Recovering Conductances of Resistor Networks in a Punctured Disk

Yulia Alexandr*   Brian Burks†   Patricia Commins‡

November 27, 2018

Abstract

The response matrix of a resistor network is the linear map from the potential at the boundary vertices to the net current at the boundary vertices. For circular planar resistor networks, Curtis, Ingerman, and Morrow have given a necessary and sufficient condition for recovering the conductance of each edge in the network uniquely from the response matrix using local moves and medial graphs. We generalize their results for resistor networks on a punctured disk. First we discuss additional local moves that occur in our setting, prove several results about medial graphs of resistor networks on a punctured disk, and define the notion of z-sequences for such graphs. We then define certain circular planar graphs that are electrically equivalent to standard graphs and turn them into networks on a punctured disk by adding a boundary vertex in the middle. We prove such networks are recoverable and are able to generalize this result to a much broader family of networks. A necessary condition for recoverability is also introduced.

1   Introduction

In this paper, we study resistor networks, electrical networks made up of only resistors. The electrical properties of these networks is characterized by the response matrix. The response matrix gives the linear map from a potential assignment at each boundary vertex to the net current flow at each vertex. A common question in the study of electrical networks is when we can uniquely recover the conductances of a resistor network given the response matrix. This is called the Dirichlet-to-Neumann problem.

This problem has been extensively studied for a class of networks called circular planar resistor networks. Curtis, Moores, and Morrow [2] developed an algorithm to recover the conductances of a particular family of these graphs called standard graphs. Curtis, Ingerman, and Morrow [1] and De Verdiere, Gitler, and Vertigan [3] were able to use this algorithm to obtain results for all circular planar resistor networks. They considered the property of criticality, a condition regarding the connections of a network. Building on the previous algorithm and using medial graphs, they proved that the networks where we uniquely recover conductances from the response matrix are exactly the critical networks. They were also able to use medial graphs to show a complete catalogue of local moves that preserve the response matrix of a circular planar resistor network and use these local moves to classify critical networks. Moreover, they were able to identify all response matrices of resistor networks with a positivity criteria for minors of the matrix.

Kenyon and Wilson [5, 6] studied the combinatorics of the response matrix for resistor networks. In particular, they were able to realize the minors of the response matrix for a circular planar resistor

---

*Wesleyan University, CT

†University of California, Berkeley, CA

‡Carleton College, MN

network as a sum over groves in the network. This explains the positivity criteria for response matrices of circular planar resistor networks found in earlier work.

However, very little is known for more complicated networks. In [7], Lam and Pylyavskyy studied the Dirichlet-to-Neumann problem for electrical networks on a cylinder. They gave a conjectural solution for general cylindrical electrical networks, and showed it held for a special class of these networks known as "purely cylyndrical" networks.

In this paper, we introduce resistor networks in a punctured disk (rnpds). These are networks embedded in a disk where exactly one boundary vertex is placed inside the disk. We first discuss new local moves that arise for these networks. We conjecture that the local moves stated in the paper are the only moves that preserve electrical equivalence. Next we define medial graphs for rnpds and prove several properties of the medial graph for irreducible rnpds. Medial graphs allow us to define spider graphs, a famliy of rnpds closely related to the standard graphs. Using spider graphs, we find several conditional local moves that can be applied to rnpds under certain conditions without changing the response matrix. Turning to the question of recoverability, we find a class of recoverable rnpds. We also give a necessary condition for recoverability or rnpds.

2   Background and Definitions

For more background on electrical networks, see [1], [3], and [4].

Defnition 1. A *resistor network* is a graph (V, E) with a specified set $B \subseteq V$ of *boundary vertices* and a real nonnegative conductance c_e, for each $e \in E$. The remaining vertices, $I = V \setminus B$, are called *internal vertices*. We will say graph to refer to a resistor network without conductances.

In this paper, we'll assume all resistor networks are connected graphs. Our convention throughout this paper will be to color boundary vertices white and internal vertices black.

Defnition 2. A *circular planar resistor network* (cprn) is a resistor network that can be embedded in a disk so that it is planar and all the boundary vertices are on the boundary of the disk. A *resistor network in a punctured disk* (rnpd) is a resistor network that can be embedded in a disk so that it is planar and all boundary vertices but one are on the boundary of the disk. See Figure 1.

Figure 1: A cprn (left) and rnpd (right)

Defnition 3. A *non-trival rnpd* is an rnpd that cannot be drawn as a cprn. Note that in a non-trivial rnpd, the interior boundary vertex of must be surrounded by a polygon. Otherwise, we are able to redraw the network so that the interior boundary vertex is on the boundary of the disk.

Defnition 4. A *potential function* for a resistor network (V, E) is a function $f : V \to \mathbb{R}_{\geq 0}$. An edge (v_1, v_2) has a *voltage* given by the difference in potential at v_1 and v_2.

We want our resistor networks to model electrical circuits. This means they need to satisfy Ohm's Law and Kirchhoff's Current Law.

Theorem 1 (Ohm's Law)**.** *If v_e is the voltage across edge e, then $v_e c_e = i_e$, where i_e is the current across edge e.*

By Ohm's Law, a potential function induces a current across each edge. Note that current always flows from the vertex with the higher potential to the vertex with the lower potential.

Theorem 2 (Kirchhoff's Current Law)**.** *For each node in an electrical circuit, the sum of the currents entering the node must be the same as the sum of the currents leaving the node.*

The analogue of Kirchhoff's Current Law for resistor networks is that for each internal vertex in a resistor network, the sum of the currents entering the vertex must be the same as the sum of the currents leaving the vertex. If we use Ohm's Law to calculate current, only certain potential functions satisfy Kirchhoff's current Law.

Theorem 3 ([1])**.** *Given a function $\phi : B \to \mathbb{R}_{\geq 0}$ for a resistor network, there is a unique choice of potential function f where $f|_B = \phi$ and the currents induced by f from Ohm's Law satisfy Kirchhoff's Current Law.*

Defnition 5. Let $\Gamma = (V, E)$ be a resistor network with m vertices, n of which are boundary vertices. Index all boundary vertices $v \in B$ as $\{1, \cdots, n\}$ and all internal vertices $v \in I$ as $\{n+1, \cdots, m\}$. Then, we define the *Kirchhoff matrix* of Γ as follows:

$$
K_{i,j} = \begin{cases} \displaystyle\sum_{e=(i,j)} c_e & i \neq j \\ -\displaystyle\sum_{\substack{e=(i,k) \\ k \in [n]}} c_e & i = j \end{cases}
$$

Note that the Kirchhoff matrix can be represented as:

$$
K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}
$$

where A is a symmetric $n \times n$ matrix, B is an $n \times (m-n)$, and C is a symmetric $(m-n) \times (m-n)$ matrix.

Defnition 6. The *response matrix*, $\Lambda(\Gamma)$ of a resistor network Γ is the Schur complement $A - BC^{-1}B^T$ of the Kirchhoff matrix.

We can represent a potential assignment to the boundary vertices as a vector $u \in \mathbb{R}^n_{\geq 0}$. $\Lambda(\Gamma)$ gives the linear map from such an assignment to the vector of currents flowing into each boundary vertex (where a negative value represents current flowing out of the vertex).

Example 1. Consider the following cprn:
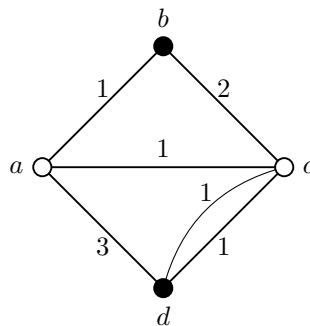
We see that the vertex set is $V = \{a, b, c, d\}$. The set of boundary vertices is $B = \{a, c\}$ and the set of internal vertices is $I = \{b, d\}$. Thus, in order to compute the Kirchhoff matrix, we label a and c by 1 and 2, respectively, and b and d by 3 and 4, respectively. Then, the Kirchhoff matrix is:

$$
K = \begin{bmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 2 & 2 \\ 1 & 2 & -3 & 0 \\ 3 & 2 & 0 & -5 \end{bmatrix}
$$

Now, as described above, one can compute the response matrix:

$$
\begin{bmatrix} -5 & 1 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}^{T} = \begin{bmatrix} -\frac{43}{15} & \frac{7}{15} \\ \frac{7}{15} & -\frac{43}{15} \end{bmatrix}
$$

Defnition 7. Graphs Γ_1 and Γ_2 are *electrically equivalent*, if, for every assignment of conductances for Γ_1, there exists an assignment of conductances to Γ_2 such that the resulting electrical networks have the same response matrix, and vice versa.

We are interested in three properties of networks that have been shown to be the same for the cprn case: irreducibility, recoverability, and criticality.

Defnition 8. We call a graph *reducible* if it is electrically equivalent to a graph with fewer edges, and *irreducible* otherwise.

Note that a network may have the same response matrix as a network with fewer edges while still being irreducible.

Defnition 9. A resistor network is *recoverable* if given its response matrix and and graph, we are able to uniquely determine its conductances.

Defnition 10. A *connection* of a resistor network $\Gamma = (V, E)$ is a tuple (P, Q) s.t. $P = \{p_1, \cdots, p_k\}$ and $Q = \{q_1, \cdots, q_k\}$ are subsets of B and there exists k disjoint paths that do not pass through boundary vertices connecting p_i to $q_{\pi(i)}$, $\forall i$ and for some $\pi \in S_k$. For a cprn, we require that $p_1, \cdots, p_k, q_k, \cdots, q_1$ occur in clockwise order around the disk's boundary. For an rnpd, we have the same requirement, with the exception that the interior boundary vertex can appear in either Q or P, at any index.

Remark 1. If Γ is a cprn and (P, Q) is a connection, then the only possible $\pi \in S_k$ connecting P and Q is the identity permutation. Otherwise, the paths will not be disjoint.

Defnition 11. A cprn is *critical* if the removal of any edge in the network breaks a connection.

Curtis, Ingerman, and Morrow [1] proved that these three definitions are equivalent in the cprn case. In this paper, we investigate to what extent this equivalence holds in the rnpd case.

3   Local Moves

There are five local transformations on cprns that do not change the response matrix [3]. We list these below. Note that the networks we draw below are components of a possibly larger network. The grey vertices indicate where the components in the pictures may be connected to the rest of the graph. That is, they are vertices that may be either boundary vertices or internal vertices and may have other edges incident to them. In the cases where the arrow goes only in one direction, we may still apply the transformation in the opposite direction, but there is not a unique way to do so.
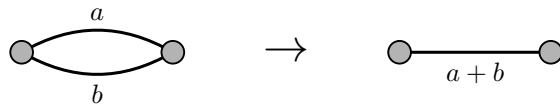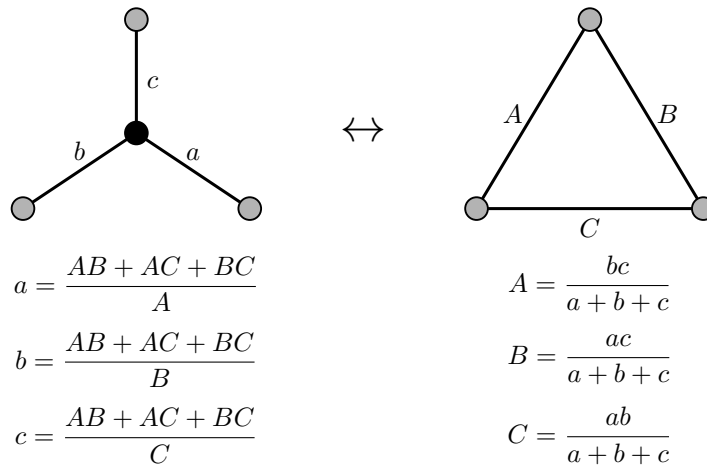
- Loop removal

- Pendant removal

- Series transformation

- Parallel transformation

- Y-Δ move

$$a = \frac{AB + AC + BC}{A} \qquad\qquad A = \frac{bc}{a + b + c}$$

$$b = \frac{AB + AC + BC}{B} \qquad\qquad B = \frac{ac}{a + b + c}$$

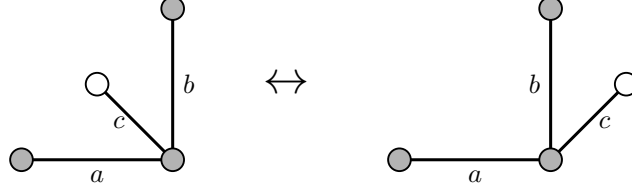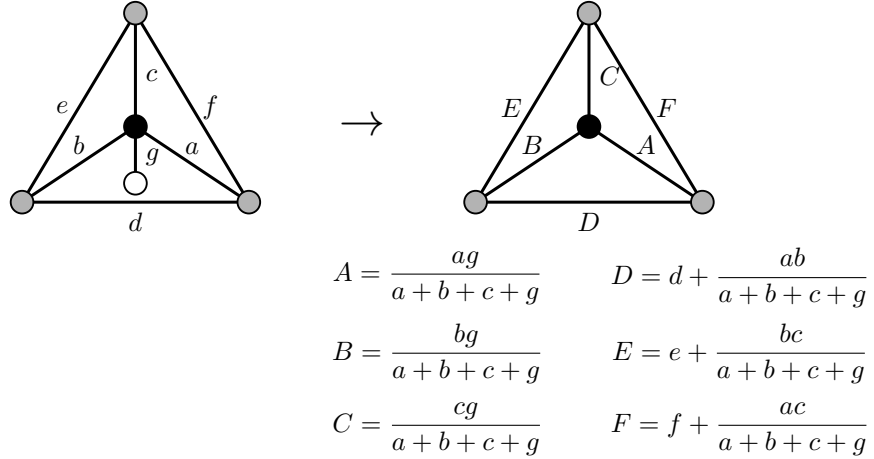$$c = \frac{AB + AC + BC}{C} \qquad\qquad C = \frac{ab}{a + b + c}$$
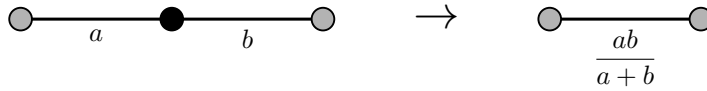
All of these moves are valid for rnpds. We also have two new local moves for rnpds that do not exist for cprns:

- Antenna Jumping

  We define an *antenna* to be an interior boundary vertex of degree one. If we have such a vertex and it is adjacent to vertex v, then *antenna jumping* is performed by jumping the antenna over an edge incident to v and into a different face. The conductances are unchanged by this move.


- Antenna Absorption

$$A = \frac{ag}{a+b+c+g} \qquad D = d + \frac{ab}{a+b+c+g}$$

$$B = \frac{bg}{a+b+c+g} \qquad E = e + \frac{bc}{a+b+c+g}$$

$$C = \frac{cg}{a+b+c+g} \qquad F = f + \frac{ac}{a+b+c+g}$$

Conjecture 1. Any two electrically equivalent graphs associated to rnpds are connected by the local moves stated in this section.

4 Medial Graphs and Z-sequences

Defnition 12 ([1]). For an cprn Γ with n boundary vertices, $v_1, ..., v_n$ clockwise, place $2n$ points, $t_1, ..., t_{2n}$, around the boundary of the disk in clockwise order such that for even i, t_{i-1} and t_i are on either side of $v_{i/2}$. That is, between v_j and v_{j+1} we have, in clockwise order, $v_j, t_{2j}, t_{2j+1}, v_{j+1}$ where indices are modulo n for the v_i's and modulo $2n$ for the t_i's. For each edge e in Γ, let m_e be its midpoint. Then the *medial graph*, $M(\Gamma)$, of Γ is the graph with vertices $\{t_i \mid 1 \le i \le 2n\} \cup \{m_e \mid e \in E\}$ and edges $\{(m_e, m_f) \mid e, f$ edges in Γ on the same face and sharing a vertex$\} \cup \{(m_e, m_e) \mid m_e$ a spike into a face of $\Gamma\} \cup \{(t_i, m_e) \mid t_i, e$ on the same face, i odd, e an edge connected to $v_{(i+1)/2}\} \cup \{(t_i, m_e) \mid t_i, e$ on the same face, i even, e an edge connected to $v_{i/2}\}$.

Example 2. The medial graph of the cprn in Figure 1 is drawn below.

6
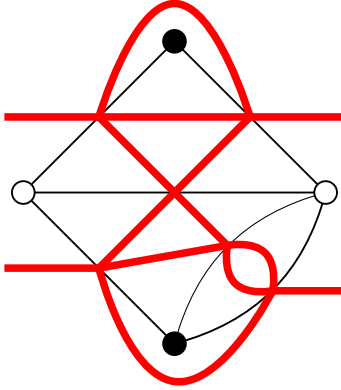
Remark 2. For an edge e of cprn Γ, m_e has degree 4 in $M(\Gamma)$. This means that if we approach m_e via some edge in $M(\Gamma)$, we can turn right, turn left, or go straight.

Defnition 13. We define an equivalence relation $\equiv$ on the edges of $M(\Gamma)$ as follows. For every e an edge of Γ with corresponding vertex m_e in $M(\Gamma)$, let the four medial edges adjacent to m_e be $v_{e,1}, v_{e,2}, v_{e,3}$, and $v_{e,4}$ in clockwise order. Then $v_{e,1} \equiv v_{e,3}$ and $v_{e,2} \equiv v_{e,4}$. Then the *medial strands* of $M(\Gamma)$ are the equivalence classes of edges of $M(\Gamma)$ induced by $\equiv$. In other words, a medial strand is a path in $M(\Gamma)$ where we always go straight through a vertex m_e. We frequently smooth the corners of a strand in depictions of the medial graph.

**Example 3.** The medial graph depicted in Example 2 has two medial strands, as shown below.
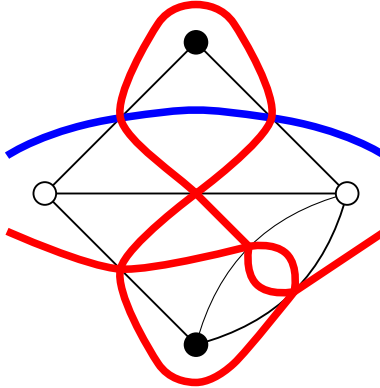
Remark 3. Each vertex t_i has degree 1 in $M(\Gamma)$. This means that each t_i is an endpoint of exactly one medial strand.

Defnition 14. [1] Let Γ be a cprn with n boundary vertices, $v_1, ..., v_n$ clockwise, and let $t_1, ..., t_{2n}$ be the endpoints of medial strands. Label the strand beginning at t_1 as with a 1. The remaining strands are labeled 2 through n so that if $i < j$, the endpoints of the strand i are t_a and t_b, and the endpoints of strand j are t_c and t_d, we have either $a < c, d$ or $b < c, d$. In other words, the first endpoint of strand i appears clockwise from t_1 before the first endpoint of j. For each $i \in \{1, 2, \cdots, 2n\}$ let z_i be the number associated with the strand with an endpoint at t_i. The sequence $z_1, ..., z_{2n}$ is called the *z-sequence* of Γ.

We can now make some analogous definitions for rnpds.

Defnition 15. For an rnpd Γ we can define the *medial graph* $M(\Gamma)$ of Γ to be as for a cprn, treating the interior boundary vertex as an internal vertex. We obtain *medial strands* as in the cprn case.

Example 4. The medial graph of the rnpd in Figure 1 is
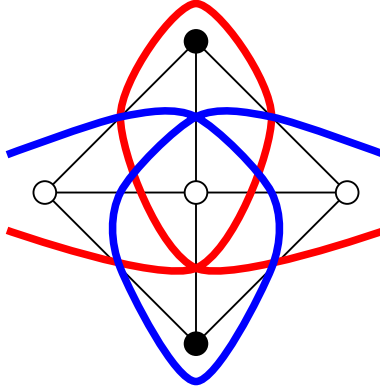
Defnition 16. For an rnpd Γ, we can define the *z-sequence* by beginning with the construction for cprns to get some permutation of the multiset $\{1, 1, 2, 2, ..., n, n\}$. For each i, label one strand endpoint of s_i as s_i^+ and the other s_i^- such that the strand from s_i^- to s_i^+ moves clockwise around b. Additionally, if a strand contains a self-intersection, put a bar over its endpoints' labels. The z-sequence of Γ is the sequence of labels of strand endpoints, starting at t_1 going clockwise.

Example 5. Consider the rnpd from Example 4. If the boundary vertex on the left is v_1 and the boundary vertex on the right is v_2, then the z-sequence for this rnpd is $\overline{1^+}, \overline{2^-}, \overline{2^+}, \overline{1^-}$.

Defnition 17. A *motion* on a triangular face ABC of a medial graph is defined by the following (local) transformation:
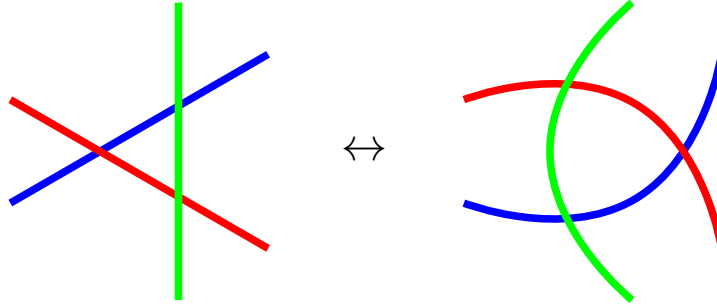

Remark 4. Medial graph motions correspond directly to $Y - \Delta$ moves, so the graphs of two rnpds are $Y - \Delta$ equivalent if and only if their medial graphs are equivalent by motions.

Defnition 18. Suppose s' and t' are segments of two medial strands s and t such that s' and t' both have endpoints a and b, s' and t' do not intersect themselves or each other between a and b, and the medial edges in s and t adjacent to a and b but not on s' and t' are outside the region enclosed by s' and t'. Define a *medial lens* to be the region enclosed by s and t between a and b.

Defnition 19. Define a *medial loop* to be the region enclosed by a medial strand segment whose endpoints are the same.

Defnition 20. Define a *medial circle* to be a medial strand that does not have endpoints.

Example 6. In the following example medial graph, the green strand is a medial circle. It bounds two lenses with the blue strand. The red strand bounds a medial loop. The blue and red strands bound a medial lens.
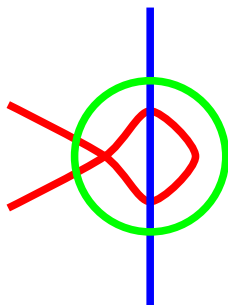
With these definitions in hand, we may state the first of two main theorems of this section.

Theorem 4. *An rnpd is irreducible by $Y - \Delta$ moves, series reductions, parallel reductions, pendant removal, self-edge removal, and antenna jumping if and only if the following conditions all hold:*

(a) *$M(\Gamma)$ contains no medial circles.*

(b) *$M(\Gamma)$ contains at most one self-intersecting strand. Furthermore, if such a strand exists, it intersects itself only once, with the loop it creates containing the interior boundary vertex.*

(c) *Any two distinct medial strands of $M(\Gamma)$ intersect at most twice. Furthermore, if two distinct medial strands intersect twice, creating a lens, the lens they create contains the interior boundary vertex.*

We first prove the backwards direction.

Proof. Edges in series and parallel induce a medial lens face in the medial graph. Pendants and self-edges induce a medial loop face in the medial graph. Thus, no graph satisfying (b) and (c) can be reduced only by series reductions, parallel reductions, self-edge removal, and pendant removal.

Medial circles remain medial circles under medial graph motions and antenna jumping. Similarly, medial lenses and medial loops not containing b remain medial lenses and medial loops not containing b under motions and antenna jumping. Thus, if a graph satisfies properties (a), (b), and (c), it will always satisfy the properties after $Y - \Delta$ moves and antenna jumping, and hence will never be able to be reduced by series reductions, parallel reductions, pendant removal, or self-edge removal. □

To prove the forwards direction, we require several lemmas.

Lemma 1. In the medial graph of an rnpd irreducible by $Y - \Delta$ moves, series reductions, parallel reductions, pendant removal, and self-edge removal:

(a) Any medial lens contains the interior boundary vertex.

(b) Any medial loop contains the interior boundary vertex.

(c) Any medial circle contains the interior boundary vertex.

Proof. Suppose our medial graph has a medial lens. We follow the algorithm in Lemma 6.2 of [1], noting that we may still perform motions on any triangular face in the medial graph that does not contain the boundary vertex. For a lens that does not contain the boundary vertex, none of its faces contain the boundary vertex. Following the algorithm in Lemma 6.2 will never cause the lens to contain new vertices. Thus, for rnpds, any medial graph with a medial lens not containing the boundary vertex is equivalent by motions to one with a lens (still not containing the boundary vertex) as a face.

We can apply the same argument for medial loops, in which case we obtain a medial loop face, and medial circles, in which case we obtain a medial circle face.

Lens faces may be reduced by a series or parallel transformation (depending on whether a vertex is inside the lens) and medial loop faces may be reduced by pendant removal or loop removal (depending on whether a vertex is inside the loop). In both cases, we have a contradiction because our graph was reduced. Medial circle faces cannot exist. □

Lemma 2. Let p, q, r be pairwise-intersecting medial strands in an rnpd Γ irreducible by $Y - \Delta$ moves, series reductions, parallel reductions, pendant removal, and self-edge removal. Define v_{qr} be an intersection of strands q and r and define v_{pr}, and v_{pq} similarly. Suppose that the segments of strands p, q, and r between v_{pr} and v_{pq}, v_{pq} and v_{qr}, and v_{pr} and v_{qr} form the sides of a triangle T. That is, these strand segments do not intersect themselves or each other except for at v_{qr}, v_{pr}, and v_{pq}. If T does not contain b, then we can apply a sequence of motions on faces of T to obtain a network where T is a face of the medial graph.

Proof. We prove this inductively on the number of faces inside T. If T only has one face, no motions are required.

If T has more than one face, some strand s intersects the boundary of T twice. If these intersections occur on the same strand, there would be a lens inside T. Since T does not contain b, this contradicts Lemma 1. So, without loss of generality, we can assume these intersections occur on strands q and r and that the intersection points are v_{qs} and v_{rs}. Then q, r, and s enclose a region U with vertices v_{qr}, v_{qs}, and v_{rs}. Since U is strictly inside T, U has fewer faces than T. Thus, by induction, we can perform motions so that there are no intersections inside U. These motions will not increase the number of faces in T. Next we can perform a motion to move U outside T, decreasing the number of faces in T by one. By induction, we can perform another sequence of motions to make T a face of the medial graph. $\qquad\square$

Lemma 3. In an rnpd irreducible by $Y - \Delta$ moves, series reductions, parallel reductions, pendant removal, self-edge removal, and antenna jumping with no medial circles, there exists at most one medial loop. That is, there exists at most once medial strand with a self-intersection, and if such a strand exists, it intersects itself only once.

Proof. For the sake of contradiction, suppose we have at least two medial loops. Let ℓ be a medial loop contains no other medial loops within its interior. By the previous lemma, the interior boundary vertex b must be inside ℓ.

Claim 1: We can use motions to make ℓ a face in the medial graph.

Any strand segment s in ℓ with boundary on ℓ contains no self-intersection. Then it divides ℓ into a lens and a triangle T. The lens must then contain the interior boundary vertex b, so T does not. By Lemma 2 we can use motions to make T empty. These motions do not increase the number of strand segments in ℓ. After we make T empty, a motion on T decreases the number of strand segments in ℓ. Applying this process repeatedly, we can make ℓ a face, proving Claim 1.

By Claim 1, up to electrical equivalence, we may assume ℓ is a face in the medial graph. This face must correspond to the vertex b of the rnpd by part (b) of Lemma 1. Since every medial loop contains b, every medial loop contains ℓ. Without loss of generality, let ℓ' be such that ℓ is the only loop contained in ℓ'.

Claim 2: We can use motions to make ℓ' contain only ℓ.

Let the self-intersection of ℓ' be a. As in the proof of Claim 1, any strand segment s in ℓ' with endpoints s_1 and s_2 on ℓ' contains no self-intersection. Since s cannot intersect ℓ or create a lens, s_1, s_2, and a create a triangle T that does not contain b.

By Lemma 2 we can use motions to make T empty. These motions do not increase the number of strand segments in ℓ'. After we make T empty, a motion on T decreases the number of strand segments in ℓ'. Applying this process repeatedly, we can make ℓ' contain only ℓ, proving Claim 2.

Now we're in one of three situations, shown below.

Case 1A: Case 1B: Case 2:
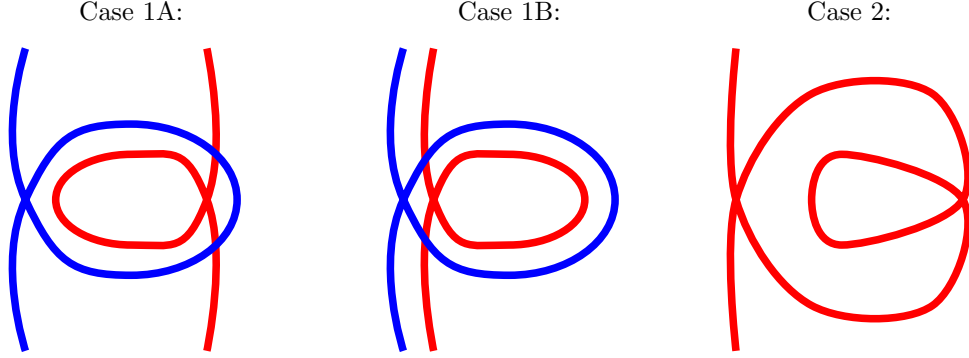
In cases 1A and 1B, antenna jumping yields a medial lens not containing b (note the parallel edges in 1A and the parallel edges after a $Y - \Delta$ move in 1B). In case 2, antenna jumping yields a medial loop not containing b (note the self-edge). This is a contradiction with Lemma 1, so we have proven the lemma. □

To prove the next Lemma, we need to introduce some notation.

Defnition 21. Let Γ be a graph with vertex set V. Let $V' \subseteq V$, V'' be all vertices in V' or adjacent to V', E' be the edges of Γ with both endpoints in V', and E'' be the edges of Γ with at least one endpoint in V'. Then the *strong restriction* of Γ to V' is $\Gamma' = (V', E')$ and the *weak restriction* of Γ to V' is $\Gamma'' = (V'', E'')$. The boundary vertices of Γ' are $(B \cap V') \cup \{v \in V' \mid v$ is adjacent to a vertex in $V \setminus V'\}$. The boundary vertices of Γ'' are $(B \cap V'') \cup (V'' \setminus V')$.

Remark 5. Both the strong restriction and weak restrictions of Γ to V' are *subgraphs* of Γ.

Remark 6. All graphs Γ are equal to the strong restriction to their vertex set. Assuming connectedness, they are also equal to the weak restriction to their interior vertex set.

Lemma 4. The medial graph of an irreducible rnpd does not contain any medial circles.

Proof. For the sake of contradiction, suppose there exists an irreducible rnpd Γ whose medial graph $M(\Gamma)$ contains a medial circle c. Without loss of generality, assume c contains no medial circles in its interior. Let Γ' be the strong restriction of Γ to the vertices inside c. Then $M(\Gamma')$ is the restriction of $M(\Gamma)$ to medial edges strictly contained in c. Note that $M(\Gamma')$ contains no medial circles and at most one medial loop. We consider several disjoint cases:

Case 1: The medial circle c contains no self-intersection.
If a strand segment inside c does not have a self-intersection, then the strand creates a two lenses with c. Only one of those lenses can contain the boundary vertex, so by Lemma 2 we find Γ is not reduced. This is a contradiction, so every strand segment inside c with endpoints on c contains a self-intersection.
Since $M(\Gamma')$ contains at most one medial loop, c contains at most one strand segment. The graph is connected, so c can't be empty. The medial graph containing only c and a medial loop as segments is shown below.
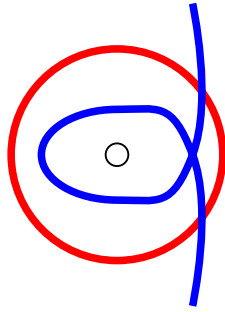
This is not a possible configuration as its network is reducible.

Case 2: The medial circle c contains exactly one self-intersection.
The medial circle c cannot form a figure-eight, as that creates two disjoint loops. Thus it must have the following form:
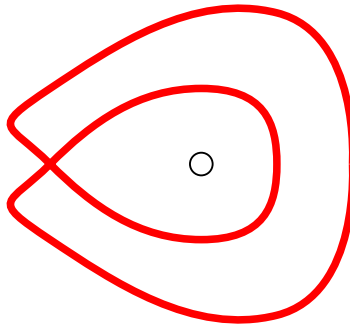

Just as in case 1, any strand segment inside c without self-intersections creates a lens not containing b. From earlier, c can strictly contain at most one medial loop which can be its only strand. This yields three configurations up to motions for strands in c:
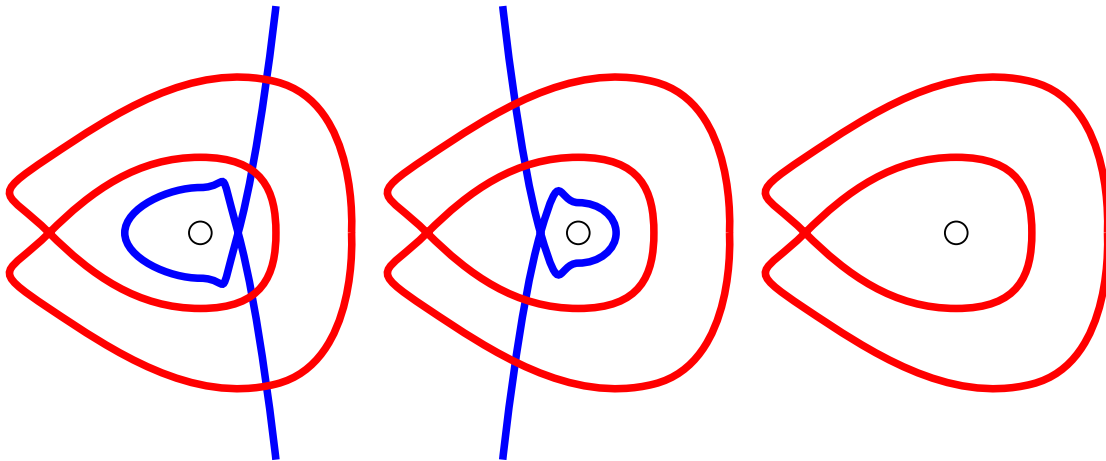
The first two are reducible. The third can't exist, as the medial graph of a connected graph is connected and c is not the full medial graph of any rnpd.

Case 3: The medial circle c contains at least two self-intersections.

Let p be a point on c. Then without loss of generality, moving along the to reach the first intersection e_1 of the strand segment, we create a counterclockwise loop. This must contain b. Then, continuing along c to the second intersection e_2 we cannot go clockwise, as that creates a lens not containing b. Thus we must go counterclockwise, and a segment of c mus look as below.
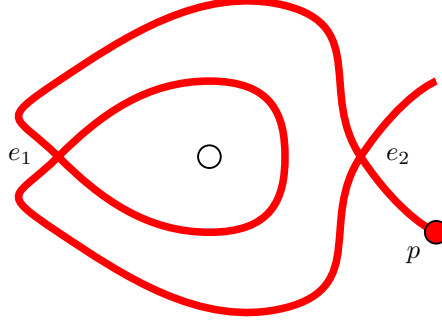
Now, let ℓ be the shown strand segment of c. Then any not-self-intersecting strand segment s in ℓ intersecting the boundary of ℓ at endpoints s_1 and s_2 either creates a lens with ℓ not containing b or a triangle T with s_1, s_2, and e_2. The first case is reducible. In the second we may apply motions on regions in T to make T empty. Then a motion inside T decreases the number of strand segments intersecting ℓ. Thus, we may assume every strand segment s in ℓ contains a self-intersection.
Since the loop with self-intersecting vertex e_1 is strictly contained in c, no other loops strictly in c exist. Thus, no strands may intersect ℓ. But this is then reducible. $\qquad\square$

Lemma 5. In an rnpd irreducible by $Y - \Delta$ moves, pendant removal, self-edge removal, parallel reductions, series reductions, and antenna jumping, any two medial strands intersect at most twice.

Proof. Assume for contradiction we have two medial strands p and q that intersect (at least) three times. Without loss of generality, assume p contains no self-intersection. As we move along q let our three intersections be v_1, v_2, and v_3. Then p and q form two lenses - one with endpoints v_1 and v_2, and one with endpoints v_2 and v_3. Since these lie on opposite sides of p, they have disjoint interiors. Thus at least one medial lens does not contain the interior boundary vertex, a contradiction. $\qquad\square$

With this series of lemmas, we can now prove Theorem 4.

Proof. Let Γ be an rnpd irreducible by our moves. Part (a) holds by Lemma 4. Part (a) and Lemma 3 imply that $M(\Gamma)$ contains at most one medial loop. By Lemma 1, if there is a medial loop, then it must contain the interior boundary vertex. This proves part (b) of the theorem. Finally, by Lemma 5, any two distinct strands of $m(\Gamma)$ intersect at most twice. Lemma 1 tells us that any two strands that do intersect twice must contain the interior boundary vertex in the lens they create, so we have proven part (c) or the theorem. $\qquad\square$

Conjecture 2. A graph is irreducible by $Y - \Delta$ moves, pendant removal, self-edge removal, parallel reductions, series reductions, antenna jumping, and antenna absorption if and only if it has three medial strands which pairwise intersect twice, there is a medial strand that self-intersects, and it is irreducible by $Y - \Delta$ moves, pendant removal, self-edge removal, parallel reductions, series reductions, and antenna jumping.

Theorem 5. *Two rnpds irreducible by $Y - \Delta$ moves, pendant removal, self-edge removal, parallel reductions, series reductions, and antenna jumping have the same z-sequence if and only if their graphs related by $Y - \Delta$ moves.*

Proof. Note that motions don't change the relative order of strand endpoints, or whether strands go around the interior boundary vertex clockwise or counterclockwise, so the z-sequence is preserved by motions and thus $Y - \Delta$ moves.

For the other direction, we induct on the number of strands. If only one strand exists, the theorem holds trivially.

Suppose we have we have two rnpds Γ and Γ' with multiple strands and the same z-sequence. Each strand in Γ which does not intersect itself divides Γ into two regions. For strand i without a self-intersection, call the region which does not contain the boundary vertex $\Gamma_{i,1}$, and the other $\Gamma_{i,2}$. Choose s such that $\Gamma_{s,1}$ contains the fewest possible number of regions. We can define $\Gamma'_{s,1}$, and $\Gamma'_{s,2}$ as we did for Γ. Since we chose s where $\Gamma_{s,1}$ is minimal, every strand in region $\Gamma_{s,1}$ intersects s.

Let G be the graph we obtain when we perform $Y - \Delta$ moves on Γ to minimize the number of regions on the side of strand s not containing the interior boundary vertex. Define G_1 and G_2 as the regions partitioned by strand s in G where G_1 is the side corresponding to $\Gamma_{s,1}$. Define G', G'_1, and G'_2 similarly from Γ'.

We claim that G_1 contains no medial strand intersections. Suppose for contradiction that p and q are strand segments that lie within G_1 and intersect at v_0 in G_1. Let the endpoint of p on s' be v_1. Let the endpoint of q on s' be v_2. Let the triangle formed by v_0, v_1, and v_2 be T. By motions on triangular faces in T, we may make T empty without increasing the number of regions in G_1. Then, a motion on T moves T to G_2, decreasing the number of regions in G_1 by one, a contradiction. Similarly, G'_1 contains no medial strand intersections.

Furthermore, G strongly restricted to vertices in G_2 is an irreducible graph whose z-sequence may be computed from from the z-sequence of G by deleting strand s and relabeling the remaining strands. Since we may determine from the z-sequence of G' which strand segments of G'_1 intersect s, G weakly restricted to vertices G_1 and G' weakly restricted to vertices in G'_1 are isomorphic graphs. As above, G' strongly restricted to G'_2 is a graph irreducible by $Y - \Delta$ moves, pendant removal, self-edge removal, parallel reductions, series reductions, and antenna jumping whose z-sequence may be computed from from the z-sequence of Γ' by deleting strand s and relabeling the remaining strands. Then G_2 and G'_2 have the same z-sequence. By the inductive hypothesis, G_2 and G'_2 are $Y - \Delta$ equivalent. This means G and G', and therefore Γ and Γ', are $Y - \Delta$ equivalent as well. $\qquad\square$

5 Conditional Local Moves and Spider Graphs

5.1 Spider Graphs

Defnition 22. The *standard graphs*, denoted by Σ_n for $n \geq 2$, are certain critical cprns with z-sequence $1, 2, ..., n, 1, 2, ..., n$. See Section 7 of [1] for a detailed construction.

In [2], it was shown that each Σ_{4m+3} is recoverable. An algorithm that transforms any critical cprn into some Σ_{4m+3} was introduced in [1] and was central to the proof that critical cprns are recoverable. We define a family of cprns that are electrically equivalent to standard graphs. These graphs will be called 4-periodic graphs. To do this, we'll first need to define some special types of edges.

Defnition 23. An edge is a *boundary edge* if it connects two boundary vertices. An edge is a *boundary spike* if it is connected to a boundary vertex of degree 1.

We will denote the 4-periodic graphs as Π_n for $n \geq 3$. The construction is as follows: For $n \geq 3$, we start with an n-gon whcich we will consider to be layer 1. Then we extend our polygon to have exactly $\lfloor \frac{n+1}{4} \rfloor$ layers, where each layer is a concentric n-gon connected to the n-gon from the previous layer by n edges between respective vertices (see Fig. 1). For $n \equiv 3 \mod 4$ we will have $\frac{n+1}{4}$ layers. For $n \equiv 0 \mod 4$ we will have $\frac{n}{4}$ layers and add $\frac{n}{2}$ consecutive boundary spikes. For $n \equiv 1 \mod 4$, we will have $\frac{n-1}{4}$ layers and add all n boundary spikes. For $n \equiv 2 \mod 4$, we will have $\frac{n-2}{4}$ layers, add all n boundary spikes, and then connect $\frac{n}{2}$ consecutive boundary vertices. In each case, we let all outermost vertices be boundary vertices (see Figure 2).
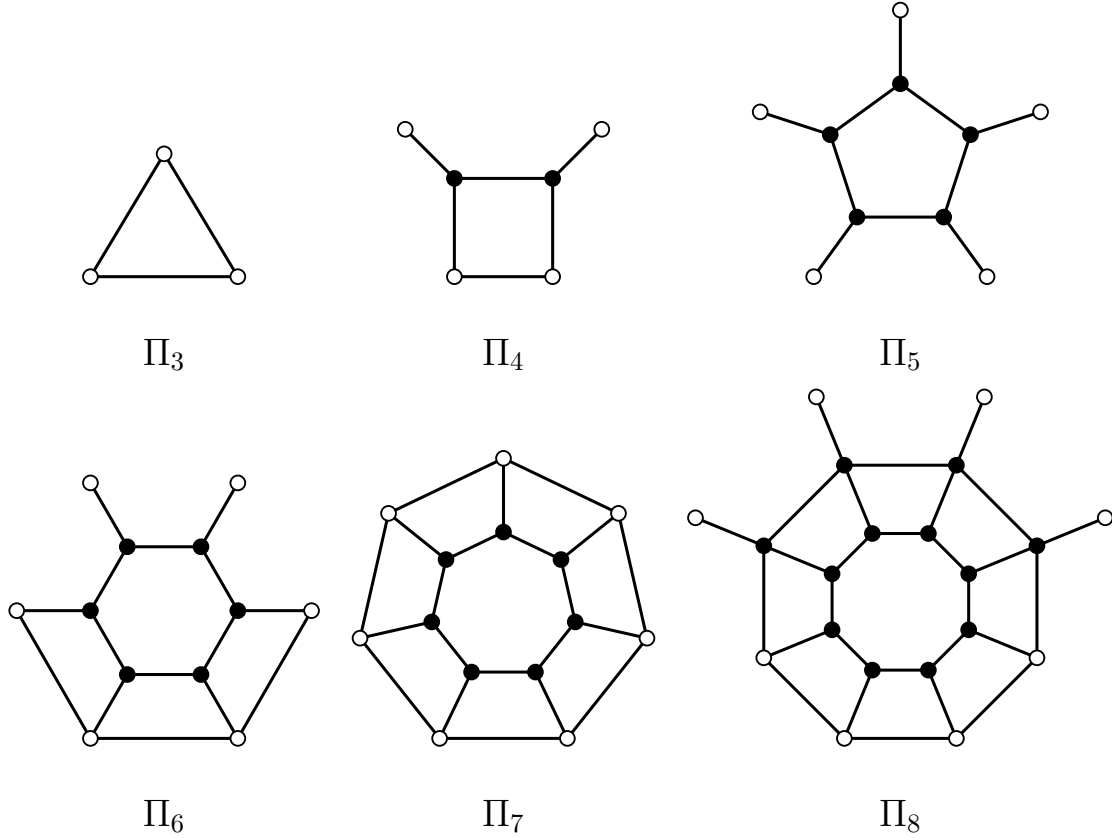
14

Figure 2: The first six 4-periodic cprns, Π_n for $3 \le n \le 8$.

Defnition 24. In Π_n for n even, we can see that as we go around the boundary of the disk, we have a set of boundary vertices with boundary spikes and then a set of boundary vertices connected by boundary edges. Let the first and last of the boundary vertices connected by boundary edges be v and w. For $n \equiv 0 \mod 4$, each of these vertices is of degree 2. We will refer to the edges connected to v and w that are not boundary edges as *boundary pseudo-spikes*. For $n \equiv 2 \mod 4$, each of these vertices is of degree 3. The vertices have a boundary edge, an edge going inward, and an edge connected to an internal vertex that is connected to a boundary spike. We will refer to the last of these edges as a *pseudo-boundary edge*.

Lemma 6. Let Π_n be a 4-periodic graph and let ℓ be the number of layers in this graph. Let s be a medial strand in Π_n with endpoints s_1 and s_2 such that the center face of the graph is to the right of the strand when going from s_1 to s_2. Then the number of strand endpoints between s_1 and s_2 to the left of the strand is:

1. $4\ell - 2$ if $n \equiv 3 \mod 4$.

2. $4\ell - 1$ if $n \equiv 0 \mod 4$.

3. 4ℓ if $n \equiv 1 \mod 4$.

4. $4\ell + 1$ if $n \equiv 2 \mod 4$.

Proof. Suppose the lemma is true for case (3) for a graph with $\ell - 1$ layers. This means that given any strand s, the number of strand endpoints between s_1 and s_2 is $4\ell - 4$ in such a graph. Choose
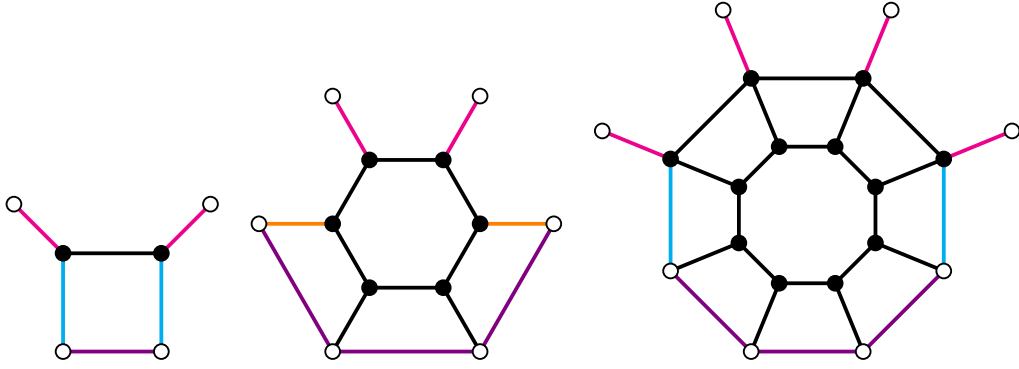
15

Figure 3: Types of edges in Π_4, Π_6, and Π_8. Boundary edges appear in purple, boundary spikes in pink, pseudo-boundary edges in light blue, and boundary pseudo-spikes in orange.

any strand s. This strand partitions the graph into two pieces. One of these pieces contains most of the center face. Add extra vertices and edges to the graph so that the graph still has $\ell - 1$ layers, but the underlying polygon now has 4 more sides. We will assume without loss of generality that these vertices and edges are added in the piece of the graph that contains most of the center face. This means it won't affect s at all. Now we add a layer to get to the next graph that falls under case (3). We can observe that adding the layer forces us to have two extra sides of the polygon between s_1 and s_2, which corresponds to exactly 4 extra strand endpoints (see Figure 4). Thus, the number of strand endpoints between s_1 and s_2 is $4\ell - 4 + 4 = 4\ell$, as desired, and we have proven case (3) of the lemma. The proof for case (1) is similar.
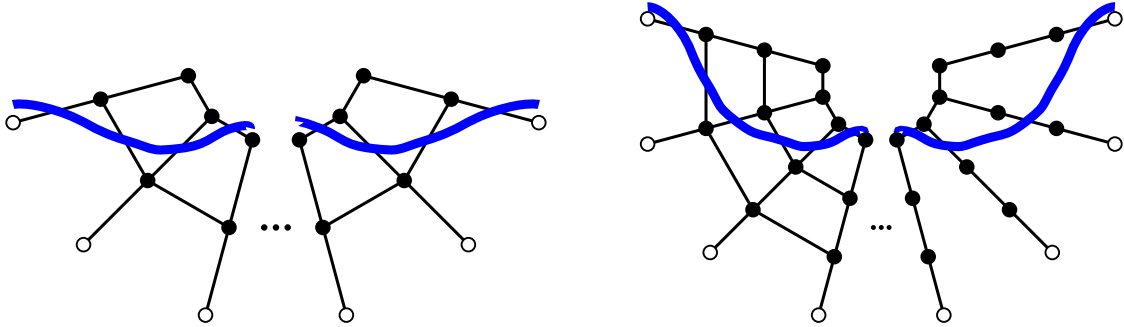

Figure 4: Adding a layer adds two extra boundary spikes between strand endpoints.

Now suppose that $n \equiv 0 \mod 4$. We claim that each strand must have one end crossing a boundary spike and the other end crossing an boundary edge or pseudo-boundary edge. To prove this, suppose for contradiction that the strand s starts by crosses a boundary spike on both ends (the other case is similar). Then the strand looks exactly like a strand from case (3) and there must be 4ℓ strand endpoints between s_1 and s_2 to the left. This means that there are at least $\frac{4\ell - 2}{2} + 2$ boundary spikes in the graph ($4\ell - 2$ internal strands and 2 spikes that S crosses). Since $n \equiv 0 \mod 4$, we get the following relation:

$$\frac{4\ell - 2}{2} + 2 = 2\ell + 1 = 2\left\lfloor \frac{n+1}{4} \right\rfloor + 1 = 2 \cdot \frac{n}{4} + 1 = \frac{n+2}{2} > \frac{n}{2}$$

This is a contradiction, since the number of boundary spikes in such graphs is exactly $\frac{n}{2}$. Now we will prove that in any such graph, any strand that has one end crossing a boundary spike and one

16

end crossing a boundary edge or pseudo-boundary edge has exactly $2\ell - 1$ strand endpoints between its endpoints. Assume that the lemma holds for a graph with $\ell - 1$ layers. Let s be a strand in the graph. We can increase the size of the polygons in each layer as we did for the proof of case (3) without affecting s. Again, adding a layer adds exactly two sides of the polygon between s_1 and s_2, and consequently 4 more strand endpoints. The proof for case (4) is similar, where we use boundary edges instead of boundary spikes and boundary spikes and boundary pseudo-spikes instead of boundary edges and pseudo-boundary edges.

$\square$

Corollary 1. For $n \geq 3$, Π_n is electrically equivalent to Σ_n.

Proof. Let $n \equiv 1 \mod 4$. We first note that the number of medial strands in Π_n is n. Also, the number of endpoints is $2n$, as every boundary vertex corresponds to exactly two endpoints. Take any strand s with endpoints s_1, s_2. By Lemma 6, we know that the number of strands between s_1 and s_2 is 4ℓ on one side. Since $n \equiv 1 \mod 4$, we know that $4\ell = 4\lfloor \frac{n+1}{4} \rfloor = 4 \cdot \frac{n-1}{4} = n - 1$, so there are $n - 1$ strand endpoints between s_1 and s_2. Since there are $2n - 2$ endpoints other than s_1 and s_2, it follows that the number of endpoints between s_1 and s_2 on the other side is also $n - 1$. Since s was an arbitrary strand, the graph has the same z-sequence as the standard graph. Also note that the medial graph is lenseless. This means Π_n is electrically equivalent to Σ_n. The cases when $n \equiv 2 \mod 4$, $n \equiv 3 \mod 4$, and $n \equiv 0 \mod 4$ are similar. $\square$

Defnition 25. A *spider graph*, Ξ_n, is defined for each $n \geq 3$. Ξ_n is constructed by placing a boundary vertex inside the center face of Π_n, and connecting it to each vertex on the boundary of that face.

Ξ_3 Ξ_4 Ξ_5

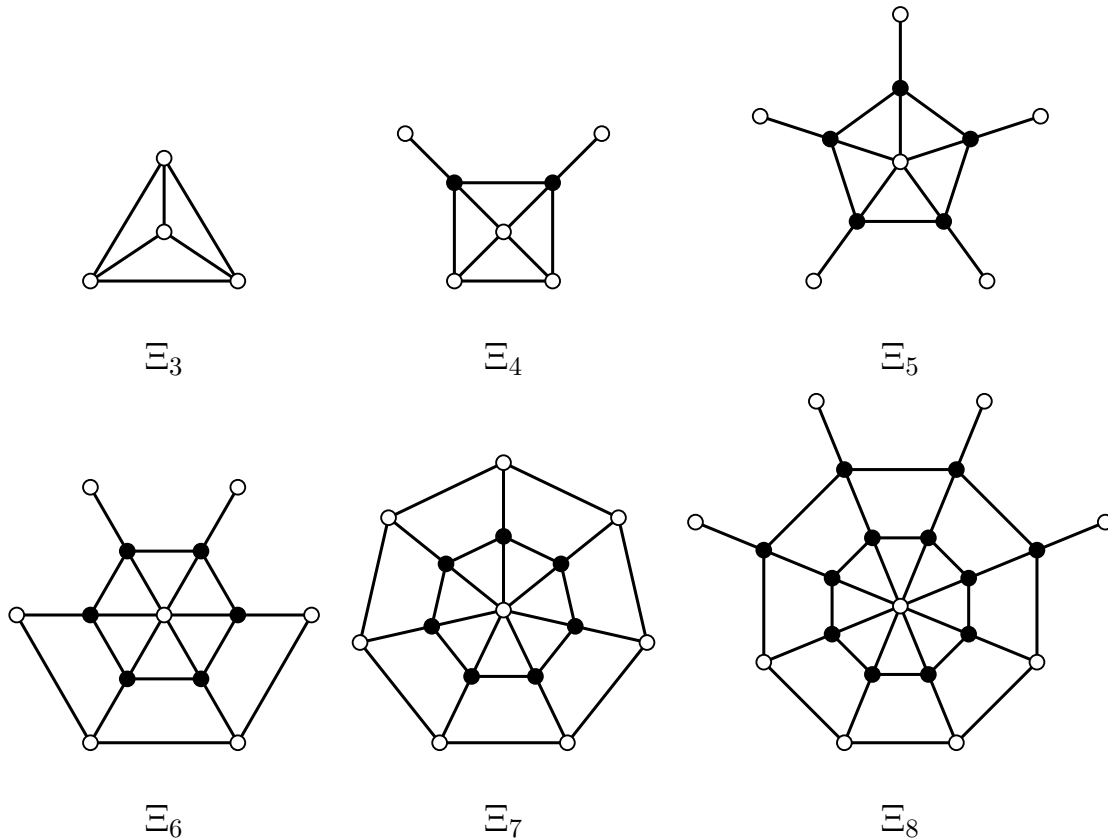Ξ_6 Ξ_7 Ξ_8

Figure 5: The first six spider graphs, Ξ_n for $3 \leq n \leq 8$.
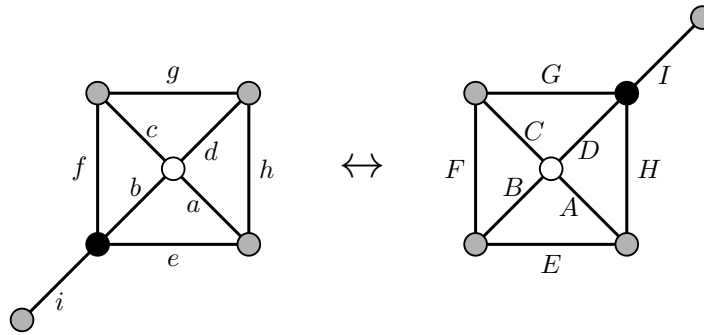
5.2   Conditional Local Moves

In addition to the 7 local moves described in section 3, there are also *conditional local moves* we can perform on rnpds. These are similar to the local moves in that they do not change the response matrix, but the formulas are not subtraction free. Thus, we can only perform the conditional local moves as long as they do not give us any negative conductances.

We have the following conditional local moves (see Appendix A for formulas):
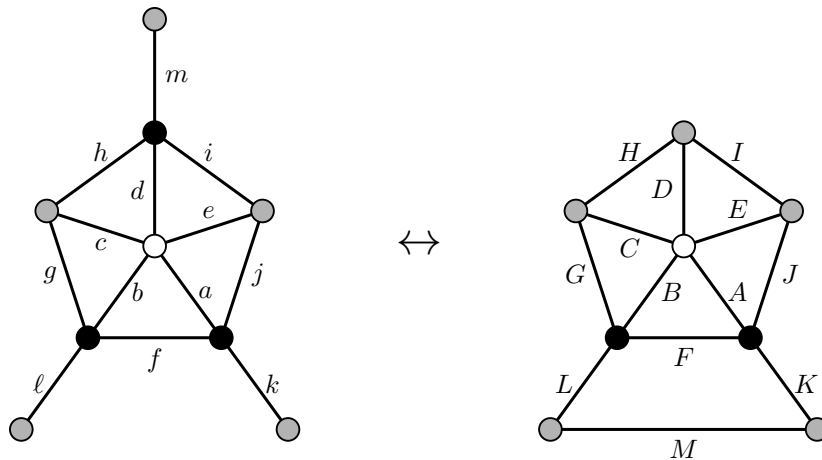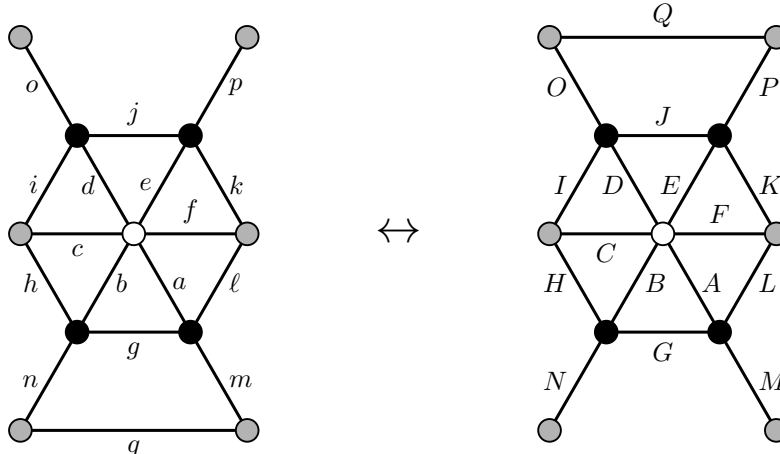
- Triangle Conditional Local Move
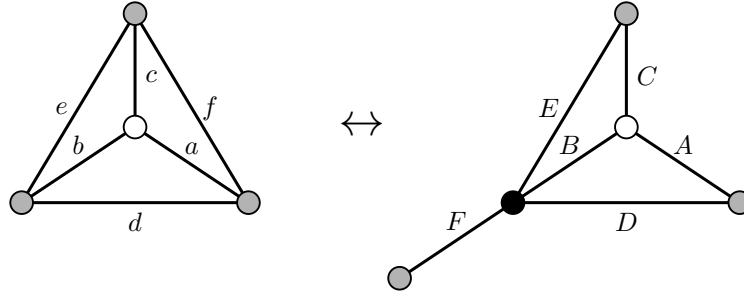
- Square Conditional Local Move

- Pentagon Conditional Local Move

Conjecture 3. We conjecture a hexagon conditional local move as well:

We have conjectural formulas for this move (see Appendix A). We were able to check that these formulas preserve every entry of the response matrix except for the entry in the row and column corresponding to the interior boundary vertex. The computations were too large to check for this entry. However, we did check that the response matrices were the same for several choices of values for the variables in one of the networks.

Conjecture 4. Notice that each of the conditional local moves listed above is a move on a subgraph of Π_n. We conjecture that there exists a conditional local move for each $n \geq 3$ where applying the local move on Π_n does the following:

- For $n \equiv 3 \mod 4$, we change a boundary edge from the outermost layer into a boundary spike on the opposite side of the graph.

- For $n \equiv 0 \mod 4$, we change a boundary spike from the outermost partial layer into a boundary spike on the opposite side of the graph.

- For $n \equiv 1 \mod 4$, we change a boundary spike from the outermost partial layer into a boundary edge on the opposite side of the graph.

- For $n \equiv 2 \mod 4$, we change a boundary edge from the outermost partial layer into a boundary edge on the opposite side of the graph.

It also seems that the formulas for these moves have a combinatorial interpretation in terms of groves or a grove-like structure (see [7]). For example the numerator in the formula for G, H, and I in the pentagon conditional local move is a sum of groves.

5.3   Properties of Spider Graphs

Defnition 26. Let P, Q be ordered subsets of B in the network Γ. As in [1], we define $\Lambda(P; Q)$ to be the submatrix of $\Lambda(\Gamma)$ with rows corresponding to elements of P and columns corresponding to an elements of Q.

Lemma 7. Assume Γ be a spider graph with boundary vertices B and interior boundary vertex b. Let $P, Q = \{p_1, \cdots, p_k\}, \{q_1, \cdots, q_k\}$ be subsets of the $B - \{b\}$ such that $p_1, p_2, \cdots, p_k, q_k, \cdots q_2, q_1$ are in clockwise order. If (P, Q) is a connection, as defined in section 2, then $(-1)^k \det \Lambda(P; Q) > 0$. Otherwise, $\det \Lambda(P; Q)$ is zero.

Proof. This lemma has been proven for cprns in Theorem 4.2 of [1]. Since the disjoint paths in the connection (P, Q) cannot pass through boundary vertices and we have restricted P and Q to not

contain b, we are not able to use the center face of the spider graph in our paths between P and Q. We therefore have the same set of possible connections as the 4-periodic graphs do. Other than properties true for all planar graphs, the proof of Theorem 4.2 relied only on the signs of the permutations for all possible paths between P and Q. Since these permutations are the same for spider graphs and 4-periodic graphs – cprns – the same proof holds. □

Theorem 6. *Let Ξ_n be a spider graph. Consider the z-sequence for Ξ_n. Let $S = (s^+, s^-)$ be an arbitrary strand. Then if n is odd, then there are exactly $n-3$ strand endpoints between s^+ and s^-, where we always measure the distance moving clockwise. If n is even, then for all but two strands, there are exactly $n-3$ strands between s^+ and s^-. One will have $n-2$ strands between its endpoints, and the other will have $n-4$.*

Proof. First assume n is odd. We note that for odd n, each strand in the medial graph of Ξ_n will first cross a boundary spike or a boundary edge and end by crossing the same type of edge. We know that the number of strand endpoints between s^+ and s^- in Π_n is exactly $n-1$. Note that adding a boundary vertex and edges in the middle of the graph forces us to have two more strand endpoints between s^- and s^+ (see Figure 6). Therefore, there are two fewer strand endpoints between s^+ and s^- and the number of strands between s^+ and s^- is $n-3$.

When $n \equiv 0 \mod 4$, we have $\frac{n}{2} - 1$ boundary edges, $\frac{n}{2}$ boundary spikes, and 2 pseudo-boundary edges. When $n \equiv 2 \mod 4$ we have $\frac{n}{2}$ boundary edges, $\frac{n}{2} - 1$ boundary spikes, and 2 boundary pseudo-spikes. Recall that in every Π_n for even n, each medial strand has one of its endpoints crossing a boundary edge or pseudo-boundary edge and another crossing a boundary spike or boundary pseudo-spike. When we add the star graph in the center to get Ξ_n, we get two additional strand endpoints between each s^- and s^+, as in the odd case. This means that from the viewpoint of s^-, s^+ is shifted by exactly one boundary vertex. Thus, all but two strands will still have one endpoint crossing a boundary edge or pseudo-boundary edge and the other crossing a boundary spike or boundary pseudo-spike. As in the odd case, there are now two more strand endpoints between s^+ and s^-, and are therefore $n-3$ strand endpoints between s^+ and s^-. We now look at the two strands that do not have this property. First, let $n \equiv 0 \mod 4$. One of these strands has both its endpoints crossing the pseudo-boundary edges (call this s_1), so there are $n-2$ strand endpoints between s_1^+ and s_1^-. The other strand has both its endpoints crossing the rightmost and leftmost boundary spikes (call this s_2), so there are $n-4$ strand endpoints between s_2^+ and s_2^-. Now, let $n \equiv 2 \mod 4$. One of these strands has both its endpoints crossing the boundary pseudo-spikes (call this s_1), so there are $n-2$ strand endpoints between s_1^+ and s_1^-. The other strand has both its endpoints crossing the rightmost and leftmost boundary edges (call this s_2), so there are $n-4$ strand endpoints between s_2^+ and s_2^-. □
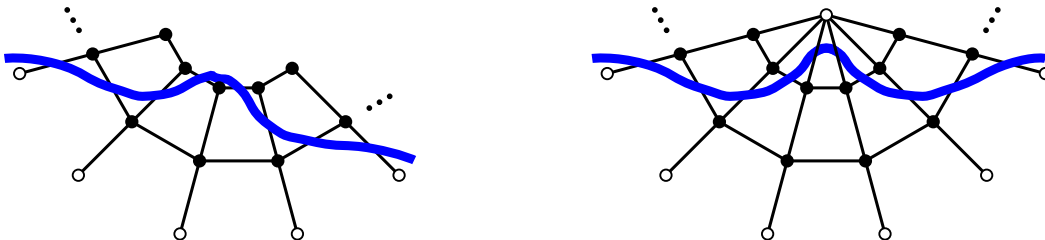

Figure 6: Adding the star graph to the center adds two more strand endpoints between s^- and s^+.

6 Recoverability Conditions

6.1 Recoverable Rnpds from Critical Cprns

Lemma 8 (Lemma 11.3 from [1])**.** Let Γ be a critical cprn. Then Γ has a boundary edge or a boundary spike.

Lemma 9 (Lemma 11.1 from [1])**.** Let Γ be a critical cprn. Then Γ remains critical after deleting a boundary edge.

Lemma 10 (Lemma 11.2 from [1])**.** Let Γ be a critical cprn. Then Γ remains critical after contracting a boundary spike.

Lemma 11. Let Γ be an rnpd with interior boundary vertex b, and let e be a boundary edge or a boundary spike. If deleting or contracting $e = pq$ breaks some connection (P, Q) s.t. $b \notin P \cup Q$, then we can derive the conductance of e from $\Lambda(\Gamma)$.

Proof. Corollaries 4.3 and 4.4 of [1] prove this for the cprn case. However, this remains true for rndps if we restrict P and Q to not include the interior boundary vertex. □

Defnition 27. A *star graph* is a central boundary vertex with only boundary spikes or pendants attached. *Inserting a star graph into a face of a graph* means adding a boundary vertex inside the face and connecting it to any number of vertices on the face.

Example 7. Consider the following graph:
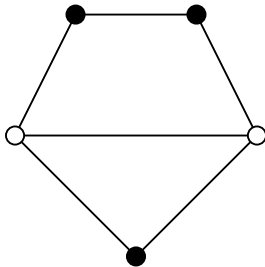
One way to insert a star graph into the upper face is illustrated below:
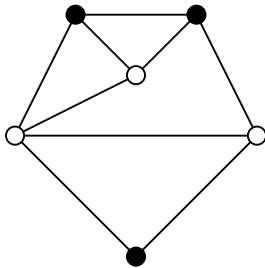
Theorem 7. *Assume Γ is a critical cprn. If Γ' be the rnpd obtained from inserting a star graph into any face of Γ, then Γ' is recoverable.*

Proof. Let $\Gamma' = (V, E)$ and $S = (V_S, E_S)$ be the star graph that was inserted into Γ to obtain Γ'. Assume we are given $\Lambda(\Gamma')$. First, note that $(V \setminus \{b\}, E \setminus E_S) = \Gamma$, which is known to be critical. So, by Lemma 8, we always have a boundary edge or boundary spike with which to begin the following process. First, pick a boundary edge or boundary spike, e, to delete or contract. Notice that deleting or contracting e must break a connection (P, Q) in Γ by the definition of critical. Since paths in connections cannot pass through other boundary vertices, adding back in S does not repair the broken

connection, and (P, Q) is broken in Γ'. Thus, by Lemma 11, we know the conductance of e. Knowing this allows us to derive the response matrix of the graph after contracting or deleting e, which we will call Γ'' (see section 8 of [1]). Now, by Lemmas 9 and 10, our Γ'' without S is still critical. Then, we can continue this process, deleting and contracting boundary edges and spikes one by one, until we are left with S. In other words, we will eventually know the conductance of every edge in Γ' except for those in G, but will know the response matrix of G. However, at that point, S will be a cprn with only boundary vertices, and hence its response matrix is exactly its Kirchhoff matrix. Therefore, Γ' is recoverable. □

Example 8. Note the Ξ_n is Π_n with a star graph inserted into a face. So, all the spider graphs are recoverable.

Example 9. By Theorem 5, all irreducible graphs that have the same z-sequence as a spider graph are recoverable.

6.2 Necessary Condition for Recoverablilty of Rnpds

Before stating the necessary condition, we first need to define the following algorithm for obtaining a cprn from a certain type of rnpd. Suppose we're given an rnpd with the interior boundary vertex b.

Algorithm 1.

1. Turn all the neighbors of b into boundary vertices.

2. Remove b from the graph (with all the edges incident to it).

3. If the resulting graph is a cprn, stop.

4. Otherwise, apply the above steps to each boundary vertex (in any order) that can't be placed on the disk until it results in a cprn.

Recall that in any rnpd, the interior boundary vertex has to be inside a polygon (one of the faces of a cprn).

Theorem 8. *Assume Γ' is an rnpd where the internal boundary vertex is the center of a star graph in the interior of some face f of the cprn Γ. Apply Algorithm 1 iteratively starting at the internal boundary vertex, and call the resulting cprn Γ''. Then Γ'' is critical if Γ' is recoverable.*

Proof. Assume Γ'' is not critical. Then the number of edges can be reduced. Note that Γ'' is isomorphic to a subgraph of Γ', but not necessarily to a subnetwork (not respecting the type of vertices). So Γ'' may have more boundary vertices. Having more boundary vertices may only decrease the number of possible moves. This implies that the number of edges in Γ' could be reduced to begin with, and thus Γ' is not recoverable. □
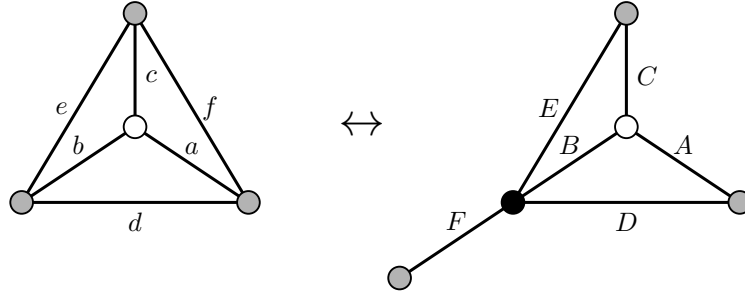
Acknowledgements

Appendix A   Formulas for Conditional Local Moves

Here we list the conditional local moves with formulas.

- Triangle Conditional Local Move

$$a = \frac{AB + AD + AE + AF + BD}{B + D + E + F} \qquad\qquad A = \frac{ea - bf}{e}$$

$$b = \frac{BF}{B + D + E + F} \qquad\qquad B = \frac{b(bf + de + df + ef)}{de}$$

$$c = \frac{BC + BE + CD + CE + CF}{B + D + E + F} \qquad\qquad C = \frac{cd - bf}{d}$$

$$d = \frac{DF}{B + D + E + F} \qquad\qquad D = \frac{bf + de + df + fe}{e}$$

$$e = \frac{EF}{B + D + E + F} \qquad\qquad E = \frac{bf + de + df + fe}{d}$$

$$f = \frac{DE}{B + D + E + F} \qquad\qquad F = \frac{bf + de + df + fe}{f}$$

- Square Conditional Local Move
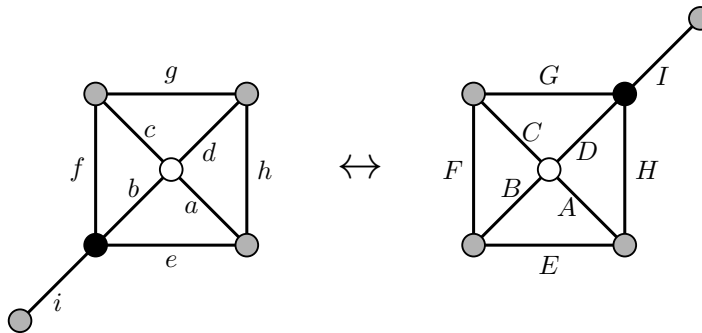
In this case the move is symmetric, so we only include formulas for one direction.

$$A = \frac{abg + aeg + afg + agi + beg - def}{g(b + e + f + i)}$$

$$B = \frac{bi}{b + e + f + i}$$

$$C = \frac{bch + bfh + ceh + cfh + chi - def}{h(b + e + f + i)}$$

$$D = \frac{d(bgh + def + efg + efh + egh + fgh + ghi)}{gh(b + e + f + i)}$$
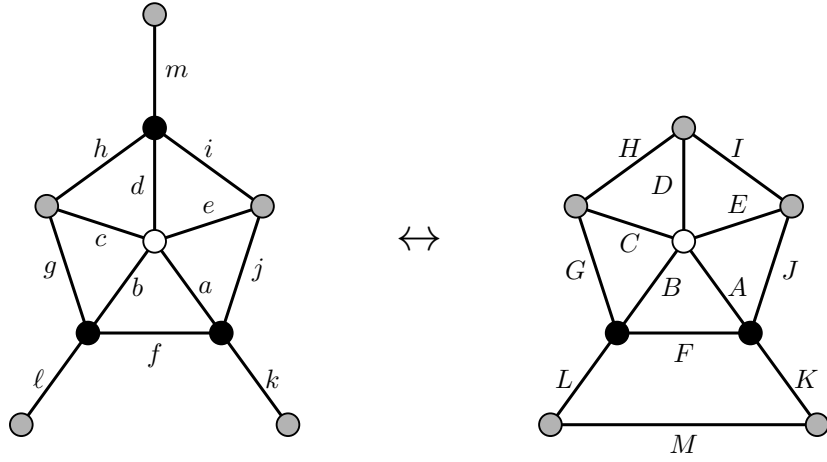
$$E = \frac{ei}{b + e + f + i}$$

$$F = \frac{fi}{b + e + f + i}$$

$$G = \frac{bgh + def + efg + efh + egh + fgh + ghi}{h(b + e + f + i)}$$

$$H = \frac{bgh + def + efg + efh + egh + fgh + ghi}{g(b + e + f + i)}$$

$$I = \frac{bgh + def + efg + efh + egh + fgh + ghi}{ef}$$

- Pentagon Conditional Local Move

Let $\alpha = ABM + AFM + AGM + ALM + BFM + BJM + BKM + FGM + FJM + FKL + FKM + FLM + GJM + GKM + JLM + KLM$, $\beta = FKL + FKM + FLM + KLM$, $\gamma = fgj(d + h + i + m)$, and $\delta = abhi + afhi + aghi + bfhi + bhij + fghi + fhij + ghij$.

$$a = \frac{A(\beta + BKM + GKM)}{\alpha}$$

$$b = \frac{B(\beta + ALM + JLM)}{\alpha}$$

$$c = C + \frac{ABGIM + AFGIM + BFGIM + BGIJM + BGIKM - DFGJM}{I\alpha}$$

$$d = D + \frac{D(DFGJM + HFGJM + IFGJM)}{IH\alpha}$$

$$e = E + \frac{ABHJM + AFHJM + AGHJM + AHJLM + BFHJM - DFGJM}{H\alpha}$$

24

$$f = \frac{(\beta + BKM + GKM)(\beta + ALM + JLM)}{LK\alpha}$$

$$g = \frac{G(\beta + ALM + JLM)}{\alpha}$$

$$h = H + \frac{DFGJM + HFGJM + IFGJM}{I\alpha}$$

$$i = I + \frac{DFGJM + HFGJM + IFGJM}{H\alpha}$$

$$j = \frac{J(\beta + BKM + GKM)}{\alpha}$$

$$k = \frac{\beta + BKM + GKM}{FL}$$

$$\ell = \frac{\beta + ALM + JLM}{KF}$$

$$m = \frac{HI\alpha + DFGJM + HFGJM + IFGJM}{FGJM}$$

$$A = a + \frac{a(\delta + ahi\ell + hij\ell)}{\gamma}$$

$$B = b + \frac{b(\delta + bhik + ghik)}{\gamma}$$

$$C = c + \frac{g(dfhj - abhi - afhi - bfhi - bhij - ihbk)}{\gamma}$$

$$D = \frac{dfgjm}{\gamma}$$

$$E = e + \frac{j(dfgi - abhi - afhi - aghi - ahi\ell - bfhi)}{\gamma}$$

$$F = \frac{f(\gamma + \delta + ahi\ell + hij\ell)(\gamma + \delta + bhik + ghik)}{\gamma(\gamma + \delta + ahi\ell + bhik + fhik + fhi\ell + ghik + hij\ell + hik\ell)}$$

$$G = g + \frac{g(\delta + bhik + ghik)}{\gamma}$$

$$H = \frac{fghjm}{\gamma}$$

$$I = \frac{fgijm}{\gamma}$$

$$J = j + \frac{j(\delta + ahi\ell + hij\ell)}{\gamma}$$

$$K = \frac{k(\gamma + \delta + ahi\ell + hij\ell)}{\gamma + \delta + ahi\ell + bhik + fhik + fhi\ell + ghik + hij\ell + hik\ell}$$

$$L = \frac{\ell(\delta + bhik + ghik)}{\gamma + \delta + ahi\ell + bhik + fhik + fhi\ell + ghik + hij\ell + hik\ell}$$

$$M = \frac{ghik\ell}{\gamma + \delta + ahi\ell + bhik + fhik + fhi\ell + ghik + hij\ell + hik\ell}$$

- Conjectured Hexagon Conditional Local Move
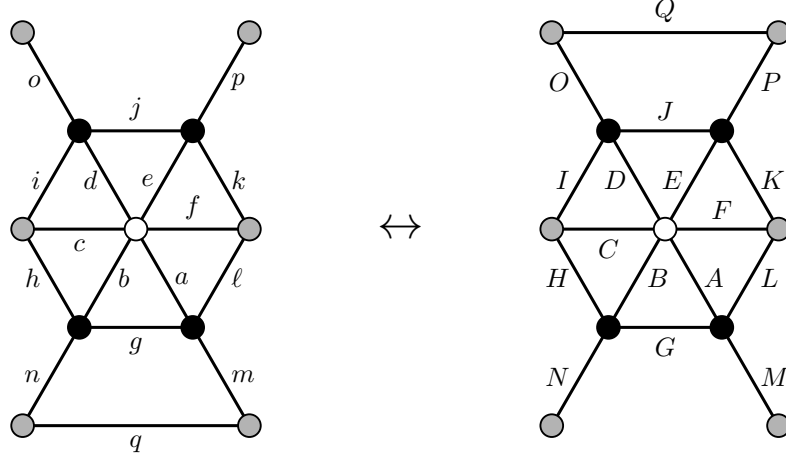
In this case the move is symmetric, so we only include formulas for one direction.

Let $\alpha = gmn + gmq + gnq + mnq, \beta = abq + agq + ahq + anq + bgq + b\ell q + bmq + ghq + g\ell q + h\ell q + hmq + \ell nq, \gamma = degh\ell q + dghj\ell q + dghk\ell q + eghi\ell q + eghj\ell q + ghij\ell q + ghik\ell q + ghijk\ell q$, and $\delta = dgh\ell pq + ghi\ell pq + egh\ell oq + ghk\ell oq + ghj\ell oq + ghj\ell pq + ghlopq$.

$$A = \frac{a(\alpha + bmq + hmq)}{\alpha + \beta}$$

$$B = \frac{b(\alpha + anq + \ell nq)}{\alpha + \beta}$$

$$C = c + \frac{hjk(abq + agq + bgq + b\ell q + bmq) - degh\ell q - dghj\ell q - dghk\ell q - dgh\ell pq - eghj\ell q}{jk(\alpha + \beta)}$$

$$D = d + \frac{d(\gamma + dgh\ell pq + ghi\ell pq)}{ijk(\alpha + \beta)}$$

$$E = e + \frac{e(\gamma + egh\ell oq + ghk\ell oq)}{ijk(\alpha + \beta)}$$

$$F = f + \frac{ij\ell(abq + agq + ahq + anq + bgq) - degh\ell q - dghj\ell q - eghi\ell q - eghj\ell q - egh\ell oq}{ij(\alpha + \beta)}$$

$$G = \frac{(\alpha + bmq + hmq)(\alpha + anq + \ell nq)}{mn(\alpha + \beta)}$$

$$H = \frac{h(\alpha + anq + \ell nq)}{\alpha + \beta}$$

$$I = i + \frac{\gamma + dgh\ell pq + ghi\ell pq}{jk(\alpha + \beta)}$$

$$J = \frac{(ijk(\alpha + \beta) + \gamma + dgh\ell pq + ghi\ell pq)(ijk(\alpha + \beta) + \gamma + egh\ell oq + ghk\ell oq)}{ik(\alpha + \beta)(ijk(\alpha + \beta) + \gamma + \delta)}$$

$$K = k + \frac{k(\gamma + egh\ell oq + ghk\ell oq)}{ijk(\alpha + \beta)}$$

$$L = \frac{\ell(\alpha + bmq + hmq)}{\alpha + \beta}$$

$$M = \frac{\alpha + bmq + hmq}{gn}$$

$$N = \frac{\alpha + anq + \ell nq}{gm}$$

$$O = \frac{o(ijk(\alpha + \beta) + \gamma + dgh\ell pq + ghi\ell pq)}{ijk(\alpha + \beta) + \gamma + \delta}$$

$$P = \frac{p(ijk(\alpha + \beta) + \gamma + egh\ell oq + ghk\ell oq)}{ijk(\alpha + \beta) + \gamma + \delta}$$

$$Q = \frac{hgj\ell opq}{ijk(\alpha + \beta) + \gamma + \delta}$$

References

[1] E.B. Curtis, D. Ingerman, and J.A. Morrow. "Circular Planar Graphs and Resistor Networks". In: *Linear Algebra and its Applications* 283 (1998), pp. 115 –150.

[2] E.B. Curtis, E. Mooers, and J.A. Morrow. "Finding the Conductors in Circular Networks from Boundary Measurments". In: *Linear Algebra and its Applications* 28 (1994), pp. 781–814.

[3] Y.C. de Verdiere, I. Gitler, and D. Vertigan. "Reseaux electriques planaires II". In: *Comment. Math. Helv.* 71.1 (1996), pp. 144–167.

[4] R. Kenyon. "The Laplacian on planar graphs and graphs on surfaces". In: *ArXiv e-prints* (Mar. 2012). arXiv: `1203.1256 [math.PR]`.

[5] R. Kenyon and D. Wilson. "Boundary partitions in trees and dimers". In: *Trans. Amer. Math. Soc.* 363.3 (2011), 13251364.

[6] R. Kenyon and D. Wilson. "Combinatorics of tripartite boundary connections for trees and dimers". In: *Electron. J. Combin.* 16.1 (2009).

[7] T. Lam and P. Pylyavskyy. "Inverse problem in cylindrical electrical networks". In: *ArXiv e-prints* (Apr. 2011). arXiv: `1104.4998 [math.CO]`.