	·	·	840	1260	1
χ^{4111}	·	·	672	1008	1
χ^{322}	·	·	168	252	1
χ^{31111}	·	·	84	126	1
χ^{43}	·	·	·	2976	-1
χ^{331}	·	·	·	396	-1
χ^{3211}	·	·	·	396	-1
χ^{2221}	·	·	·	96	-1
χ^{22111}	·	·	·	48	-1
χ^{211111}	·	·	·	48	-1
$\chi^{1111111}$	·	·	·	6	-1

	$\nu_{(1^8)}$	$\nu_{(2^1,1^6)}$	$\nu_{(2^2,1^4)}$	$\nu_{(2^3,1^2)}$	$\nu_{(2^4)}$	$\mathbb{Z}_2$
χ^{8}	40320	564480	2116800	2116800	264600	1
χ^{71}	·	60480	453600	680400	113400	-1
χ^{611}	·	6720	50400	75600	12600	-1
χ^{62}	·	·	64512	193536	48384	1
χ^{521}	·	·	8064	24192	6048	1
χ^{5111}	·	·	6048	18144	4536	1
χ^{422}	·	·	1344	4032	1008	1
χ^{41111}	·	·	672	2016	504	1
χ^{53}	·	·	·	42240	21120	-1
χ^{431}	·	·	·	5376	2688	-1
χ^{4211}	·	·	·	4544	2272	-1
χ^{332}	·	·	·	960	480	-1
χ^{3221}	·	·	·	896	448	-1
χ^{32111}	·	·	·	512	256	-1
χ^{311111}	·	·	·	432	216	-1
χ^{22211}	·	·	·	128	64	-1
$\chi^{2111111}$	·	·	·	48	24	-1
χ^{44}	·	·	·	·	11904	1
χ^{3311}	·	·	·	·	1584	1
χ^{2222}	·	·	·	·	384	1
χ^{221111}	·	·	·	·	192	1
$\chi^{11111111}$	·	·	·	·	24	1

Eigenvalues of $\nu_{(2^k,1^{n-2k})}$ acting on the non-kernel eigenspace afforded by χ^{λ}.

CHAPTER VI

The original family $\nu_{(k,1^{n-k})}$

Let us return to the context of Theorem I.1.1. Here $W = \mathfrak{S}_n$ and $\mathcal{O} = \mathcal{O}_{(k,1^{n-k})}$ is the W-orbit containing the subspace $X_{(k,1^{n-k})}$, so we wish to analyze the elements

$$\nu_{\mathcal{O}} = \nu_{(k,1^{n-k})} := \sum_{w \in W} \mathrm{noninv}_k(w) \cdot w$$

where $\mathrm{noninv}_k(w)$ is the number of increasing sequences of length k contained in w.

1. Proof of Theorem I.1.1

Recall the statement of Theorem I.1.1.

Theorem I.1.1. *The operators from the family* $\{\nu_{(k,1^{n-k})}\}_{k=1,2,\ldots,n}$ *pairwise commute.*

Proof. Fix k and ℓ. One has $\nu_{(k,1^{n-k})}\nu_{(\ell,1^{n-\ell})} = \sum_{w \in \mathfrak{S}_n} d_w^{k,\ell} \cdot w$, where

$$d_w^{k,\ell} = \sum_{\substack{1 \le i_1 < \cdots < i_k \le n \\ 1 \le j_1 < \cdots < j_\ell \le n}} \left| \left\{ (u,v) \in \mathfrak{S}_n \times \mathfrak{S}_n : \begin{array}{r} uv = w, \\ u(i_1) < \cdots < u(i_k), \\ v(j_1) < \cdots < v(j_\ell) \end{array} \right\} \right|. \tag{24}$$

We want to show that $d_w^{k,\ell} = d_w^{\ell,k}$ for any permutation w in $\mathfrak{S}_n$.

Let us reformulate this coefficient $d_w^{k,\ell}$ a bit. First get rid of v using $v = u^{-1}w$. Second, if one names the sequences of lengths k and ℓ

$$\begin{aligned} K &:= (u(i_1), \ldots, u(i_k)) \\ L &:= (j_1, \ldots, j_\ell) \\ \text{so } w(L) &:= (w(j_1), \ldots, w(j_\ell)), \end{aligned}$$

then the condition on u in (24) is that both sequences K and $w(L)$ appear from left-to-right as subsequences inside $(u_1, u_2, \ldots, u_n)$. In other words, u lies in the set $\mathcal{L}(P_{K,w(L)})$ of all linear extensions of the poset $P_{K,w(L)}$ on $[n] = \{1, 2, \ldots, n\}$ defined as the transitive closure of the order relations

$$\begin{array}{ccc} u(i_1) & < \cdots < & u(i_k) \\ w(j_1) & < \cdots < & w(j_\ell). \end{array}$$

Example 1.1. If $n = 10$, $k = 6$, $\ell = 4$ and

$$\begin{aligned} w &= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 1 & 7 & 3 & 5 & 2 & 6 & 8 & 10 & 4 \end{bmatrix} \\ K &= (1,3,5,6,8,10) \\ w(L) &= (7,3,2,8) \end{aligned}$$

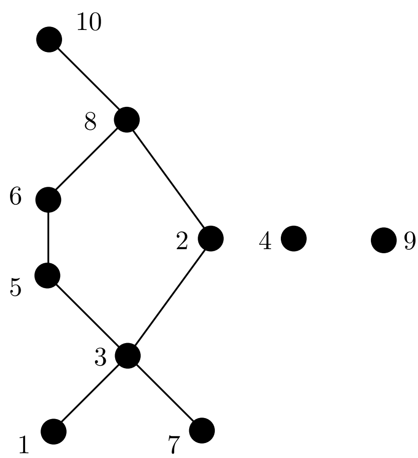

FIGURE 1. The poset $P_{K,w(L)}$ with $K = (1, 3, 5, 6, 8, 10)$ and $w(L) = (7, 3, 2, 8)$.

then the poset $P_{K,w(L)}$ for this example is shown in Figure 1.

Lastly note that when w is written in two-line notation, K is a k-subsequence of the top line, while $w(L)$ is an ℓ-subsequence of the bottom line. Thus one has

$$d_w^{k,\ell} = \sum_{\text{such } K,L} \#\mathcal{L}(P_{K,w(L)})$$

and we wish to show that $d_w^{k,\ell} = d_w^{\ell,k}$ for all permutations w in $\mathfrak{S}_n$.

First we fix the intersection and union of the underlying sets of K and $w(L)$

$$I := K \cap w(L)$$
$$J := K \cup w(L)$$

and define a new coefficient

$$d_{(w,I,J)}^{k,\ell} := \sum_{\substack{K,L:\\ K\cap w(L)=I\\ K\cup w(L)=J}} \#\mathcal{L}(P_{K,w(L)}). \tag{25}$$

Thus it suffices to show that for each fixed pair $I \subseteq J \subseteq [n]$, one has $d_{(w,I,J)}^{k,\ell} = d_{(w,I,J)}^{\ell,k}$.

One can reduce to the case where $J = K \cup w(L) = [n]$ as follows. If m lies in the complement $[n] \setminus J$, then m is incomparable to all other elements in $P_{K\cup w(L)}$ (such as $m = 4$ or $m = 9$ in the previous example), and one finds

$$\left|\mathcal{L}(P_{K,w(L)})\right| = n \cdot \left|\mathcal{L}(\hat{P}_{K,w(L)})\right|$$

where $\hat{P}_{K,w(L)}$ is the poset on $[n] \setminus \{m\}$ in which the element m has been removed. Thus we will assume without loss of generality that $J := K \cup w(L) = [n]$.

We reformulate further. Think of the fixed elements in $I := K \cap w(L)$ as a set of $i := |I|$ vertical dividers that partition the remaining elements $[n] \setminus I$ in the top and bottom of w into $i+1$ divisions:

$$w = \begin{bmatrix} t^{(1)} & | & t^{(2)} & | & \cdots & | & t^{(i+1)} \\ b^{(1)} & | & b^{(2)} & | & \cdots & | & b^{(i+1)} \end{bmatrix}$$

Note that the sequences $t^{(j)}$ and $b^{(j)}$ need not have the same length, and any of them are allowed to be empty sequences.

EXAMPLE 1.2. If one has $k = 7$, $\ell = 5$,

$$w = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 1 & 7 & 3 & 5 & 2 & 6 & 8 & 10 & 6 \end{bmatrix}.$$

and $I = \{3, 6\}$, then the divided w looks like

(26)
$$w = \begin{array}{c} \qquad\qquad\quad {\scriptstyle 3} \qquad\quad\;\; {\scriptstyle 6} \qquad\qquad\qquad \\ \left[\begin{array}{ccc|cc|cccc} 1 & 2 & & 4 & 5 & 7 & 8 & 9 & 10 \\ 9 & 1 & 7 & 5 & 2 & 8 & 10 & 4 & \end{array}\right] \end{array}.$$

One may as well relabel 26 to look like the following divided permutation w' of $[n'] := [8]$ where $n' := n - |I|$:

$$w' = \left[\begin{array}{ccc|cc|cccc} 1 & 2 & & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 1 & 5 & 4 & 2 & 6 & 8 & 3 & \end{array}\right].$$

Note that the remaining choice of

$$K' := K \setminus I$$
$$L' := w(L) \setminus I$$

gives a disjoint decomposition $[n] \setminus I = K' \sqcup L'$. So the number of linear extensions of $P_{K,w(L)}$ becomes the product, running over each of the $i+1$ divisions $(t^{(j)}, b^{(j)})$ in w, of the number of shuffles of the two sequences $K' \cap t^{(j)}$ and $L' \cap b^{(j)}$. Therefore, to count the linear extensions that make up $d^{k,\ell}_{(w,I,J)}$ in 25, it is equivalent to sum over the decompositions of $[n'] := K' \sqcup L'$ that have

$$k' := |K'| = k - i$$
$$l' := |L'| = \ell - i$$

and for each such decomposition, sum up the product of the number of shuffles of $K' \cap t^{(j)}$ with $L' \cap b^{(j)}$, that is, the product

(27)
$$\prod_{j=1}^{i+1} \binom{|K' \cap t^{(j)}| + |L' \cap b^{(j)}|}{|K' \cap t^{(j)}|, |L' \cap b^{(j)}|}.$$

Call such a choice of decomposition and the shuffles, a decomposition-shuffle of the divided permutation w' of $[n']$, and call the total number of them $d^{k',\ell'}_{w'}$. Thus we wish to show that $d^{k',\ell'}_{w'} = d^{\ell',k'}_{w'}$ for every divided permutation w' of $[n']$ and every pair (k', ℓ') with $k' + \ell' = n'$. This will be done by induction on n'.

First, note that one can reorder the elements in any of the $t^{(j)}, b^{(j)}$ arbitrarily; this does not affect the possible choices of a decomposition $[n'] = K' \sqcup L'$ nor does it affect the product (27).

So without loss of generality, assume that the numbers appear in integer order in each $t^{(j)}$ and $b^{(j)}$; in particular, the largest number n' will appear last within both its division on the top of w' and its division on the bottom. For example, the divided permutation w' from 26 would be reordered to

$$w' = \left[\begin{array}{ccc|cc|cccc} 1 & 2 & & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 7 & 2 & 4 & 3 & 6 & 8 & \end{array}\right].$$

We now count $d_{w'}^{k',\ell'}$ by classifying the decompositions-shuffles according to the entry m (if any) that appears directly after the entry n', within the shuffle that contains n'.

Case 1. The decomposition-shuffle has no such value of m, that is, n' appears last in the shuffle for its division.

Let w'' be obtained from w' by removing n' from both the top and bottom.

Subcase 1a. The decomposition $[n'] := K' \sqcup L'$ has n' in K'. It is straightforward to check that these decomposition-shuffles are counted by $d_{w''}^{k'-1,\ell'}$.

Subcase 1b. The decomposition $[n'] := K' \sqcup L'$ has n' in L'. Similarly, it is straightforward to check that these decomposition-shuffles are counted by $d_{w''}^{k',\ell'-1}$.

Putting together the two subcases, the decomposition/shuffles in this Case 1 are counted by the sum $d_{w''}^{k'-1,\ell'} + d_{w''}^{k',\ell'-1}$.

Note that when considering the corresponding decomposition/shuffles counted by $d_{w'}^{\ell',k'}$ (where the roles of k', ℓ' have been reversed, but w' is the same), those that fall in this Case 1 will analogously be counted by the sum $d_{w''}^{\ell'-1,k'} + d_{w''}^{\ell',k'-1}$, where w'' is the same permutation derived from w'. By induction,

$$\begin{aligned} d_{w''}^{\ell'-1,k'} &= d_{w''}^{k',\ell'-1} \\ d_{w''}^{\ell',k'-1} &= d_{w''}^{k'-1,\ell'} \end{aligned}$$

and hence these two sums are the same.

Case 2. The decomposition/shuffle has such a value m (i.e. something appearing directly after n' within the shuffle that contains n').

Then n', m must appear in opposite sets within the decomposition $[n'] := K' \sqcup L'$, due to the fact that n' appears last in its division.

Subcase 2a. The decomposition puts $m \in K'$ and $n' \in L'$.

Since m appears directly after n' in its shuffle, both m, n' must appear in the same division, i.e. $m \in K' \cap t^{(j)}$ and $n' \in L' \cap b^{(j)}$ for some j.

This time let w'' be obtained from w' by removing all occurrences of n', m and replacing the division $\left[\begin{smallmatrix} t^{(j)} \\ b^{(j)} \end{smallmatrix}\right]$ with two divisions separated by a divider labelled (n', m),

$$\left[\begin{array}{c|c} t' & t'' \\ b^{(j)} & - \end{array}\right]$$

in which t', t'' are the elements that appeared before and after m within $t^{(j)}$.

EXAMPLE 1.3. If $n' = 8$ and $m = 6$ in the above example, with $n' \in K$ and $m \in L$, one would replace the third division

$$\begin{bmatrix} t^{(3)} \\ b^{(3)} \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 3 & 6 & 8 & \end{bmatrix}$$

obtaining

$$w'' = \begin{array}{c} \qquad\qquad\qquad\qquad\qquad\quad {\scriptstyle (8,6)} \\ \left[\begin{array}{ccc|cc|c|c} 1 & 2 & & 3 & 4 & 5 & 7 \\ 1 & 5 & 7 & 2 & 4 & 3 & - \end{array}\right] \end{array}.$$

It is not hard to check that the decomposition-shuffles in Subcase 2a are then counted by $d_{w''}^{k'-1,\ell'-1}$. Note that by induction on n' one has

$$d_{w''}^{k'-1,\ell'-1} = d_{w''}^{\ell'-1,k'-1}.$$

Hence the decomposition-shuffles in the same Subcase 2a (with the same value of m) when the roles of k', ℓ' are reversed will have the same cardinality.

Subcase 2b. The decomposition puts $m \in L'$ and $n' \in K'$.

Same as Subcase 2a, with an analogous construction of w'' from w' by introducing one new divider labelled (n', m).

Thus in each case, reversing the roles of k', ℓ' leads to cases with the same cardinalities. Hence $d_{w'}^{k',\ell'} = d_{w'}^{\ell',k'}$, completing the proof. $\square$

PROBLEM 1.4. Find a more enlightening (non-inductive?) proof of Theorem I.1.1!

It turns out at that one also has pairwise commutativity for the family of BHR operators $\{b_{(k,1^{n-k})}\}_{k\in[n]}$ (this follows by combining Proposition IV.2.1 and [**60**, Main Theorem 2.1]), which are closely related to the operators $\nu_{(k,1^{n-k})}$ by Corollary IV.2.2. Perhaps this fact can be used as a starting point to prove Theorem I.1.1?

2. The kernel filtration and block-diagonalization

There is a way to get a good start on simultaneously diagonalizing the commuting family $\{\nu_{(k,1^{n-k})}\}$, by looking at a filtration that comes from their kernels.

As in the proof of Theorem I.4.3, since one can find a nested chain of representative subspaces for the W-orbits $\mathcal{O}_{(k,1^{n-k})}$ as k varies, Proposition II.6.2 implies the following inclusions of kernels:

(28)
$$\begin{array}{ccccccccccccc} 0 = & \ker \pi_{(n)} & \subset & \ker \pi_{(n-1,1)} & \subset & \ker \pi_{(n-2,1^2)} & \subset \cdots \subset & \ker \pi_{(2,1^{n-2})} & \subset & \ker \pi_{(1^n)} & \subset \mathbb{R}\mathfrak{S}_n \\ & \| & & \| & & \| & & \| & & \| & \\ & \ker \nu_{(n)} & & \ker \nu_{(n-1,1)} & & \ker \nu_{(n-2,1^2)} & & \ker \nu_{(2,1^{n-2})} & & \ker \nu_{(1^n)} & \end{array}$$

Since Theorem I.1.1 says that the $\nu_{(k,1^{n-k})}$ pairwise commute, they preserve each others kernels, and hence (28) gives an $\mathbb{R}[W \times \mathbb{Z}_2]$-module filtration of $\mathbb{R}\mathfrak{S}_n$ which is preserved by each of the $\nu_{(k,1^{n-k})}$. Denote the filtration factors for $j = 1, 2, \ldots, n$ by

$$\begin{aligned} F_{n,j} &= \ker \nu_{(n-j-1,1^{j+1})} / \ker \nu_{(n-j,1^j)} \\ &= \ker \pi_{(n-j-1,1^{j+1})} / \ker \pi_{(n-j,1^j)} \end{aligned}$$

with the convention that $F_{n,n} = \mathbb{R}\mathfrak{S}_n / \ker \pi_{(1^n)}$. One knows from the self-adjointness of each $\nu_{(k,1^{n-k})}$ that there exists an (orthogonal) direct sum decomposition

$$\mathbb{R}\mathfrak{S}_n = \bigoplus_{j=1}^{n} V_{n,j} \tag{29}$$

of $\mathbb{R}[W \times \mathbb{Z}_2]$-modules in which

$$V_{n,j} = \ker \nu_{(n-j-1,1^{j+1})} \cap \ker \nu^{\perp}_{(n-j,1^j)} \cong F_{n,j}$$

and hence (29) gives a simultaneous block diagonalization of the operators $\{\nu_{(k,1^{n-k})}\}_{k\in[n]}$.

At this point, we can use some of the equivariant BHR theory to analyze the $\mathbb{R}[W\times\mathbb{Z}_2]$-module structure of each $V_{n,j}$ or $F_{n,j}$: one has $\ker \nu_{(n-j,1^j)} = \ker b_{(n-j,1^j)}$ for a certain BHR operator $b_{(n-j,1^j)}$, whose kernel was analyzed in Example IV.6.3. This shows that

$$V_{n,j} \cong F_{n,j} \cong \bigoplus_{\substack{\lambda \text{ has exactly} \\ j \text{ parts of size } 1}} \mathrm{WH}_{\mathcal{O}_{X_\lambda}} \otimes (\chi^-)^{\otimes n-\ell(\lambda)}. \tag{30}$$

This shows that the dimension of $V_{n,j}$ is the number of permutations w in $\mathfrak{S}_n$ having j fixed points, in light of Proposition IV.7.2. However it is not very explicit as decomposition into $\mathbb{R}[W \times \mathbb{Z}_2]$-modules since we do not have solution for Problem IV.7.10 in general.

It turns outs that with a little work, we can provide a much more explicit description of $V_{n,j}$. The representation theory of $W = \mathfrak{S}_n$ asserts an $\mathbb{R}W$-module decomposition into irreducibles

$$\mathbb{R}\mathfrak{S}_n = \bigoplus_Q \chi^{\mathrm{shape}(Q)} \tag{31}$$

where Q runs over all *standard Young tableaux* of size n, and where shape(Q) is the partition whose *Ferrers diagram* gives the shape of Q. Although we will not need it here, one can also incorporate the $\mathbb{Z}_2$-action in (31) and give an explicit $\mathbb{R}[W \times \mathbb{Z}_2]$-module decomposition

$$\mathbb{R}\mathfrak{S}_n = \bigoplus_Q \chi^{\mathrm{shape}(Q)} \otimes (\chi^-)^{\otimes\, \mathrm{maj}(Q)} \tag{32}$$

where maj(Q) is the *major index* statistic on standard Young tableaux; this follows from Springer's theory of regular elements [**61**], the fact that w_0 is a regular element of $\mathfrak{S}_n$ [**49**, Lemma 8.4], and the formula for the fake degree polynomials in type A_{n-1} in terms of major indices [**42**].

Instead our goal in the next few subsections, culminating in Theorem VI.10.5, will be to provide a similar decomposition of $V_{n,j}$, as a sum of irreducible $\mathbb{R}[W\times\mathbb{Z}_2]$-modules of the form

$$\sum_Q \chi^{\mathrm{shape}(Q)} \otimes \chi^{\epsilon(Q)} \tag{33}$$

where Q runs over a certain class of standard Young tableaux that depends on j, and $\epsilon(Q)$ is a $\pm$ sign that depends upon Q. Here is an outline of how this goal is achieved.

- Step 1. Relate the bottom kernel $F_{n,0} = \ker \pi_{(n-1,1)}$ in the filtration to the homology of the *complex of injective words*, by showing that $\pi_{(n-1,1)}$ is a sign-twisted version of the top boundary map in this complex. This is achieved in Proposition VI.4.2.
- Step 2. Use homological techniques to describe this homology as an $\mathbb{R}[W \times \mathbb{Z}_2]$-module recursively. This is achieved in 41.

Step 3. Show that a description of $F_{n,0}$ automatically leads to one for $F_{n,j}$. This is achieved in Theorem VI.8.5.

Step 4. Solve this recursion for $F_{n,0}$ and $F_{n,j}$ in the form of (33). This is achieved in Theorem VI.10.5.

3. The (unsigned) maps on injective words

DEFINITION 3.1. Given a finite alphabet A, and an integer i in the range $0 \leq i \leq |A|$ let $A^{\langle i \rangle}$ denote the set of *injective words* of length i with letters taken from the alphabet A, that is, those words which use each letter at most once.

For a set M let $\mathbb{R}^M$ denote an $\mathbb{R}$-vector space with basis indexed by M. Given integers i, j with $0 \leq i \leq j \leq |A|$, define a map

$$\pi_{A,j,i} : \mathbb{R}^{A^{\langle j \rangle}} \longrightarrow \mathbb{R}^{A^{\langle i \rangle}}$$

that sends an injective word $a = (a_1, \ldots, a_j)$ of length j to the sum $\sum b$ of its subwords $b = (a_{k_1}, \ldots, a_{k_i})$, $1 \leq k_1 < \cdots < k_i \leq j$, of length i.

Note that the $\mathbb{R}$-linear maps $\pi_{A,j,i}$ are actually maps of $\mathbb{R}[\mathfrak{S}_A \times \mathbb{Z}_2]$-modules, when we consider $\mathbb{R}^{A^{\langle i \rangle}}$ as an $\mathbb{R}[\mathfrak{S}_A \times \mathbb{Z}_2]$-module in the following fashion:

- $\mathfrak{S}_A$ permutes the letters A, and
- the non-identity element of $\mathbb{Z}_2$ sends a word $a = (a_1, a_2, \ldots, a_i)$ to its *reversed word* $a^{\text{rev}} := (a_i, \ldots, a_2, a_1)$.

Our goal in the next few subsections is to begin by understanding the kernel of the first of the maps $\pi_{A,j,i}$, for which we use an abbreviated notation:

$$\pi_A := \pi_{A,|A|,|A|-1}.$$

The kernel of this map will turn out be closely related to the homology of the complex of injective words on A; see §VI.4.

REMARK 3.2. In fact, the maps $\pi_{A,|A|,i}$ are simply instances of the maps $\pi_{\mathcal{O}}$ where $W = \mathfrak{S}_A$ and $\mathcal{O}$ is the W-orbit of intersection subspaces where i of the coordinates are set equal.

4. The complex of injective words

DEFINITION 4.1. The *complex of injective words on* A is the chain complex $(K_A, \partial_{A,\cdot})$ having i^{th} chain group $K_{A,i} := \mathbb{R}^{A^{\langle i+1 \rangle}}$ and whose i^{th} boundary map

$$\partial_{A,i} : K_{A,i} \longrightarrow K_{A,i-1}$$

is a *signed* version of the map $\pi_{A,i+1,i}$:

$$\partial_{A,i}(a_0, a_1, \ldots, a_i) := \sum_{m=0}^{i} (-1)^m (a_0, \ldots, \widehat{a_m}, \ldots, a_i).$$

One can check that the complex $(K_A, \partial_{A,\cdot})$ becomes a complex of $\mathbb{R}[\mathfrak{S}_A \times \mathbb{Z}_2]$-modules only after we slightly twist our previously-defined $\mathbb{Z}_2$-action: one must now have the non-identity element of $\mathbb{Z}_2$ send an injective word a of length ℓ to $(-1)^{\lfloor \frac{\ell}{2} \rfloor} \cdot a^{\text{rev}}$.

There is a very simple relation between the maps π_A and $\partial_{A,|A|-1}$, once one identifies their source and targets with the group algebra $\mathbb{R}\mathfrak{S}_A$ appropriately. Define an $\mathbb{R}$-linear map $i_A : \mathbb{R}^{A^{\langle |A|-1 \rangle}} \longrightarrow \mathbb{R}\mathfrak{S}_A$ that sends an injective word u of length

$|A|-1$ that is missing exactly one letter a from A to the permutation of the set A which starts with the letter a and continues with the word u. The following proposition is straightforward.

PROPOSITION 4.2. *The map* $i_A : \mathbb{R}^{A^{\langle |A|-1\rangle}} \longrightarrow \mathbb{R}\mathfrak{S}_A$ *is an* $\mathbb{R}$*-linear isomorphism that makes the following diagram commute:*

$$\begin{tikzcd}
\mathbb{R}\mathfrak{S}_A \arrow[r, "\pi_A"] \arrow[d, "\mathrm{sgn}"'] & \mathbb{R}^{A^{\langle |A|-1\rangle}} \arrow[r, "i_A"] & \mathbb{R}\mathfrak{S}_A \arrow[d, "\mathrm{sgn}"] \\
\mathbb{R}\mathfrak{S}_A \arrow[r, "\partial_{A,|A|-1}"] & \mathbb{R}^{A^{\langle |A|-1\rangle}} \arrow[r, "i_A"] & \mathbb{R}\mathfrak{S}_A
\end{tikzcd}$$

where $\mathbb{R}\mathfrak{S}_n \xrightarrow{\mathrm{sgn}} \mathbb{R}\mathfrak{S}_n$ *is the involutive map that scales the basis element corresponding to a permutation* w *in* $\mathfrak{S}_A$ *by the sign of* w.

In particular, as subspaces of $\mathbb{R}\mathfrak{S}_A$*, the kernels of the two maps* π_A *and* $\partial_{A,|A|-1}$ *are sent to each other by the map* sgn.

5. Pieri formulae for $\mathfrak{S}_n$ and $\mathfrak{S}_n \times \mathbb{Z}_2$

We quickly review here the Pieri rules from the representation theory of $\mathfrak{S}_n$ and $\mathfrak{S}_n \times \mathbb{Z}_2$ that we will need, and introduce a more compact notation for induction products of characters.

Recall that for a finite group G, the irreducible complex representations $\mathrm{Irr}(G)$ are determined by their characters χ. Therefore, we will often speak of irreducible characters when we speak of elements of $\mathrm{Irr}(G)$.

The irreducible characters $\mathrm{Irr}(\mathfrak{S}_n)$ are indexed χ^λ by partitions λ of n, with $\chi^{(n)} = \mathbf{1}$ and $\chi^{(1^n)} = \mathrm{sgn}$. Since $\mathbb{Z}_2$ is abelian, its irreducible characters are both of degree 1:

$$\mathrm{Irr}(\mathbb{Z}_2) = \{\chi^+ = \mathbf{1}, \chi^-\}.$$

Therefore, the product group $\mathfrak{S}_n \times \mathbb{Z}_2$ has irreducible characters

$$\mathrm{Irr}(\mathfrak{S}_n \times \mathbb{Z}_2) = \{\chi^{\lambda,+} := \chi^\lambda \otimes \chi^+, \quad \chi^{\lambda,-} := \chi^\lambda \otimes \chi^- : \lambda \text{ a partition of } n\}.$$

Given this setup, the following is a corollary to Proposition VI.4.2. Recall that λ^T denotes the conjugate partition of λ.

COROLLARY 5.1. *For* $A = [n]$*, as* $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$*-modules, one has*

$$\ker \pi_A \cong \bigoplus_\alpha \chi^{\lambda_\alpha, \epsilon_\alpha}$$

if and only if

$$\ker \partial_{A,|A|-1} \cong \bigoplus_\alpha \chi^{\lambda_\alpha^T, \epsilon_\alpha}.$$

PROOF. The map $\mathbb{R}\mathfrak{S}_n \xrightarrow{\mathrm{sgn}} \mathbb{R}\mathfrak{S}_n$ has these effects:

- For the $\mathbb{R}\mathfrak{S}_n$-module structure, it tensors with the sgn-character, which on irreducibles does the following:

$$\chi^\lambda \mapsto \mathrm{sgn} \otimes \chi^\lambda = \chi^{\lambda^T}.$$

- For the $\mathbb{R}\mathbb{Z}_2$-module structure it is equivariant, since the non-identity element of $\mathbb{Z}_2$ acts by $w \mapsto ww_0$, when we are thinking of $\mathbb{R}\mathfrak{S}_n$ as the domain of π_A, but this non-identity element introduces an extra sign in front when we are thinking of $\mathbb{R}\mathfrak{S}_n$ as the domain of $\partial_{A,|A|-1}$ within the complex of injective words:

$$\begin{tikzcd}
w \arrow[rr, mapsto, "i_A \circ \pi_A"] \arrow[d, mapsto, "\mathrm{sgn}"'] & & ww_0 \arrow[d, mapsto, "\mathrm{sgn}"'] \\
\mathrm{sgn}(w) \cdot w \arrow[r, mapsto, "i_A \circ \partial_{A,|A|-1}"] & & \begin{array}{l}\mathrm{sgn}(ww_0)\cdot ww_0 \\ = (-1)^{\lfloor \frac{n}{2} \rfloor}\,\mathrm{sgn}(w)\cdot ww_0\end{array}
\end{tikzcd}$$

$\square$

The Young subgroup embedding $\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2} \hookrightarrow \mathfrak{S}_{n_1+n_2}$ leads to the usual *induction product* of characters χ_i in $\mathrm{Irr}(\mathfrak{S}_{n_i})$, for $i = 1, 2$, defined by

$$\chi_1 * \chi_2 := \mathrm{Ind}_{\mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}}^{\mathfrak{S}_{n_1+n_2}} \chi_1 \otimes \chi_2.$$

The *Pieri formulae* give two important special cases of the the irreducible expansion for the induction product of two $\mathfrak{S}_n$-irreducibles:

$$\begin{aligned} \chi^\mu * \chi^{(j)} &= \sum_{\substack{\lambda: \\ \lambda/\mu \text{ is a horizontal} \\ \text{strip of size } j}} \chi^\lambda \\ \chi^\mu * \chi^{(1^j)} &= \sum_{\substack{\lambda: \\ \lambda/\mu \text{ is a vertical} \\ \text{strip of size } j}} \chi^\lambda \end{aligned} \tag{34}$$

One can define an induction product of characters χ_i in $\mathrm{Irr}(\mathfrak{S}_{n_i} \times \mathbb{Z}_2)$ for $i = 1, 2$ by

$$\chi_1 * \chi_2 := \mathrm{Res}_{\mathfrak{S}_{n_1+n_2} \times \mathbb{Z}_2}^{\mathfrak{S}_{n_1+n_2} \times (\mathbb{Z}_2)^2} \mathrm{Ind}_{\mathfrak{S}_{n_1} \times \mathbb{Z}_2 \times \mathfrak{S}_{n_2} \times \mathbb{Z}_2}^{\mathfrak{S}_{n_1+n_2} \times (\mathbb{Z}_2)^2} (\chi_1 \otimes \chi_2).$$

where the restriction map above comes from the *diagonal* embedding $\mathbb{Z}_2 \hookrightarrow (\mathbb{Z}_2)^2$ that sends $x \mapsto (x, x)$.

PROPOSITION 5.2. *For any partition μ and signs ϵ_1, ϵ_2 in $\{+, -\}$, one has*

$$\begin{aligned} \chi^{\mu,\epsilon_1} * \chi^{(j),\epsilon_2} &= \sum_{\substack{\lambda: \\ \lambda/\mu \textit{ is a horizontal} \\ \textit{strip of size } j}} \chi^{\lambda,\epsilon_1\epsilon_2} \\ \chi^{\mu,\epsilon_1} * \chi^{(1^j),\epsilon_2} &= \sum_{\substack{\lambda: \\ \lambda/\mu \textit{ is a vertical} \\ \textit{strip of size } j}} \chi^{\lambda,\epsilon_1\epsilon_2} \end{aligned} \tag{35}$$

PROOF. More generally, for any embedding of finite groups $G_1 \times G_2 \hookrightarrow G$, and an abelian group A, along with characters χ_i in $\mathrm{Irr}(G_i)$ for $i = 1, 2$, and characters ϵ_1, ϵ_2 in $\mathrm{Irr}(A)$, we claim

$$\begin{aligned} &\mathrm{Res}_{G \times A}^{G \times A^2} \mathrm{Ind}_{G_1 \times A \times G_2 \times A}^{G \times A^2} (\chi_1 \otimes \epsilon_1 \otimes \chi_2 \otimes \epsilon_2) \\ &= \bigoplus_{\chi \in \mathrm{Irr}(G)} \left\langle \mathrm{Ind}_{G_1 \times G_2}^{G} \chi_1 \otimes \chi_2, \chi \right\rangle_G \cdot \chi \otimes \epsilon_1 \epsilon_2. \end{aligned}$$

This comes, for example, using Frobenius reciprocity to calculate the inner product with an irreducible $\chi \otimes \epsilon$ in $\mathrm{Irr}(G \times A)$:

$$\begin{aligned}
&\Big\langle \ \mathrm{Res}^{G\times A^2}_{G\times A} \mathrm{Ind}^{G\times A^2}_{G_1\times A\times G_2\times A} (\chi_1 \otimes \epsilon_1 \otimes \chi_2 \otimes \epsilon_2) \ , \ \chi \otimes \epsilon \ \Big\rangle_{G\times A} \\
&\quad = \Big\langle \ \chi_1 \otimes \epsilon_1 \otimes \chi_2 \otimes \epsilon_2 \ , \ \mathrm{Res}^{G\times A^2}_{G_1\times A\times G_2\times A} \mathrm{Ind}^{G\times A^2}_{G\times A} \chi \otimes \epsilon \ \Big\rangle_{G_1\times A\times G_2\times A} \\
&\quad = \big\langle \ \chi_1 \otimes \chi_2 \ , \ \mathrm{Res}^{G}_{G_1\times G_2} \chi \ \big\rangle_{G_1\times G_2} \cdot \Big\langle \ \epsilon_1 \otimes \epsilon_2 \ , \ \mathrm{Ind}^{A^2}_{A} \epsilon \ \Big\rangle_A \\
&\quad = \Big\langle \ \mathrm{Ind}^{G}_{G_1\times G_2} \chi_1 \otimes \chi_2 \ , \ \chi \ \Big\rangle_G \cdot \Big\langle \ \mathrm{Res}^{A^2}_{A} \epsilon_1 \otimes \epsilon_2 \ , \ \epsilon \ \Big\rangle_A \\
&\quad = \begin{cases} \Big\langle \ \mathrm{Ind}^{G}_{G_1\times G_2} \chi_1 \otimes \chi_2 \ , \ \chi \ \Big\rangle_G & \text{if } \epsilon = \epsilon_1\epsilon_2 \\ 0 & \text{otherwise.} \end{cases} \qquad \square
\end{aligned}$$

6. Some derangement numerology

The nullity of either map π_A or $\partial_{A,|A|-1}$ turns out to be the number of *derangements* (that is, permutations with no fixed points) in $\mathfrak{S}_n$. We review here some easy, classical, enumerative results about derangements, along with a few somewhat more recent results about even and odd derangements, relevant for the $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-module structures on the kernels; see also Chapman [**18**], Mantaci and Rakotandrajao [**45**], Gordon and McMahon [**32**, §4].

Definition 6.1. For $n \geq 1$, let d_n, d_n^+, d_n^- denote, respectively, the total number of derangements in $\mathfrak{S}_n$, the number whose sign is positive, and the number whose sign is negative. (The table in Figure 2 lists the first few values.)

n	d_n	d_n^+	d_n^-
0	1	1	0
1	0	0	0
2	1	0	1
3	2	2	0
4	9	3	6
5	44	24	20
6	265	130	135

Figure 2. The first few values of d_n, d_n^+ and d_n^-: the total number of derangements, even derangements and odd derangements in $\mathfrak{S}_n$, respectively.

Proposition 6.2. *The numbers d_n, d_n^+, d_n^- satisfies the initial conditions*

$$d_0 = d_0^+ = 1, d_0^- = 0$$
$$d_1 = d_1^+ = d_1^- = 0$$

as well as the following recurrences and identities:

(36a) $$d_n = (n-1)(d_{n-1} + d_{n-2}) \text{ for } n \geq 2;$$

(36b) $$d_n^+ = (n-1)(d_{n-1}^- + d_{n-2}^-) \text{ for } n \geq 2;$$

$$d_n^- = (n-1)(d_{n-1}^+ + d_{n-2}^+) \textit{ for } n \geq 2; \tag{36c}$$

$$d_n = nd_{n-1} + (-1)^n \textit{ for } n \geq 1; \tag{36d}$$

$$d_n^+ - d_n^- = (-1)^{n-1}(n-1) \textit{ for } n \geq 0; \tag{36e}$$

$$d_n = \binom{n}{2} 2d_{n-2} + (-1)^n(n-1) \textit{ for } n \geq 2; \tag{36f}$$

$$n! = \sum_{j=0}^{n} \binom{n}{j} d_{n-j}, \tag{36g}$$

as $\binom{n}{j}d_{n-j}$ *(resp.* $\binom{n}{j}d_{n-j}^+$, $\binom{n}{j}d_{n-j}^-$*) counts the number of permutations (resp. even, odd permutations) having exactly* j *fixed points.*

Proof. Recurrences 36a, 36b, 36c follow from the fact that given a derangement w in $\mathfrak{S}_n$, erasing n from the cycle structure of w results in one of two possibilities.

- A derangement $\hat{w}$ in $\mathfrak{S}_{n-1}$ having opposite sign to w. From $\hat{w}$ one can uniquely recover w by specifying the value $w(n)$ in $[n-1]$.
- A permutation in $\mathfrak{S}_{n-1}$ with exactly one fixed point. After removing this fixed point $w(n)$, one obtains a derangement $\hat{w}$ in $\mathfrak{S}_{n-2}$ having opposite sign to w. And again, from $\hat{w}$ one can uniquely recover w by specifying the value $w(n)$ in $[n-1]$.

Recurrence 36d follows from 36a by induction on n. The base cases where $n = 0, 1$ are easily checked. In the inductive step where $n \geq 2$, recurrence 36a implies

$$d_n - nd_{n-1} = -(d_{n-1} - (n-1)d_{n-2}) = -(-1)^{n-1}$$

where the second equality uses induction.

Recurrence 36e follows from 36b and 36c by induction on n. The base cases where $n = 0, 1$ are easily checked. In the inductive step where $n \geq 2$, recurrences 36b and 36c imply

$$\begin{aligned} d_n^+ - d_n^- &= (n-1)\left((d_{n-1}^- - d_{n-2}^-) - (d_{n-1}^+ - d_{n-2}^+)\right) \\ &= (n-1)\left((d_{n-2}^+ - d_{n-2}^-) - (d_{n-1}^+ - d_{n-1}^-)\right) \\ &= (n-1)\left((-1)^{n-3}(n-3) - (-1)^{n-2}(n-2)\right) \\ &= (n-1)(-1)^{n-1} \end{aligned}$$

where the third equality uses induction.

Recurrence 36f is a rewriting of the first iterate of recurrence 36d:

$$\begin{aligned} d_n &= nd_{n-1} + (-1)^n \\ &= n\left((n-1)d_{n-2} + (-1)^{n-1}\right) + (-1)^n \\ &= \binom{n}{2} \cdot 2d_{n-2} + (-1)^{n-1}(n-1) \end{aligned}$$

The assertions in 36g all come from the fact that every permutation w in $\mathfrak{S}_n$ having j fixed points gives rise to a derangement $\hat{w}$ on its set of $n-j$ non-fixed points; this $\hat{w}$ has the same sign as w. ∎

7. $(\mathfrak{S}_n \times \mathbb{Z}_2)$-structure of the first kernel

We begin with a proposition showing that the kernel of the top boundary map $\partial_{[n],n-1}$ in the complex of injective words satisfies the representation-theoretic analogues of the derangement number recurrences in 36d and 36f. For the $\mathbb{R}\mathfrak{S}_n$-module structure, this was observed in [**51**, §2]; for the $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-module structure it appears to be new.

PROPOSITION 7.1. *Considered as a virtual character of $\mathfrak{S}_n$,*

$$\ker \partial_{[n],n-1} = \ker \partial_{[n-1],n-2} * \chi^{(1)} + (-1)^n \chi^{(n)}. \tag{37}$$

Considered as a virtual character of $\mathfrak{S}_n \times \mathbb{Z}_2$,

$$\ker \partial_{[n],n-1} = \ker \partial_{[n-2],n-3} * \left(\chi^{(2),-} + \chi^{(1^2),+}\right) + (-1)^{n-1}\chi^{(n-1,1),+}. \tag{38}$$

PROOF. (cf. Proof of [**51**, Propositions 2.1, 2.2]) The complex of injective words $(K_{[n]}, \partial_{[n],\cdot})$ is known to be the augmented cellular chain complex corresponding to a regular CW-complex of dimension $n-1$, homotopy equivalent to a bouquet of spheres of dimension $n-1$; see Farmer [**24**], Björner and Wachs [**13**]. Consequently, its homology $\widetilde{\mathrm{H}}_\bullet(K_{[n]})$ is concentrated in dimension $n-1$, and coincides with $\ker \partial_{[n],n-1}$.

On the other hand, the Hopf trace formula (see [**77**, Theorem 2.3.9]) gives the following identity of virtual characters for any finite group acting on $K_{[n]}$:

$$\sum_{i \geq -1} (-1)^i \widetilde{\mathrm{H}}_i(K_{[n]}) = \sum_{i \geq -1} (-1)^i K_{[n],i}.$$

From this we conclude that

$$\ker \partial_{[n],n-1} = \sum_{i \geq -1} (-1)^{n-i-1} K_{[n],i}. \tag{39}$$

Using this expression (39), the two recurrences in the proposition will follow after deriving recurrences for $\mathfrak{S}_n$ and $\mathfrak{S}_n \times \mathbb{Z}_2$-structures on the chain groups $K_{[n],i}$.

The recurrence as characters of $\mathfrak{S}_n$ takes the form

$$K_{[n],i} = \begin{cases} \chi^{(n)} & \text{if } i = -1 \\ K_{[n-1],i-1} * \chi^{(1)} & \text{if } i \geq 0. \end{cases}$$

This is because as $\mathbb{R}\mathfrak{S}_n$-modules one has the general description

$$\begin{aligned} K_{[n],i} &= \mathbb{R}^{[n]^{\langle i+1 \rangle}} \\ &\cong \chi^{(n-i-1)} * \underbrace{\chi^{(1)} * \cdots * \chi^{(1)}}_{i+1 \text{ factors}}. \end{aligned}$$

The recurrence as characters of $\mathfrak{S}_n \times \mathbb{Z}_2$ takes the form

$$K_{[n],i} = \begin{cases} \chi^{(n),+} & \text{if } i = -1 \\ \chi^{(n-1),+} * \chi^{(1),+} & \text{if } i = 0 \\ K_{[n-2],i-2} * \left(\chi^{(2),-} + \chi^{(1^2),+}\right) & \text{if } i \geq 1. \end{cases}$$

To understand this, note that the $\mathbb{Z}_2$-action reversing the positions in injective words of length $i+1$ decomposes according to the cycle structure of the reversing permutation w_0 in $\mathfrak{S}_{i+1}$. This allows one to describe the chain groups via the induction product as follows: as $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-modules,

$$\begin{aligned} K_{[n],i} &= \mathbb{R}^{[n]^{\langle i+1\rangle}} \\ &\cong \chi^{(n-i-1),+} * \begin{cases} \underbrace{\mathbb{R}\mathfrak{S}_2 * \cdots * \mathbb{R}\mathfrak{S}_2}_{\frac{i+1}{2} \text{ factors}} & \text{if } i \text{ is odd,} \\ \underbrace{\mathbb{R}\mathfrak{S}_2 * \cdots * \mathbb{R}\mathfrak{S}_2}_{\frac{i}{2} \text{ factors}} * \mathbb{R}\mathfrak{S}_1 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Also note that

$$\begin{aligned} \mathbb{R}\mathfrak{S}_1 &\cong \chi^{(1),+} \\ \mathbb{R}\mathfrak{S}_2 &\cong \chi^{(2),-} + \chi^{(1^2),+}. \end{aligned}$$

It only remains to explain the last term on the right in (38), arising from the following computation:

$$\begin{aligned} (-1)^{n-1} K_{[n],0} + (-1)^n K_{[n],-1} &= (-1)^{n-1} \left(\chi^{(n-1),+} * \chi^{(1),+} - \chi^{(n),+} \right) \\ &= (-1)^{n-1} \chi^{(n-1,1),+} \end{aligned}$$ □

Combining this with Corollary VI.5.1 immediately gives the following version of the same recurrences, which are again analogues of the derangement recurrences 36d and 36f.

COROLLARY 7.2. *Considered as a virtual character of $\mathfrak{S}_n$,*

$$\ker \pi_{[n]} = \ker \pi_{[n-1]} * \chi^{(1)} + (-1)^n \chi^{(1^n)}. \tag{40}$$

Considered as a virtual character of $\mathfrak{S}_n \times \mathbb{Z}_2$,

$$\ker \pi_{[n]} = \ker \pi_{[n-2]} * \left(\chi^{(2),+} + \chi^{(1^2),-} \right) + (-1)^{n-1} \chi^{(2,1^{n-2}),+}. \tag{41}$$ □

COROLLARY 7.3. *The kernels of the two maps $\pi_{[n]}$ and $\partial_{[n],n-1}$ both have dimension d_n, the number of derangements in $\mathfrak{S}_n$. Furthermore, both have the dimension of their $\mathbb{Z}_2$-isotypic components equal to d_n^+, d_n^-, the number of even, odd derangements in $\mathfrak{S}_n$, respectively.*

PROOF. The first assertion follows upon comparison of the recurrence (37) with 36d.

For the second assertion, note that (38) implies that the dimensions $\hat{d}_n^+, \hat{d}_n^-$ of the $\mathbb{Z}_2$-isotypic components of the kernel of $\pi_{[n]}$ satisfy for $n \geq 2$ the recurrences

$$\begin{aligned} \hat{d}_n^+ &= \binom{n}{2} \hat{d}_{n-2}^- + \binom{n}{2} \hat{d}_{n-2}^+ + (-1)^{n-1}(n-1) \\ \hat{d}_n^- &= \binom{n}{2} \hat{d}_{n-2}^+ + \binom{n}{2} \hat{d}_{n-2}^-. \end{aligned}$$

Subtracting these gives for $n \geq 2$,

$$\hat{d}_n^+ - \hat{d}_n^- = (-1)^{n-1}(n-1) = d_n^+ - d_n^-$$

where the last equality is 36e. One can directly verify that this holds also for $n = 0, 1$. On the other hand, by the first assertion of the corollary, one has

$$\hat{d}_n^+ + \hat{d}_n^- = d_n = d_n^+ + d_n^-$$

and hence one concludes that $\hat{d}_n^\epsilon = d_n^\epsilon$ for $\epsilon = +, -$. □

8. $(\mathfrak{S}_n \times \mathbb{Z}_2)$-structure of the kernel filtration

We continue our study of the $(\mathfrak{S}_n \times \mathbb{Z}_2)$-structure of the filtration factors $F_{n,j}$ in (28). The following proposition is a straightforward special case of 4, and will be used in the proof of Lemma VI.8.3 and Theorem VI.8.5.

PROPOSITION 8.1. *For any finite set A and $0 \leq i \leq j \leq k \leq |A|$, one has*

$$\pi_{A,j,i} \circ \pi_{A,k,j} = \binom{k-i}{j-i} \pi_{A,k,i}.$$

We first use this proposition in the proof of a technical lemma.

DEFINITION 8.2. Given a set A, let $\binom{A}{j}$ denote the collection of all j-element subsets J of A. Given J in $\binom{A}{j}$, define an $\mathbb{R}$-bilinear concatenation product

$$\mathbb{R}\mathfrak{S}_J \times \mathbb{R}\mathfrak{S}_{A \setminus J} \longrightarrow \mathbb{R}\mathfrak{S}_A$$

by sending (u, v) to $u \bullet v := (u_1, \ldots, u_j, v_1, \ldots, v_{n-j})$; that is the permutation sending i to u_i if $1 \leq i \leq j$ and i to v_{i-j} if $j+1 \leq i \leq n$.

LEMMA 8.3. *For any J in $\binom{[n]}{j}$,*

$$\ker \pi_{[n]\setminus J} \subset \operatorname{im}\left(\pi_{[n],n,n-j}\right).$$

PROOF. Let u, v be permutations in $\mathfrak{S}_J, \mathfrak{S}_{[n]\setminus J}$, respectively. Then their concatenation product $u \bullet v$ in $\mathbb{R}\mathfrak{S}_n$ has the following image under $\pi_{[n],n,n-j}$:

$$\begin{aligned}
\pi_{[n],n,n-j}(u \bullet v) &= \sum_{\substack{\text{subwords } \hat{u},\hat{v} \text{ of } u,v: \\ \ell(u)+\ell(v)=n-j}} \hat{u} \bullet \hat{v} \\
&= \sum_{\text{subwords } \hat{u} \text{ of } u} \hat{u} \bullet \pi_{[n],n,n-j-\ell(\hat{u})}(v).
\end{aligned}$$

Now assume x lies in $\ker \pi_{[n]\setminus J}$. Since $\pi_{[n]\setminus J} := \pi_{[n]\setminus J,n-j,n-j-1}$, Proposition VI.8.1 shows that x also lies in $\ker \pi_{[n],n,n-j-\ell}$ for every $\ell \geq 1$. Therefore,

$$\pi_{[n],n,n-j}(u \bullet x) = \sum_{\text{subwords } \hat{u} \text{ of } u} \hat{u} \bullet \pi_{[n],n,n-j-\ell(\hat{u})}(x) = x$$

as only the empty subword $\hat{u} = \varnothing$ can contribute in the sum above. Thus x lies in $\operatorname{im} \pi_{[n],n,n-j}$. $\square$

We will also need one simple general linear algebra fact.

PROPOSITION 8.4. *Given any linear maps $A \xrightarrow{f} B \xrightarrow{g} C$, the map f induces an isomorphism*

$$A/\ker(f) \cong \operatorname{im}(f)$$

which restricts to an isomorphism

$$\ker(g \circ f)/\ker(f) \cong f(\ker(g \circ f)) = \operatorname{im}(f) \cap \ker(g). \quad \square$$

THEOREM 8.5. *For each $j = 0, 1, \ldots, n$, the map*

$$\pi_{[n],n,n-j} : \mathbb{R}\mathfrak{S}_n \longrightarrow \mathbb{R}^{[n]^{\langle n-j \rangle}} = \bigoplus_{J \in \binom{[n]}{j}} \mathbb{R}\mathfrak{S}_{[n]\setminus J}$$

induces an $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$*-module isomorphism*

$$F_{n,j} \xrightarrow{\sim} \bigoplus_{J \in \binom{[n]}{j}} \ker(\pi_{[n]\setminus J}) \quad \cong \quad \ker \pi_{[n-j]} * \chi^{(j),+}. \tag{42}$$

PROOF. Since $\pi_{[n],n,n-j}$ is a map of $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-modules, one needs only show it is an $\mathbb{R}$-linear isomorphism. We prove this by a dimension-counting argument, beginning with a chain of equalities and inequalities justified below:

$$\begin{aligned}
n! &\overset{(1)}{=} \sum_{j=0}^{n} \binom{n}{j} d_{n-j} \\
&\overset{(2)}{=} \sum_{j=0}^{n} \sum_{J \in \binom{[n]}{j}} \dim_{\mathbb{R}} \ker(\pi_{[n]\setminus J}) \\
&\overset{(3)}{\leq} \sum_{j=0}^{n} \dim_{\mathbb{R}} \operatorname{im}(\pi_{[n],n,n-j}) \cap \ker(\pi_{[n],n,n-j}) \\
&\overset{(4)}{=} \sum_{j=0}^{n} \dim_{\mathbb{R}} \ker(\pi_{[n],n-j,n-j-1} \circ \pi_{[n],n,n-j}) / \ker(\pi_{[n],n,n-j}) \\
&\overset{(5)}{=} \sum_{j=0}^{n} \dim_{\mathbb{R}} \underbrace{\ker(\pi_{[n],n,n-j-1}) / \ker(\pi_{[n],n,n-j})}_{F_{n,j}:=} \\
&\overset{(6)}{=} n!.
\end{aligned}$$

Equality (1) is 36g. Equality (2) comes from Corollary VI.7.3. Inequality (3) comes from the inclusion

$$\bigoplus_{J \in \binom{n}{j}} \ker(\pi_{[n]\setminus J}) \quad \subseteq \quad \operatorname{im}(\pi_{[n],n,n-j}) \cap \ker(\pi_{[n],n,n-j}) \tag{43}$$

implied by Lemma VI.8.3. Equality (4) comes from Proposition VI.8.4 applied to the composition

$$\mathbb{R}\mathfrak{S}_n \xrightarrow{f:=\pi_{[n],n,n-j}} \bigoplus_{J \in \binom{[n]}{j}} \mathbb{R}\mathfrak{S}_{[n]\setminus J} \xrightarrow{g:=\pi_{[n],n-j,n-j-1}} \bigoplus_{K \in \binom{[n]}{j-1}} \mathbb{R}\mathfrak{S}_{[n]\setminus K}.$$

Equality (5) comes from Proposition VI.8.1. Equality (6) comes from telescoping the dimensions of the factors $F_{n,j}$ in the filtration 28 of $\mathbb{R}\mathfrak{S}_n$.

One concludes that the inequality (3) is actually an equality. Hence the set inclusion 43 must actually be an equality of sets. Since Equality (4) was mediated by the map $f := \pi_{[n],n,n-j}$, the desired conclusion follows. □

Combining Theorem VI.8.5 with Corollary VI.7.3 and 36g immediately implies the following.

COROLLARY 8.6. *The factor* $F_{n,j}$ *in the filtration* (28) *has dimension equal to the number* $\binom{n}{j} d_{n-j}$ *of permutations with exactly* j *fixed points. Furthermore, its* $\mathbb{Z}_2$*-isotypic components have dimensions* $\binom{n}{j} d^{+}_{n-j}$, $\binom{n}{j} d^{-}_{n-j}$ *equal to the number of even, odd permutations with exactly* j *fixed points.*

9. Desarrangements and the random-to-top eigenvalue of a tableaux

There is a well-known $\mathbb{R}\mathfrak{S}_n$-module decomposition of the group algebra

$$\mathbb{R}\mathfrak{S}_n \cong \bigoplus_Q \chi^{\text{shape}(Q)}$$

where Q runs over all *standard Young tableaux* of size n. The next two sections refine this in two ways. The current section first reviews Désarménien and Wachs [**22**] notion of desarrangements, as well as some of the unpublished work [**50**]. In particular, it is shown how to assign to each tableau Q an integer $\text{eig}(Q)$ such that Q contributes the $\mathfrak{S}_n$-irreducible $\chi^{\text{shape}(Q)}$ to the kernel filtration factor $F_{n,\text{eig}(Q)}$. In the next section, we refine this further to give the $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-structure, defining a sign $\epsilon(Q)$ such that Q contributes the $\mathfrak{S}_n \times \mathbb{Z}_2$-irreducible $\chi^{\text{shape}(Q),\epsilon(Q)}$ to $F_{n,\text{eig}(Q)}$.

We begin by recalling some well-known definitions about ascents/descents in permutations and tableaux.

Definition 9.1. For a permutation w in $\mathfrak{S}_n$, say that i in $\{1, 2, \ldots, n-1\}$ is an *ascent* (resp. *descent*) of w if $w(i) < w(i+1)$ (resp. $w(i) > w(i+1)$). We will furthermore decree that n is always an ascent of any w in $\mathfrak{S}_n$.

For a standard Young tableau Q of size n, say that i in $\{1, 2, \ldots, n-1\}$ is an *ascent* (resp. *descent*) of Q if $i+1$ appears weakly to the north and east (resp. south) of i, using English notation for tableaux. Again we decree that n is always an ascent of Q for any Q of size n.

Let SYT_n denote the set of all standard Young tableaux of size n, and say that such a tableau Q has $\text{size}(Q) := n$.

Recall that the *Robinson-Schensted algorithm* is a bijection

$$\mathfrak{S}_n \longrightarrow \{(P, Q) \in \text{SYT}_n^2 : \text{shape}(P) = \text{shape}(Q)\}.$$

This algorithm has many wonderful properties, and relations to Schützenberger's *jeu-de-taquin*. We refer the reader to Sagan's book [**54**, Chapter 3] for background on some of these. For a $w \in \mathfrak{S}_n$ we denote by $P(w)$ and $Q(w)$ the standard Young tableaux such that under the Robinson-Schensted algorithm we have $w \mapsto (P(w), Q(w))$. Among the wonderful properties mentioned above is the fact that when $w \mapsto (P(w), Q(w))$, then w shares the same set of ascents and descents as $Q(w)$.

Proposition 9.2. *For Q in SYT_n (resp. w in $\mathfrak{S}_n$), there exists a unique value j lying in $\{0, 1, 2, \ldots, n-2, n\}$ such that*

- *$1, 2, \ldots, j-1$ are ascents in Q, and*
- *if Q has at least one descent then the first ascent among $j+1, j+2, \ldots, n$ in Q occurs at a value $j+k$ with k even.*

The value of k is unique provided that Q has at least one descent.

We denote these unique values by $\text{eig}(Q) := j$ and $k(Q) := k$; and we set $k(Q) := 0$ if Q has no descents.

Proof. We give the proof in the case of tableaux; the case for permutations is similar, and also follows from the property of the Robinson-Schensted algorithm mentioned above.

If Q is empty, so that $n = 0$, then one is forced to take $(j, k) = (0, 0)$.

If Q is non-empty then it contains a unique maximal subtableaux of the form

$$\begin{array}{|c|c|c|c|c|}
\hline
1 & 2 & \multicolumn{1}{c}{\cdots} & \ell{-}1 & \ell \\
\hline
\ell{+}1 & \multicolumn{4}{c}{} \\
\cline{1-1}
\ell{+}2 & \multicolumn{4}{c}{} \\
\cline{1-1}
\multicolumn{1}{c}{\vdots} & \multicolumn{4}{c}{} \\
\cline{1-1}
\ell{+}m & \multicolumn{4}{c}{} \\
\cline{1-1}
\end{array}$$

for which $\ell + m$ is an ascent. Here $\ell \geq 1$ and one allows the possibility that $m = 0$ or that $\ell + m = n$, the size of Q. Then one is forced to choose

$$(j, k) = \begin{cases} (\ell, m) & \text{if } m \text{ is even,} \\ (\ell - 1, m + 1) & \text{if } m \text{ is odd.} \end{cases} \qquad \square$$

DEFINITION 9.3. Say w is a *desarrangement* if $\operatorname{eig}(w) = 0$.
Say Q is a *desarrangement tableau* if $\operatorname{eig}(Q) = 0$.

We will use the notion of *jeu-de-taquin slides* on skew tableaux; again see [**54**, Chapter 3]. Given a standard Young tableau Q, its *(Schützenberger) demotion* will be the tableau $\operatorname{demote}(Q)$ obtained by replacing the entry 1 in its northwest corner with a jeu-de-taquin hole, doing jeu-de-taquin to slide the hole out, and subtracting 1 from all of the entries in the resulting tableau.

PROPOSITION 9.4. *For* $1 \leq j \leq n - 1$ *the map* $Q \longmapsto (\operatorname{demote}^j(Q), \operatorname{shape}(Q))$ *gives a bijection*

$$\{Q \in \mathrm{SYT}_n : \operatorname{eig}(Q) = j\} \longrightarrow \{(\hat{Q}, \mu)\}$$

in which on the right side, $\hat{Q}$ *is a desarrangement tableaux of size* $n - j$*, and* μ *is a partition of* n*, such that the skew shape* $\mu/\operatorname{shape}(Q)$ *is a horizontal* j*-strip.*

PROOF. We describe the inverse map. Start with $(\hat{Q}, \mu)$ and do outward jeu-de-taquin slides on $\hat{Q}$ into the cells of $\mu/\operatorname{shape}(Q)$, from left-to-right. Then add j to all of the entries in the result. Properties of jeu-de-taquin [**54**, Exercise 3.12.6] imply that the sliding will have created j empty cells in the first row, which one now fills with the values $1, 2, \ldots, j$. The resulting tableau Q will have $\operatorname{eig}(Q) = j$. $\square$

The assertion of the next theorem appears for $j = 0$ in [**51**, Proposition 2.3], and for $j > 0$ in the unpublished work [**50**].

THEOREM 9.5. *As* $\mathbb{R}\mathfrak{S}_n$*-modules, the* j^{th} *filtration factor* $F_{n,j}$ *from* (28) *has irreducible decomposition*

$$\bigoplus_{Q \in \mathrm{SYT}_n : \operatorname{eig}(Q) = j} \chi^{\operatorname{shape}(Q)}.$$

PROOF. Temporarily denote by $U_{n,j}$ the direct sum appearing above. We first prove it is isomorphic to $F_{n,j}$ for $j = 0$, so that Q runs over desarrangement tableaux in the direct sum, by checking that $U_{n,0}$ satisfies recurrence (40); cf. [**51**, proof of Proposition 2.3].

When n is even, we must show that

$$U_{n,0} = U_{n-1,0} * \chi^{(1)} + \chi^{(1^n)}.$$

The usual Pieri formula 34 shows that the term $U_{n-1,0} * \chi^{(1)}$ on the right give rises to all desarrangement tableau of size n which are obtained by adding one cell labelled n from a desarrangement tableau of size $n-1$. The only desarrangement tableau of size n it will not produce is the desarrangement tableau

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline n-1 \\ \hline n \\ \hline \end{array}$$

that appears on the left, accounted for by the term $\chi^{(1^n)}$ on the right.

When n is odd, we must show that

$$U_{n,0} + \chi^{(1^n)} = U_{n-1,0} * \chi^{(1)}.$$

Again the term $U_{n-1,0} * \chi^{(1)}$ on the right give rises to all desarrangement tableaux of size n which are obtained by adding one cell labelled n from a desarrangement tableau of size $n-1$. Because n is odd, this accounts for all of the terms in $U_{n,0}$ on the left. However, it also produces one extra *non-desarrangement tableau*, namely

$$\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \multicolumn{1}{c}{\vdots} \\ \hline n-1 \\ \hline n \\ \hline \end{array}$$

accounted for by $\chi^{(1^n)}$ on the left.

For $j \geq 1$, it suffices to check that $U_{n,j}$ satisfies the relation

$$U_{n,j} = U_{n-j,0} * \chi^{(n-j)}$$

which one knows is satisfied by $F_{n,j}$ by forgetting the $\mathbb{Z}_2$-action in (42). This follows from the Pieri formula 34 and Proposition VI.9.4. $\square$

10. Shaving tableaux

The goal here is to define a sign $\epsilon(Q) = \pm 1$ so that a standard Young tableau Q having $\text{eig}(Q) = j$ contributes the irreducible $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-module $\chi^{\text{shape}(Q),\epsilon(Q)}$ to the filtration factor $F_{n,j}$. The idea is to define the sign first for a very special class of desarrangement tableaux, which will form the base case when extending the sign inductively to all desarrangement tableaux, and finally extend the sign using the demotion operator to all tableaux.

Definition 10.1 (Shaven desarrangement tableaux). Define the following three kinds of *shaven* desarrangement tableaux in Q in SYT_n.

(1) When $n = 0$, define the empty tableau $\varnothing$ to be shaven.
(2) When n is even and at least 4, define the following desarrangement tableau $Q_-^{(n)}$ to be shaven:

$$Q_-^{(n)} := \begin{array}{|c|c|} \hline 1 & n-1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \hline n-2 & \multicolumn{1}{c}{} \\ \cline{1-1} n & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

(3) When n is odd and at least 3, define the following desarrangement tableau $Q_+^{(n)}$ to be shaven:

$$Q_+^{(n)} := \begin{array}{|c|c|} \hline 1 & n \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \hline n-2 & \multicolumn{1}{c}{} \\ \cline{1-1} n-1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Call any other desarrangement tableau not in one of these three special forms $\varnothing, Q_+^{(n)}, Q_-^{(n)}$ an *unshaven* desarrangement tableau.

Unshaven desarrangements can be "shaved" down to shaven desarrangements due to the following proposition.

Proposition 10.2. *Any unshaven desarrangement tableau Q has size at least 2, and the tableau $\hat{Q}$ obtained from Q by removing the largest two entries $\{n-1, n\}$ is again a desarrangement tableau.*

Proof. An unshaven desarrangement tableau Q in SYT_n must be non-empty, so that $n \geq 1$. Since there are no desarrangements of size 1, one must have $n \geq 2$. If the tableau $\hat{Q}$ obtained by removing its two largest entries $\{n-1, n\}$ is *not* a desarrangement tableau, then $n-1$ must be the first ascent of Q, and be even.

This forces Q to be the shaven desarrangement tableau $Q_+^{(n)}$ for some odd $n \geq 3$, a contradiction. $\square$

DEFINITION 10.3. Define the sign $\epsilon(Q)$ for Q in SYT_n inductively as follows.

- In the base case, Q is one of the three kinds of shaven desarrangement tableaux from Definition VI.10.1, for which we decree

$$\epsilon(Q) := \begin{cases} +1 & \text{if } Q = \varnothing \\ +1 & \text{if } Q = Q_+^{(n)} \\ -1 & \text{if } Q = Q_-^{(n)} \end{cases}$$

- If Q is an unshaven desarrangement tableaux, let $\hat{Q}$ be the tableau obtained from Q by removing the largest two entries $\{n-1, n\}$ (so that $\hat{Q}$ is again a desarrangement tableau by Proposition VI.10.2) and define inductively

$$\epsilon(Q) := \begin{cases} +\epsilon(\hat{Q}) & \text{if } \{n-1, n\} \text{ form an ascent in } Q \\ -\epsilon(\hat{Q}) & \text{if } \{n-1, n\} \text{ form a descent in } Q \end{cases}$$

- If Q is not a desarrangement tableaux, so $j := \mathrm{eig}(Q) > 0$, define inductively

$$\epsilon(Q) := \epsilon(\mathrm{demote}^j(Q)).$$

EXAMPLE 10.4. We compute the sign $\epsilon(Q)$ for this tableau Q in SYT_{15}: One can check that $j := \mathrm{eig}(Q) = 3$, so Q has the same sign as the desarrangement tableaux obtained by applying the demotion operator 3 times From this unshaven desarrangement tableau $\mathrm{demote}^j(Q)$, one can

first "shave" the descent pair $\{11, 12\}$,
then the descent pair $\{9, 10\}$,
then the ascent pair $\{7, 8\}$,

leaving as a result the shaven desarrangement tableau Since there were two descent pairs shaved, the original tableau Q has sign

$$\epsilon(Q) = (-1)^2 \cdot \epsilon\left(Q_-^{(6)}\right) = -1.$$

THEOREM 10.5. *As an $\mathbb{R}[\mathfrak{S}_n \times \mathbb{Z}_2]$-module the j^{th} filtration factor $F_{n,j}$ from* (28) *has irreducible decomposition*

$$\bigoplus_{\substack{Q \in \mathrm{SYT}_n: \\ \mathrm{eig}(Q) = j}} \chi^{\mathrm{shape}(Q), \epsilon(Q)}.$$

PROOF. We follow roughly the same plan as in the proof of Theorem VI.9.5. Temporarily denote by $U_{n,j}$ the direct sum in the theorem. Assume for the moment that we have shown $F_{n,0}$ is isomorphic to $U_{n,0}$. Then for $j > 0$, it suffices to check that $U_{n,j}$ satisfies the relation $U_{n,j} = U_{n-j,0} * \chi^{(j)}$ from (42). This follows from the $\mathbb{Z}_2$-Pieri formula (35) and the fact that demotion respects signs.

Thus it only remains to show $U_{n,0} \cong F_{n,0}$ for $j = 0$. In other words, we wish to show that the sum of $\chi^{\mathrm{shape}(Q),\epsilon(Q)}$ over all desarrangement tableaux Q satisfies the recurrence (41).

When n is odd and at least 3, we must show that

$$U_{n,0} = U_{n-2,0} * \left(\chi^{(2),+} + \chi^{(1^2),-}\right) + \chi^{(2,1^{n-2}),+}.$$

This follows because most of the desarrangements Q in SYT_n which appear on the left are unshaven, with $\{n-1, n\}$ forming either an ascent or descent. The $\mathbb{Z}_2$-Pieri formula (35) shows that these terms are counted with appropriate sign $\epsilon(Q)$ by a term of $U_{n-2,0} * \chi^{(2),+}$ or $U_{n-2,0} * \chi^{(1^2),-}$ on the right. The only term on the left which is shaven is $Q_+^{(n)}$, and is accounted for by the extra summand $\chi^{(2,1^{n-2}),+}$ on the right.

When n is even, we must show that

$$U_{n,0} + \chi^{(2,1^{n-2}),+} = U_{n-2,0} * \left(\chi^{(2),+} + \chi^{(1^2),-}\right)$$

This again follows because most of the desarrangements Q in SYT_n which appear on the left are unshaven, with $\{n-1, n\}$ forming either an ascent or descent, in which case the $\mathbb{Z}_2$-Pieri formula (35) shows that they are counted with appropriate sign $\epsilon(Q)$ by a term of $U_{n-2,0} * \chi^{(2),+}$ or $U_{n-2,0} * \chi^{(1^2),-}$ on the right. But there are two other terms $\chi^{(2,1^{n-2}),+} + \chi^{(2,1^{n-2}),-}$ on the right, which will be generated from the term inside $U_{n-2,0}$ for the desarrangement tableaux having a single column of length $n-2$. Correspondingly on the left, there are two other terms $\chi^{(2,1^{n-2}),-} + \chi^{(2,1^{n-2}),+}$, the first coming from the unique *shaven* desarrangement of size n, namely $Q_-^{(n)}$, and the second coming from the extra summand on the left. □

11. Fixing a small value of k and letting n grow.

With the help of `Sage` [**68**] we computed the decomposition of the $\mathfrak{S}_n$-modules afforded by the eigenspaces of the operators $\nu_{(k,1^{n-k})}$. We present this data in Figure 4 through Figure 6, as follows:

- to enhance the presentation, every zero has been replaced by a dot;
- each row of the table corresponds to a subspace E in a decomposition of $\mathbb{R}\mathfrak{S}_n$ into $\mathfrak{S}_n$-modules,
- the horizontal lines partition the rows into blocks of rows whose corresponding subspaces E contribute to $F_{n,j}$ for a fixed j, for $j = n, \ldots, 1$ reading from top to bottom,
- the entry in the column indexed by $\nu_{(k,1^{n-k})}$ is the eigenvalue of $\nu_{(k,1^{n-k})}$ on E;
- the entry in the column indexed by w_0 is the eigenvalue for the $\mathbb{Z}_2$-action on E;
- for $n \leq 5$, the entry in the column indexed by the $\mathfrak{S}_n$-irreducible χ^λ is the tableau from Theorem VI.10.5 that contributes χ^λ to the $\mathfrak{S}_n$-module afforded by E, whereas for $n > 5$ the corresponding entry is just the multiplicity of χ^λ in E.

For $n \leq 5$ the quantities $\mathrm{eig}(Q)$, $\epsilon(Q)$ and $\mathrm{shape}(Q)$ determine the placements of the tableaux in the tables, with the exception of the two tableaux marked by † in Figure 5 (they share the same eig- and ϵ-statistic). For $n > 5$ there is much more ambiguity.

We now highlight some patterns that jump out from this data.

For a fixed value of k, as n grows large, most of $\mathbb{R}\mathfrak{S}_n$ will be swallowed up in the 0-eigenspace (kernel) of $\nu_{(k,1^{n-k})}$ according to Example IV.6.3. For example, it shows that the non-zero eigenspaces $\operatorname{im} \nu_{(k,1^{n-k})}$ comprise a representation of the form $\psi * \mathbf{1}_{n-k}$ for some $\mathfrak{S}_k$-representation ψ. Hence the Pieri formula shows that

$\mathfrak{S}_n$-*module* $V_{n,j}$	$\mathfrak{S}_n$-*irreducibles* χ^λ	*eigenvalue of* $\nu_{(k,1^{n-k})}$ *on* χ^λ		
		$\nu_{(1^n)}$	$\nu_{(2,1^{n-2})}$	$\nu_{(3,1^{n-3})}$
$V_{n,n}$	$\chi^{(n)}$	$\binom{n}{1}(n-1)!$	$\binom{n}{2}(n-2)!$	$\binom{n}{3}(n-3)!$
$V_{n,n-2}$	$\chi^{(n-1,1)}$	0	$\frac{(n+1)!}{3!}$	$\frac{(n+1)!}{4!}$
	$\chi^{(n-2,1,1)}$	0	$\frac{n!}{3!}$	$\binom{n}{2}\frac{(n-1)!}{3!}$
$V_{n,n-3}$	$\chi^{(n-1,1)}$	0	0	$\frac{(n+2)!}{5!}$
	$\chi^{(n-2,2)}$	0	0	$\frac{(n+1)!}{30}$
	$\chi^{(n-2,1,1)}$	0	0	$\frac{(n+1)!}{60}$
	$\chi^{(n-3,2,1)}$	0	0	$\frac{n!}{15}$

FIGURE 3. (Conjectural) decomposition of the non-zero eigenspaces of $\nu_{(1^n)}$, $\nu_{(2,1^{n-2})}$ and $\nu_{(3,1^{n-3})}$ into irreducible $\mathbb{R}\mathfrak{S}_n$-modules together with their eigenvalues; c.f. Conjecture VI.11.1.

any irreducible χ^λ that occurs within it must have $n - \lambda_1 \leq 2k$, that is, most of its cells will live in the first part λ_1 when n grows large.

For $k=1,2,3$, one can certainly easily write down exactly which $\mathfrak{S}_n$-irreducibles occur outside the kernel of $\nu_{(k,1^{n-k})}$, segregated by the subspaces $V_{n,j}$ from 29 in which they will occur. However, even for $k = 2, 3$ it is already not immediately obvious how they will segregate further into simultaneous eigenspaces, nor is it obvious what will be their corresponding eigenvalues as a function of n. The data suggests the following conjectural table summarizing the story for $k = 1, 2, 3$. It is correct for $k = 1$, and probably not so hard to prove for $k = 2, 3$ by brute force (i.e. write down the eigenvectors explicitly), but we have not tried.

CONJECTURE 11.1. *For $\nu_{(1^n)}$, $\nu_{(2,1^{n-2})}$ and $\nu_{(3,1^{n-3})}$, all of the non-zero eigenspaces can be simultaneously described by subspaces carrying irreducible $\mathbb{R}\mathfrak{S}_n$-modules described in the second column of* Figure 3*, and having eigenvalues as shown in the remaining columns.*

The form of the eigenvalues in this last table suggests the following somewhat vague stability conjecture, in the spirit of the *representation stability* recently discussed by Church and Farb [**19**].

CONJECTURE 11.2. *There exists an infinite sequence of partitions $\lambda^1, \lambda^2, \ldots$ and positive integers $j_0^1, j_0^2, \ldots$ with the following property. For each positive integer n, there exist a positive integer $\tau(n)$ and subspaces $E_1^{(n)}, E_2^{(n)}, \ldots E_{\tau(n)}^{(n)} \subseteq \mathbb{R}\mathfrak{S}_n$ such that*

- *$E_i^{(n)}$ carries the $\mathfrak{S}_n$-irreducible indexed by the partition*

$$\lambda^i + \left(n - |\lambda^i|, 0, 0, \ldots, 0\right)$$

- $E_i^{(n)}$ *is a simultaneous eigenspace for the operators* $\nu_{(1^n)}$, $\nu_{(2,1^{n-2})}$, $\ldots$, $\nu_{(n-1,1)}$ *with eigenvalue for* $\nu_{(j,1^{n-j})}$ *described by*

$$\begin{cases} 0 & \text{if } 1 \leq j < j_0^i \\ f(E_i^{(n)}, j) \neq 0 & \text{if } j_0^i \leq j \leq n \end{cases}$$

where $f(E_i^{(n)}, j)$ *are functions for which*

$$\frac{f(E_i^{(n)}, j)}{f(E_i^{(n-1)}, j)}$$

is a rational function of n *of total degree* 1

- $\left(\bigoplus_{i=1}^{\tau(n)} E_i^{(n)}\right)^{\perp} \subseteq \mathbb{R}\mathfrak{S}_n$ *lies in the common kernel of* $\nu_{(1^n)}$, $\nu_{(2,1^{n-2})}$, $\ldots$, $\nu_{(n-1,1)}$.

12. The representation $\chi^{(n-1,1)}$

We next focus on the $\chi^{(n-1,1)}$-isotypic component for the eigenspaces of $\nu_{(k,1^{n-k})}$, reasoning using our block-diagonalization 29. This allows us to piggyback on computations of Uyemura-Reyes for the case $k = n-1$.

PROPOSITION 12.1. *For* $j = 0, 1, 2, \ldots, n-2$, *the* $\chi^{(n-1,1)}$*-isotypic component of* $\mathbb{R}\mathfrak{S}_n$ *intersects the summand* $V_{n,j}$ *in 29 in a single copy* $V_{n,j}^{(n-1,1)}$ *of the irreducible* $\chi^{(n-1,1)}$.

Consequently, each such intersection $V_{n,j}^{(n-1,1)}$ *for* $j = 0, 1, 2, \ldots, n-2$ *lies within a single eigenspace for any operator* $\nu_{(k,1^{n-k})}$, *and carries an integer eigenvalue for any of these operators.*

PROOF. There are exactly $n-1$ standard Young tableaux Q of shape $(n-1, 1)$, determined completely by their unique entry m in $\{2, 3, \ldots, n\}$ lying in the second row of the tableau. One can check that such a Q has $j := \mathrm{eig}(Q) = m-2$, and hence this accounts for exactly one copy of $\chi^{(n-1,1)}$ within $V_{n,j}$ for each $j = 0, 1, 2, \ldots, n-2$, proving the first assertion.

For the second assertion, note that this multiplicity-freeness allows one to apply Proposition I.3.1 to each of the subspaces $U = V_{n,j}^{(n-1,1)}$. $\square$

Uyemura-Reyes provided a complete set of $\chi^{(n-1,1)}$-isotypic eigenspaces for $\nu_{(n-1,1)}$ using evaluations of *discrete Chebyshev polynomials*, and computed their eigenvalues for $\nu_{(n-1,1)}$ explicitly using the Fourier transform which also yields an alternative proof of the reduction in §II.7 [**76**, §5.2.1]. Because these eigenvalues turned out to all be distinct, this implies that the entire family of operators $\nu_{(k,1^{n-k})}$, for $k = 1, 2, \ldots, n-1$, when restricted to the $\chi^{(n-1,1)}$-isotypic component of $\mathbb{R}\mathfrak{S}_n$, become polynomials in the single operator $\nu_{(n-1,1)}$. Hence they all share these same $\chi^{(n-1,1)}$-isotypic eigenspaces which he constructed.

The following conjecture about their common eigenvalues on these spaces is consistent with Uyemura-Reyes's eigenvalue calculation for $k = n-1$, and with our data up through $n = 9$, but we have not tried to prove it.

$\nu_{(1^2)}$	w_0	χ^2	χ^{11}
2	1	$\ytableaushort{12}$	·
·	-1	·	$\ytableaushort{1,2}$

$\nu_{(1^3)}$	$\nu_{(2,1)}$	w_0	χ^3	χ^{21}	χ^{111}
6	9	1	$\ytableaushort{123}$	·	·
·	4	-1	·	$\ytableaushort{12,3}$	·
·	1	-1	·	·	$\ytableaushort{1,2,3}$
·	·	1	·	$\ytableaushort{13,2}$	·

$\nu_{(1^4)}$	$\nu_{(2,1^2)}$	$\nu_{(3,1)}$	w_0	χ^4	χ^{31}	χ^{211}	χ^{22}	χ^{1111}
24	72	16	1	$\ytableaushort{1234}$	·	·	·	·
·	20	10	-1	·	$\ytableaushort{123,4}$	·	·	·
·	4	6	-1	·	·	$\ytableaushort{12,3,4}$	·	·
·	·	6	1	·	$\ytableaushort{124,3}$	·	·	·
·	·	4	1	·	·	·	$\ytableaushort{12,34}$	·
·	·	2	1	·	·	$\ytableaushort{14,2,3}$	·	·
·	·	·	1	·	·	·	$\ytableaushort{13,24}$	$\ytableaushort{1,2,3,4}$
·	·	·	-1	·	$\ytableaushort{134,2}$	$\ytableaushort{13,2,4}$	·	·

FIGURE 4. $\mathfrak{S}_n$-module decomposition, for $2 \le n \le 4$, of the eigenspaces of $\nu_{(k,1^{n-k})}$.

CONJECTURE 12.2. *The eigenvalues of* $\nu_{(k,1^{n-k})}$ *on the* $\chi^{(n-1,1)}$*-isotypic component of* $\mathbb{R}\mathfrak{S}_n$ *are*

$$(n-k)!\binom{n-r-1}{k-r-1}\binom{n+r}{k+r}$$

for $r = 1, 2, \ldots, n-1$.

$\nu_{(1^5)}$	$\nu_{(2,1^3)}$	$\nu_{(3,1^2)}$	$\nu_{(4,1)}$	w_0	χ^5	χ^{41}	χ^{311}	χ^{32}	χ^{2111}	χ^{221}	χ^{11111}
120	600	200	25	1	1 2 3 4 5	·	·	·	·	·	·
·	120	90	18	−1	·	1 2 3 4 / 5	·	·	·	·	·
·	20	40	13	−1	·	·	1 2 3 / 4 / 5	·	·	·	·
·	·	42	14	1	·	1 2 3 5 / 4	·	·	·	·	·
·	·	24	11	1	·	·	·	1 2 3 / 4 5	·	·	·
·	·	12	9	1	·	·	1 2 5 / 3 / 4	·	·	·	·
·	·	8	7	1	·	·	·	·	·	1 2 / 3 5 / 4	·
·	·	·	7	1	·	·	·	1 2 4 / 3 5	·	·	·
·	·	·	6	1	·	·	·	·	1 2 / 3 / 4 / 5	·	·
·	·	·	3	1	·	·	·	·	·	1 4 / 2 5 / 3	·
·	·	·	1	1	·	·	·	·	·	·	1 / 2 / 3 / 4 / 5
·	·	·	8	−1	·	1 2 4 5 / 3	·	·	·	·	·
·	·	·	7	−1	·	·	1 4 5 / 2 / 3 †	·	·	·	·
·	·	·	5	−1	·	·	·	1 2 5 / 3 4	·	1 2 / 3 4 / 5	·
·	·	·	3	−1	·	·	1 2 4 / 3 / 5 †	·	·	·	·
·	·	·	2	−1	·	·	·	·	1 4 / 2 / 3 / 5	·	·
·	·	·	·	1	·	1 3 4 5 / 2	1 3 5 / 2 / 4	1 3 5 / 2 4	1 5 / 2 / 3 / 4	1 3 / 2 5 / 4	·
·	·	·	·	−1	·	·	1 3 4 / 2 / 5	1 3 4 / 2 5	1 3 / 2 / 4 / 5	1 3 / 2 4 / 5	·

FIGURE 5. The $\mathfrak{S}_5$-module decomposition for the operators $\nu_{(k,1^{n-k})}$.

CHAPTER VII

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$\mathfrak{S}_n$-module decomposition of $\nu_{(k,1^{n-k})}$

We include here the $\mathfrak{S}_n$-module decomposition of the simultaneous eigenspaces for the operators $\nu_{(1^n)}$, $\nu_{(2,1^{n-2})}$, $\ldots \nu_{(n-1,1)}$ for $6 \leq n \leq 8$. See §VI.10 for an explanation of the presentation of this data; and Figure 4 and Figure 5 for the decomposition for $2 \leq n \leq 5$.

$\nu_{(1^6)}$	$\nu_{(2,1^4)}$	$\nu_{(3,1^3)}$	$\nu_{(4,1^2)}$	$\nu_{(5,1)}$	w_0	χ^6	χ^{51}	χ^{411}	χ^{42}	χ^{3111}	χ^{321}	χ^{21111}	χ^{33}	χ^{2211}	χ^{222}	χ^{111111}
4320	5400	2400	450	36	1	1	·	·	·	·	·	·	·	·	·	·
·	840	840	252	28	-1	·	1	·	·	·	·	·	·	·	·	·
·	120	300	144	22	-1	·	·	1	·	·	·	·	·	·	·	·
·	·	336	168	24	1	·	1	·	·	·	·	·	·	·	·	·
·	·	168	112	20	1	·	·	·	1	·	·	·	·	·	·	·
·	·	84	84	18	1	·	·	1	·	·	·	·	·	·	·	·
·	·	48	57	15	1	·	·	·	·	·	1	·	·	·	·	·
·	·	·	56	16	1	·	·	·	1	·	·	·	·	·	·	·
·	·	·	42	14	1	·	·	·	·	1	·	·	·	·	·	·
·	·	·	21	11	1	·	·	·	·	·	1	·	·	·	·	·
·	·	·	12	8	1	·	·	·	·	·	·	·	·	·	1	·
·	·	·	6	8	1	·	·	·	·	·	·	1	·	·	·	·
·	·	·	72	18	-1	·	1	·	·	·	·	·	·	·	·	·
·	·	·	56	16	-1	·	·	1	·	·	·	·	·	·	·	·
·	·	·	40	14	-1	·	·	·	1	·	·	·	·	·	·	·
·	·	·	35	13	-1	·	·	·	·	·	1	·	·	·	·	·
·	·	·	30	12	-1	·	·	·	·	·	·	·	1	·	·	·
·	·	·	24	12	-1	·	·	1	·	·	·	·	·	·	·	·
·	·	·	15	9	-1	·	·	·	·	·	1	·	·	·	·	·
·	·	·	14	10	-1	·	·	·	·	1	·	·	·	·	·	·
·	·	·	10	8	-1	·	·	·	·	·	·	·	·	1	·	·
·	·	·	·	10	1	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	9	1	·	·	1	1	·	·	·	·	·	·	·
·	·	·	·	8	1	·	·	·	·	1	1	·	·	·	·	·
·	·	·	·	7	1	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	6	1	·	·	·	1	·	1	·	·	1	·	·
·	·	·	·	5	1	·	·	·	·	·	·	·	·	·	1	·
·	·	·	·	4	1	·	·	1	·	·	1	·	·	·	·	·
·	·	·	·	3	1	·	·	·	·	1	·	·	·	1	·	·
·	·	·	·	2	1	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	9	-1	·	·	1	1	·	·	·	·	·	·	·
·	·	·	·	8	-1	·	·	·	·	1	1	·	·	·	·	·
·	·	·	·	7	-1	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	6	-1	·	·	·	·	·	1	·	·	1	·	·
·	·	·	·	5	-1	·	·	·	·	·	·	·	·	·	1	·
·	·	·	·	4	-1	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	3	-1	·	·	·	·	1	·	·	·	1	·	·
·	·	·	·	2	-1	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	·	1	·	·	1	2	3	3	1	·	1	2	·
·	·	·	·	·	-1	·	1	2	1	1	3	1	2	3	·	1

FIGURE 1. The $\mathfrak{S}_6$-module decomposition for the operators $\nu_{(k,1^{n-k})}$.

$\nu_{(1^7)}$	$\nu_{(2,1^5)}$	$\nu_{(3,1^4)}$	$\nu_{(4,1^3)}$	$\nu_{(5,1^2)}$	$\nu_{(6,1)}$	w_0	χ^{7}	χ^{61}	χ^{511}	χ^{52}	χ^{4111}	χ^{421}	χ^{31111}	χ^{43}	χ^{3211}	χ^{322}	χ^{211111}	χ^{331}	χ^{22111}	χ^{2221}	$\chi^{1111111}$
35280	52920	29400	7350	882	49	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	6720	8400	3360	560	40	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	840	2520	1596	364	33	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	3024	2016	432	36	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	1344	1176	312	31	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	672	840	264	29	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	336	504	192	25	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	504	216	27	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	336	168	24	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	168	120	21	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	84	76	17	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	42	70	17	1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	720	270	30	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	504	216	27	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	360	180	25	-1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	280	152	23	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	240	138	22	-1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
·	·	·	216	144	23	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	120	96	19	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	112	104	20	-1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	90	78	17	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
·	·	·	70	74	17	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	110	22	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	90	20	1	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	72	18	1	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·
·	·	·	·	63	17	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
·	·	·	·	60	17	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	54	16	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	48	15	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	42	14	1	·	·	·	·	·	·	·	1	·	·	·	1	·	·	·
·	·	·	·	40	15	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	40	14	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	36	14	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	28	12	1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	27	13	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	24	12	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	24	11	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	20	10	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	18	10	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	16	11	1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	·	15	9	1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·
·	·	·	·	12	9	1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·

FIGURE 2. $\mathfrak{S}_7$-module decomposition for the operators $\nu_{(k,1^{n-k})}$ (continues in Figure 3)

$\nu_{(1^7)}$	$\nu_{(2,1^5)}$	$\nu_{(3,1^4)}$	$\nu_{(4,1^3)}$	$\nu_{(5,1^2)}$	$\nu_{(6,1)}$	w_0	χ^7	χ^{61}	χ^{511}	χ^{52}	χ^{4111}	χ^{421}	χ^{31111}	χ^{43}	χ^{3211}	χ^{322}	χ^{211111}	χ^{331}	χ^{22111}	χ^{2221}	$\chi^{1111111}$
·	·	·	·	90	20	-1	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	72	18	-1	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·
·	·	·	·	63	17	-1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
·	·	·	·	54	16	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	48	15	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	42	14	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	40	14	-1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	36	14	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	28	12	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	27	13	-1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	24	12	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	20	10	-1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	18	10	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	16	11	-1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	·	15	9	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·
·	·	·	·	12	9	-1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·
·	·	·	·	·	11	1	·	·	1	2	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	10	1	·	·	·	·	3	3	·	·	·	·	·	·	·	·	·
·	·	·	·	·	9	1	·	·	·	·	·	·	1	·	1	2	·	·	·	·	·
·	·	·	·	·	8	1	·	·	·	·	·	·	·	2	·	·	·	3	·	·	·
·	·	·	·	·	7	1	·	·	·	·	·	1	·	·	3	·	·	·	1	·	·
·	·	·	·	·	6	1	·	·	·	·	·	·	·	·	·	3	·	·	·	1	·
·	·	·	·	·	5	1	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·
·	·	·	·	·	4	1	·	·	·	·	1	·	·	·	3	·	·	·	·	2	·
·	·	·	·	·	3	1	·	·	·	·	·	·	3	·	·	·	·	·	1	·	·
·	·	·	·	·	2	1	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	·	12	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	11	-1	·	·	2	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	10	-1	·	·	·	·	1	3	·	2	·	·	·	·	·	·	·
·	·	·	·	·	9	-1	·	·	·	·	·	·	1	·	3	·	·	·	·	·	·
·	·	·	·	·	8	-1	·	·	·	·	·	·	·	1	·	·	1	3	·	·	·
·	·	·	·	·	7	-1	·	·	·	1	·	2	·	·	1	·	·	·	1	·	·
·	·	·	·	·	6	-1	·	·	·	·	·	·	·	·	·	3	·	·	·	3	·
·	·	·	·	·	5	-1	·	·	1	·	·	1	·	·	·	·	·	2	·	·	·
·	·	·	·	·	4	-1	·	·	·	·	2	·	·	·	3	·	·	·	·	·	·
·	·	·	·	·	3	-1	·	·	·	·	·	·	1	·	·	·	·	·	3	·	·
·	·	·	·	·	2	-1	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	·	1	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1
·	·	·	·	·	·	1	·	1	2	2	3	6	3	3	7	4	2	4	3	3	·
·	·	·	·	·	·	-1	·	·	2	2	4	6	3	2	7	4	1	4	3	3	·

FIGURE 3. $\mathfrak{S}_7$-module decomposition for the operators $\nu_{(k,1^{n-k})}$ (continued from Figure 2)

$\nu_{(1^8)}$	$\nu_{(2,1^6)}$	$\nu_{(3,1^5)}$	$\nu_{(4,1^4)}$	$\nu_{(5,1^3)}$	$\nu_{(6,1^2)}$	$\nu_{(7,1)}$	w_0	χ^8	χ^{71}	χ^{611}	χ^{62}	χ^{5111}	χ^{521}	χ^{41111}	χ^{53}	χ^{4211}	χ^{422}	χ^{311111}	χ^{431}	χ^{32111}	χ^{3221}	$\chi^{2111111}$	χ^{44}	χ^{3311}	χ^{332}	χ^{221111}	χ^{22211}	χ^{2222}	$\chi^{11111111}$
322560	564480	376320	117600	18816	1568	64	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	60480	90720	45360	10080	1080	54	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	6720	23520	18480	5600	760	46	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	30240	25200	7200	900	50	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	12096	13104	4608	684	44	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	6048	9072	3744	612	42	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	2688	4872	2424	467	37	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	5040	2880	540	40	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	3024	2016	432	36	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	1512	1368	351	33	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	672	768	246	28	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	336	672	240	28	1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	7920	3960	660	44	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	5040	2880	540	40	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	3600	2340	480	38	-1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	2520	1800	405	35	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	2160	1800	420	36	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	2160	1620	380	34	-1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	1080	1080	305	31	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	1008	1152	324	32	-1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	720	792	248	28	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	560	736	244	28	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	1320	396	36	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	990	330	33	1	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	720	270	30	1	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	660	264	30	1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	630	250	29	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	540	230	28	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	440	220	28	1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	432	198	26	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	420	196	26	1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	378	182	25	1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	360	190	26	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	360	180	25	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	336	168	24	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	270	170	25	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	252	148	23	1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	240	145	23	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	216	144	23	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	180	120	21	1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	168	112	20	1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	162	118	21	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	144	126	22	1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	140	100	19	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	126	98	19	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	120	96	19	1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	·	96	93	19	1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·

FIGURE 4. $\mathfrak{S}_8$-module decomposition for the operators $\nu_{(k,1^{n-k})}$ (continues in Figure 5 and Figure 6)

$\nu_{(1^8)}$	$\nu_{(2,1^6)}$	$\nu_{(3,1^5)}$	$\nu_{(4,1^4)}$	$\nu_{(5,1^3)}$	$\nu_{(6,1^2)}$	$\nu_{(7,1)}$	w_0	χ^{8}	χ^{71}	χ^{611}	χ^{62}	χ^{5111}	χ^{521}	χ^{41111}	χ^{53}	χ^{4211}	χ^{422}	χ^{311111}	χ^{431}	χ^{32111}	χ^{3221}	$\chi^{2111111}$	χ^{44}	χ^{3311}	χ^{332}	χ^{221111}	χ^{22211}	χ^{2222}	$\chi^{11111111}$
·	·	·	·	990	330	33	-1	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	720	270	30	-1	·	·	·	·	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	630	250	29	-1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	540	230	28	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	432	198	26	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	378	182	25	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	360	190	26	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	360	180	25	-1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	270	170	25	-1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	252	148	23	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	216	144	23	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	180	120	21	-1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	162	118	21	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	144	126	22	-1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	140	100	19	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·
·	·	·	·	126	98	19	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·
·	·	·	·	120	96	19	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	·	96	93	19	-1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	132	24	1	·	·	1	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	110	22	1	·	·	·	·	3	3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	90	20	1	·	·	·	·	·	·	1	·	1	2	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	88	20	1	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	80	19	1	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	77	19	1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	72	18	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·
·	·	·	·	·	70	18	1	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	63	17	1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	60	17	1	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	56	16	1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	3	·	·	·	·	·
·	·	·	·	·	55	17	1	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	54	16	1	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·
·	·	·	·	·	48	15	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·
·	·	·	·	·	44	16	1	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	40	15	1	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	40	14	1	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	36	14	1	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·
·	·	·	·	·	32	13	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·
·	·	·	·	·	30	14	1	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	30	12	1	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	28	12	1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	27	13	1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	24	11	1	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·
·	·	·	·	·	21	11	1	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·
·	·	·	·	·	20	10	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·
·	·	·	·	·	18	12	1	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	18	10	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·
·	·	·	·	·	14	10	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·

FIGURE 5. $\mathfrak{S}_8$-module decomposition for the operators $\nu_{(k,1^{n-k})}$ (continued from Figure 4; continues in Figure 6)

$\nu_{(1^8)}$	$\nu_{(2,1^6)}$	$\nu_{(3,1^5)}$	$\nu_{(4,1^4)}$	$\nu_{(5,1^3)}$	$\nu_{(6,1^2)}$	$\nu_{(7,1)}$	w_0	χ^{8}	χ^{71}	χ^{611}	χ^{62}	χ^{5111}	χ^{521}	χ^{41111}	χ^{53}	χ^{4211}	χ^{422}	χ^{311111}	χ^{431}	χ^{32111}	χ^{3221}	$\chi^{2111111}$	χ^{44}	χ^{3311}	χ^{332}	χ^{221111}	χ^{22211}	χ^{2222}	$\chi^{11111111}$
·	·	·	·	·	156	26	-1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	132	24	-1	·	·	2	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	110	22	-1	·	·	·	·	1	3	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	90	20	-1	·	·	·	·	·	·	1	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	88	20	-1	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	84	20	-1	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	80	19	-1	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	77	19	-1	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	72	18	-1	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	1	·	·	·	·	·	·
·	·	·	·	·	70	18	-1	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	63	17	-1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	60	18	-1	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	60	17	-1	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	56	16	-1	·	·	·	·	·	·	·	1	·	·	·	2	·	·	·	·	1	·	·	·	·	·
·	·	·	·	·	55	17	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	54	16	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·
·	·	·	·	·	50	16	-1	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	48	15	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·
·	·	·	·	·	44	16	-1	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	40	15	-1	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	40	14	-1	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	35	13	-1	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	32	13	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·
·	·	·	·	·	30	14	-1	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	30	12	-1	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	2	·	·	·	·
·	·	·	·	·	28	12	-1	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	27	13	-1	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·	·
·	·	·	·	·	24	11	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	·
·	·	·	·	·	21	11	-1	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·
·	·	·	·	·	18	12	-1	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	18	10	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	3	·	·
·	·	·	·	·	14	10	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·
·	·	·	·	·	8	10	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
·	·	·	·	·	·	14	1	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	13	1	·	·	2	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	12	1	·	·	·	·	3	6	·	3	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	11	1	·	·	·	·	·	·	3	·	7	4	·	4	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	10	1	·	·	·	·	·	·	·	·	·	·	2	·	3	3	·	3	·	·	·	·	·	·
·	·	·	·	·	·	9	1	·	·	·	·	·	·	·	2	·	·	·	6	·	·	·	·	7	4	·	·	·	·
·	·	·	·	·	·	8	1	·	·	·	1	·	2	·	·	3	·	·	·	3	·	·	·	·	·	2	·	·	·
·	·	·	·	·	·	7	1	·	·	·	·	·	·	·	·	·	6	·	·	·	7	·	·	·	4	·	3	·	·
·	·	·	·	·	·	6	1	·	·	1	·	·	2	·	·	·	·	·	3	·	·	·	·	·	·	·	·	3	·
·	·	·	·	·	·	5	1	·	·	·	·	2	·	·	·	6	·	·	·	·	4	·	·	4	·	·	·	·	·
·	·	·	·	·	·	4	1	·	·	·	·	·	·	3	·	·	·	·	·	7	·	·	·	·	·	·	3	·	·
·	·	·	·	·	·	3	1	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	3	·	·	·
·	·	·	·	·	·	2	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	2	·	·	·	·	·	·	·
·	·	·	·	·	·	13	-1	·	·	2	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	12	-1	·	·	·	·	4	6	·	2	·	·	·	·	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	11	-1	·	·	·	·	·	·	3	·	7	4	·	4	·	·	·	·	·	·	·	·	·	·
·	·	·	·	·	·	10	-1	·	·	·	·	·	·	·	·	·	·	1	·	3	3	·	2	·	·	·	·	·	·
·	·	·	·	·	·	9	-1	·	·	·	·	·	·	·	2	·	·	·	6	·	·	·	·	7	4	·	·	·	·
·	·	·	·	·	·	8	-1	·	·	·	·	·	2	·	·	4	·	·	·	3	·	·	·	·	·	1	·	·	·
·	·	·	·	·	·	7	-1	·	·	·	·	·	·	·	·	·	6	·	·	·	7	·	·	·	4	·	3	·	·
·	·	·	·	·	·	6	-1	·	·	·	·	·	2	·	·	·	·	·	2	·	·	·	·	·	·	·	·	3	·
·	·	·	·	·	·	5	-1	·	·	·	·	2	·	·	·	6	·	·	·	·	4	·	·	4	·	·	·	·	·
·	·	·	·	·	·	4	-1	·	·	·	·	·	·	4	·	·	·	·	·	7	·	·	·	·	·	·	3	·	·
·	·	·	·	·	·	3	-1	·	·	·	·	·	·	·	·	·	·	3	·	·	·	·	·	·	·	3	·	·	·
·	·	·	·	·	·	2	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	1	·	·	·	·	·	·	·
·	·	·	·	·	·	·	1	·	·	2	3	7	10	7	3	15	12	3	12	13	13	1	4	14	6	6	5	5	1
·	·	·	·	·	·	·	-1	·	1	3	2	4	10	6	6	18	8	6	13	13	15	2	1	8	10	3	7	1	·

FIGURE 6. $\mathfrak{S}_8$-module decomposition for the operators $\nu_{(k,1^{n-k})}$ (continued from Figure 4 and Figure 5)

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List of Symbols

Algebra

E_i	unit basis vector indexed by i, page 27
$(\mathbb{K}\mathcal{F})^W$	algebra of W-invariants of $\mathbb{K}\mathcal{F}$, page 38
(G, U, χ)	twisted Gelfand pair (or triple), page 23
(W, S)	Coxeter system, page 20
$\mathbb{C}$	complex numbers, page 3
$\mathrm{Z}_W(w)$	centralizer of the element w in the group W, page 4
$\mathrm{Z}_W(X)$	pointwise stabilizer subgroup within W of the subspace X, page 12
χ	character of a group, page 3
χ^+	trivial character of $\mathbb{Z}_2$, page 48
χ^-	non-trivial character of $\mathbb{Z}_2$, page 48
χ^λ	irreducible character of the symmetric group corresponding to the number partition λ, page 32
$\mathrm{Fix}_w(M)$	set of elements in M fixed by the action of w, page 54
$\Gamma(\mathbb{K}W)$	Grothendieck group of all virtual $\mathbb{K}W$-modules, page 44
$\mathtt{1}$	identity element of a group, page 4
$\mathfrak{e}_\chi$	idempotent for character χ in $\mathbb{C}W$, page 16
Ind_H^G	induction of a representation from the subgroup H to the group G, page 4
$\mathrm{Irr}(G)$	irreducible representations of G over the complex numbers, page 72
$\mathbb{K}W$	group algebra of W over $\mathbb{K}$, page 2
$\mathbb{K}$	generic field, page 2
$\lambda_{\mathcal{O}}(\chi)$	eigenvalue Idempotent for character χ in $\mathbb{Q}W$, page 16
$\mathrm{Lie}(V)$	free Lie algebra over V, page 56
I_V	matrix of the identity endomorphism of V, page 10
$\mathrm{N}_W(X)$	stabilizer subgroup (not necessarily pointwise) within W of the subspace X, page 12
$\mathfrak{o}$	ring of integers within a fixed number field $\mathbb{K}$, page 3
Φ	root system, page 27
$\Phi_{\mathcal{O}}$	roots corresponding to hyperplanes in $\mathcal{O}$, page 27
$\Phi_{\mathcal{O},\pm}$	positive/negative roots in $\Phi_{\mathcal{O}}$, page 28
$\Phi_\pm$	positive/negative roots in root system Φ, page 27
$\mathbb{Q}$	rational numbers, page 3
$\mathbb{R}W$	group algebra of W over $\mathbb{R}$, page 2
$\mathbb{R}$	real numbers, page 2
sgn	sign character of symmetric group, page 17
$\mathrm{Stab}_W(m)$	stabilizer of $m \in M$ within the group W acting on M, page 44
$\mathrm{Sym}^m(U)$	m^{th} graded component of symmetric algebra over vectorspace U, page 56

$\mathfrak{S}_B$	symmetric group of all permutations of the set B, page 13
$\mathfrak{S}_n$	symmetric group on n n letters, page 1
$\mathbf{1}$	trivial character, page 16
$\mathbb{Z}W$	group algebra of W over the integers $\mathbb{Z}$, page 2
M/W	set of W-orbits of W acting on M, page 44
m^W	orbit of the element $m \in M$ under the group W acting on M, page 44
n_X	$[\mathrm{N}_W(X) : \mathrm{Z}_W(X)]$, page 13
R^X	$\sum_{u \in W^X} u \in \mathbb{C}W$, page 13
$T(V)$	tensor algebra of V, page 56
U^χ	χ-isotypic component of W-module U, page 3
U_λ	λ-eigenspace of $R[T]$-module U, page 46
W^J	Minimal length right coset representatives for the parabolic subgroup W_J, page 13
${}^J W$	Minimal length right coset representatives for the parabolic subgroup W_J, page 13
${}^X R$	$\sum_{u \in {}^X W} u \in \mathbb{C}W$, page 13
${}^X W$	set of right coset representative for $\mathrm{Z}_W(X)$ in W, page 13

Combinatorics

(X, Y)	open interval between X and Y in a poset, page 44
$[n] = \bigsqcup_i B_i$	set partition of $[n]$ with blocks B_i, page 8
$[n]$	set $\{1, \ldots, n\}$ of the first n natural numbers, page 8
$[X, Y]$	closed interval between X and Y in a poset, page 7
Δ_n	n-dimensional standard simplex, page 34
$\mathrm{Des}(w)$	descent set of w, page 56
$\mathrm{eig}(Q)$	index of filtration component to which Q contributes its irreducible, page 80
$\mathfrak{L}_\lambda(\mathbf{t})$	characteristic of $L(V)_\lambda$, page 55
$\hat{0}$	Bottom element of a poset, page 44
$\hat{1}$	Top element of a poset, page 45
$\mathrm{inv}_{\mathcal{O}}(c, c')$	number of subspace X in $\mathcal{O}$ for which $c/X = -c'/X$, page 10
$\iota_{\mathcal{O}}$	matrix in $\mathbb{Z}^{\mathcal{C} \times \mathcal{C}}$ whose (c, c')-entry equals $\mathrm{inv}_{\mathcal{O}}(c, c')$, page 10
$\lambda \vdash n$	number partition λ of n, page 9
λ^T	conjugate partition of number partition λ, page 72
$\mathrm{Lin}_{\mathcal{O}}$	analog of the linear ordering polytope for $\mathcal{O} \subseteq \mathcal{L}$, page 34
$\mathrm{maj}(Q)$	major index of the standard Young tableau Q, page 70
$\mathrm{noninv}_{\mathcal{O}}(c, c')$	number of $X \in \mathcal{O}$ for which $c/X = c'/X$, page 7
$\mathrm{noninv}_{\mathcal{O}}(w)$	number of subspaces in $\mathcal{O}$ for which the chambers indexed by $\imath$ and w lie on the same side, page 4
$\mathrm{noninv}_k(w)$	k-noninversion number of w, page 1
$\nu_{\mathcal{O}}$	matrix in $\mathbb{Z}^{\mathcal{C} \times \mathcal{C}}$ whose (c, c')-entry equals $\mathrm{noninv}_{\mathcal{O}}(c, c')$, page 7
$\nu_{(k, 1^{n-k})}$	matrix of k-noninversion numbers, page 1
$\mathcal{O}_\lambda$	set partitions of type λ, page 10
$\pi_{\mathcal{O}}$	rectangular "square root" of $\nu_{\mathcal{O}}$, page 9
$\pi_{(2^k, 1^{n-2k})}$	$\pi_{\mathcal{O}_{(2^k, 1^{n-2k})}}$, page 59
$\widetilde{\mathrm{C}}^i(\bullet; \mathbb{K})$	i^{th} reduced cochain group with coefficients in $\mathbb{K}$, page 45
$\widetilde{\mathrm{H}}^i(\bullet; \mathbb{K})$	i^{th} reduced cohomology group with $\mathbb{K}$ coefficients, page 44

$\mathrm{shape}(Q)$	shape of the standard Young tableau Q, page 70
SYT_n	set of standard Young tableaux of size n, page 80
$A^{\langle i\rangle}$	set of injective words of length i over A, page 71
a^{rev}	word a reversed, page 71
F_D	fundamental quasisymmetric function, page 56
h_m	m^{th} homogeneous symmetric function, page 55
BHR	Bidigare, Hanlon and Rockmore, page 2

Hyperplane Arrangements

$\mathcal{A}$	arrangement of hyperplanes, page 4
$\mathcal{A}/X$	localized arrangements of all H/X for $H \in \mathcal{A}$, page 7
$\angle\{\alpha, \beta\}$	angular measure in radians of the sector $H_\alpha^+ \cap H_\beta^+$, page 28
$\angle\{H, H'\}$	dihedral angle between H and H', page 29
$\mathcal{C}(\mathcal{A})$	chambers of the arrangement $\mathcal{A}$ of hyperplanes, page 4
$\ell(\mathbf{x})$	length of $\mathbf{x} \in \mathcal{F}^\ell$, page 41
$\mathcal{F}$	faces of an arrangement $\mathcal{A}$, page 37
$\hat{0}$	unique minimal element of poset, page 7
$\hat{1}$	unique maximal element of a poset, page 7
$\mathbb{K}\mathcal{F}$	left-ideal generated by $\mathcal{C}$ within $\mathbb{K}\mathcal{F}$, page 37
$\mathbb{K}\mathcal{F}$	semigroupalgebra of $\mathcal{F}$ with coefficients in $\mathbb{K}$, page 37
$\mathcal{L}(\mathcal{A})$	intersection lattice of the arrangement $\mathcal{A}$ of hyperplanes, page 4
$\mu(\cdot,\cdot)$	Möbius function, page 44
$\mathcal{O}$	set of intersection subspaces of an arrangement. Often orbit or union or orbits under group action., page 4
$\mathcal{O}_X$	orbit of subspace $X \in \mathcal{L}$ under group W, page 51
$\mathrm{OS}(\mathcal{A})$	Orlik-Solomon of $\mathcal{A}$, page 53
$\tilde{\mathbf{x}}$	reduced subword of $\mathbf{x}$, page 41
$\mathrm{supp}(x)$	support of $x \in \mathcal{F}$, page 41
$\mathrm{WH}^*(P;\mathbb{R})$	Whitney cohomology of a poset P with real coefficients, page 51
$\mathrm{WH}_{\mathcal{O}_X}$	$\mathrm{Ind}_{\mathrm{N}_W(X)}^W \widetilde{\mathrm{H}}^*((V,X);\mathbb{R}) \otimes \det{}_{V/X}$, page 51
$\mathbb{Z}\mathcal{C}$	free $\mathbb{Z}$-module with basis $\mathcal{C}$, page 7
c/X	chamber in the localized arrangement $\mathcal{A}/X$ corresponding to the chamber c, page 7
$c_{\mathbf{1}}$	chamber indexed by neutral element $\mathbf{1}$ of group, page 4
H_α	hyperplane orthogonal to the vector α, page 27
H_α^+	halfspace cut out hyperplane H_α containing α, page 28
$r(x)$	rank function on geometric lattice, page 7
$x \circ y$	x pulled by y, page 37

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