VIRTUAL RESOLUTIONS OF GENERAL POINTS IN SMOOTH FANO TORIC VARIETIES

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1. Introduction

Virtual Resolutions of products of projective spaces were introduced by Berkesch, Erman and Smith in [BES17] as a generalization of minimal free resolutions. They proved for products of projective spaces, a virtual resolution of length less than or equal to the dimension can always be found. We extend their ideas to general points living inside several smooth Fano toric varieties. Within certain spaces, not only is there always a virtual resolution of length 3, but it can be chosen to be Koszul.

2. Background

Definition 2.1. Recall that a toric variety is an irreducible variety V that contains a torus $T \cong (\mathbb{C}^*)^m$ as a Zariski open subset such that the action of T on itself extends to an algebraic action of T on V.

A general toric variety X can be constructed from a fan Σ. Let B denote the irrelevant ideal of X. If $\Sigma(1)$ is the set of rays in Σ, define the Cox ring as

$$S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)].$$

Smooth Fano toric varieties are a well-studied special class of toric varieties that are supported by Macaulay2. In this report, we name our smooth Fano toric varieties according to their catalog in Macaulay2, which ultimately comes from the classification of the Fano varieties in [Bat99]. When write SmoothFanoToricVariety(x, y), or SFTV(x, y) for short, we mean the y-th x-dimensional smooth Fano toric variety according to the Macaulay2 indexing.

Definition 2.2. [BES17, Definition 1.1] A free complex $F := [F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots]$ of Pic(X)-graded S-modules is called a ***virtual resolution*** of a Pic (X)-graded S-module M if the corresponding complex $\tilde{F}$ of vector bundles on X is a locally-free resolution of the sheaf $\tilde{M}$.

Equivalently, a free complex $F := [F_0 \longleftarrow F_1 \longleftarrow F_2 \longleftarrow \cdots]$ of Pic (X)-graded S-modules is called a ***virtual resolution*** of a Pic(X)-graded S-module if the following two properties hold:

- $\text{ann} H^i F : B^\infty = 0$ for $i > 0$, and
- $H^0 F : B^\infty = I : B^\infty$

The following example from [BES17, Example 5.10] motivates our construction.

Let $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the subscheme consisting of m general points and let I be the corresponding B-saturated S-ideal. We can construct a Koszul complex which is a virtual resolution of S/I.

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Note that $\dim H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_X(i,j)) = (i+1)(j+1)$, the generality of the points implies that $\dim H^0(\mathbb{P} \times \mathbb{P}, \mathcal{O}_Z(i,j)) = \min\{(i+1)(j+1), m\}$. Therefore, when $m = 2k$ for some $k \in \mathbb{N}$, then two independent global sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,k)$ vanish on Z. Using this pair, we obtain a virtual resolution of S/I of the form

$$S \longleftarrow S(-1,-k)^2 \longleftarrow S(-2,-2k) \longleftarrow 0.$$

When $m = 2k+1$, then there are independent global sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,k)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,k_1)$ that vanish on Z. Then

$$S \longleftarrow S(-1,-k) \oplus S(-1,-k-1) \longleftarrow S(-2,-2k-1) \longleftarrow 0$$

is a virtual resolution of S/I.

Macaulay2 has a database of smooth Fano Toric Varieties. For dimension up to 4, it uses Victor Batryev's classification from [Bat99].

3. Results

We begin with the following lemma which shows the complexes we construct are indeed virtual resolutions.

Lemma 3.1. Let X be a toric variety with Cox ring S. Let I be an ideal of a finite set of points with generators $f_1, f_2, \dots f_m$. Let K be the Koszul complex generated by some subset of the generators $f_{i_1}, f_{i_2}, \dots f_{i_k}$. If the Hilbert Polynomial of K equals the Hilbert Polynomial of the minimal free resolution of S/I, then K is a virtual resolution of S/I.

Proof. We first claim there is a map from the Koszul complex to the free resolution of S/I. To see this, consider the Koszul complex generated by the first form f_{i_1}. This maps to the minimal free resolution by inclusion. Similarly, we may do so for every f_{i_j}. Since the Koszul complex is defined by tensoring, the map is the blowout map arising from this construction.

We claim because the Hilbert Polynomials match, both K and the minimal free resolution give line bundle resolutions of S/I. Construct the mapping cone of these two complexes. Then there is an exact triangle among the Koszul complex, free resolution, and mapping cone. The equality of the Hilbert Polynomials implies the mapping cone has irrelevant homology, and therefore there is an isomorphism of homology between the two complexes. Therefore, they are both line bundle resolutions of S/I. By definition, the K is a virtual resolution of S/I. ■

3.1. SFTV(3, 3).

Let $X = \text{SmoothFanoToricVariety(3,3)}$ (abbreviated SFTV(3, 3)) The variables of the Cox ring of have $\mathbb{Z}^2$-multidegrees given by the following matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 \end{bmatrix}.$$

For $i, j \geq 0$, counting independent forms of the correct degree in the Cox ring, yields the formula

$$\dim H^0(X, \mathcal{O}_X(i,j)) = \sum_{k=0}^{i} (i+1-k)(j+1+k).$$

Let the columns of this matrix be labeled by $x_0, x_1, \dots, x_4$. Choose the multiplicity k of x_1, so that there are $i+1-k$ choices for the multiplicity of x_0 and x_2. In turn, there are $j+1+k$ choices for the multiplicities of x_3 and x_4. Summing over all choices of k, from 0 to a maximum of i, yields the formula. Note that

$$\dim H^0(X, \mathcal{O}_X(1,m)) = 3m + 4.$$

Let Z be a subscheme consisting of k general points and I the corresponding B-saturated ideal. It requires k forms to cut out the k general points. Therefore, when $k = 3m+1$, the generality of points implies that three independent global sections of $\mathcal{O}_X(1,m)$ vanish on Z. Using these three forms, we calculate its Koszul complex to be

$$S \longleftarrow S(-1,-m)^3 \longleftarrow S(-2,-2m)^3 \longleftarrow S(-3,-3m) \longleftarrow 0.$$

To show this is a virtual resolution we must calculate the Hilbert function $\dim_{\mathbb{C}}[S/I]_d$. Note that

$$\dim_{\mathbb{C}}[S/I]_d = \dim_{\mathbb{C}} S_d - 3 \cdot \dim_{\mathbb{C}}[S(-1,-m)]_d + 3 \cdot \dim_{\mathbb{C}}[S(-2,-2m)] - \dim_{\mathbb{C}}[S(-3,-3m)]$$

where if $d = (d_1, d_2)$ then $\dim_{\mathbb{C}}[S(-i,-j)]_d = \dim H^0(X, \mathcal{O}_X(d_1 - i, d_2 - j))$. Using Mathematica [Wol10] we found that this equals the desired $3m+1$.

Lemma 3.1 then implies that the Koszul complex is indeed a virtual resolution of S/I.

When $k = 3m+2$ the generality of points implies that two independent global sections f_1, f_2 of $\mathcal{O}_X(1,m)$ vanish on Z. Similarly, there are five independent global sections of $\mathcal{O}_X(1,m+1)$ that vanish on Z, but four of these sections are given by $x_3f_1, x_4f_1, x_3f_2, x_4f_2$; let the remaining section be f_3. The Koszul complex of the ideal (f_1, f_2, f_3) is

$$S \longleftarrow \begin{matrix} S(-1,-m-1) \\ \oplus \\ S(-1,-m)^2 \end{matrix} \longleftarrow \begin{matrix} S(-2,-2m) \\ \oplus \\ S(-2,-2m-1)^2 \end{matrix} \longleftarrow S(-3,-3m-1) \longleftarrow 0$$

We may similarly use this resolution to compute $\dim_{\mathbb{C}}[S/I]_d = 3m+2$, demonstrating that this Koszul complex is a virtual resolution of Z.

Finally, when $k = 3m+3$, generality of the points yields one independent global section f_1 of $\mathcal{O}_X(1,m)$ and four independent global sections x_3f_1, x_4f_1, f_2, f_3 of $\mathcal{O}_X(1,m+1)$ that vanish on Z. The Koszul complex of (f_1, f_2, f_3) is

$$S \longleftarrow \begin{matrix} S(-1,-m-1)^2 \\ \oplus \\ S(-1,-m) \end{matrix} \longleftarrow \begin{matrix} S(-2,-2m-1)^2 \\ \oplus \\ S(-2,-2m-2) \end{matrix} \longleftarrow S(-3,-3m-2) \longleftarrow 0$$

Lastly, we may use this resolution to compute that $\dim_{\mathbb{C}}[S/I]_d = 3m+3$. Therefore, by Lemma 3.1, this Koszul complex is a virtual resolution of Z.

3.2. **SFTV (3, 4).** SFTV(3, 4) is $\mathbb{Z}^2$-graded and has a Cox ring with 5 variables. The variables $x_0, \ldots, x_4$ have multidegrees

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

In fact, this is simply $\mathbb{P}^2 \times \mathbb{P}^1$. Note this is covered in [BES17], however, for the sense of completion we include it here. Note that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}(i,j)) = \binom{i+2}{2}(j+1).$$

In particular, note that when $i = 1$ this formula becomes $3j+3$. Let Z be a 0-dimensional subscheme of X containing $3m$ points. Then there exist 3 independent global sections f_1, f_2, f_3 of $\mathcal{O}_X(1, m)$ that vanish on Z. The Koszul complex of (f_1, f_2, f_3) has multigraded Betti numbers

$$\beta_{0,(0,0)} = 1, \beta_{1,(1,m)} = 3, \beta_{2,(2,2m)} = 3, \beta_{3,(3,3m)} = 1$$

Doing the usual alternating sum calculation yields that

$$\dim_{\mathbb{C}}[S/I]_d = 3m$$

as desired. For sets of $3m+1$ points, we choose two independent global sections of $\mathcal{O}_X(1, m)$ and one independent global section of $\mathcal{O}_X(1, m + 1)$. For sets of $3m + 2$ points, we choose one independent global section of $\mathcal{O}_X(1, m)$ and two independent global sections of $\mathcal{O}_X(1, m + 1)$. The respective Koszul complexes of these two sets of three forms is a virtual resolution of $I(Z)$.

3.3. **SFTV (3, 5).** SFTV(3, 5) is $\mathbb{Z}^3$-graded and has a Cox ring with 6 variables. The variables $x_0, \ldots, x_5$ have multidegrees

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Summing over all choices for the multiplicity of x_4, we obtain

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}(i, j, k)) = \sum_{n=0}^{k} (i + 1 + n)(j + 1 + n)$$

for $i, j, k \geq 0$. In particular,

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(0, m, 1)) = 3m + 5$$

Let Z be a 0-dimensional subscheme of X containing $3m+2$ points. Then there exist 3 independent global sections f_1, f_2, f_3 of $\mathcal{O}_X(0, m, 1)$ that vanish on Z. The Koszul complex of (f_1, f_2, f_3) has multigraded Betti numbers

$$\beta_{0,(0,0,0)} = 1, \beta_{1,(0,m,1)} = 3, \beta_{2,(0,2m,2)} = 3, \beta_{3,(0,3m,3)} = 1$$

We may compute the alternating sum

$$\dim_{\mathbb{C}}[S/I]_d = \dim_{\mathbb{C}} S_d - 3 \cdot \dim_{\mathbb{C}}[S(0, -m, -1)]_d + 3 \cdot \dim_{\mathbb{C}}[S(0, -2m, -2)] - \dim_{\mathbb{C}}[S(0, -3m, -3)] = 3m + 2$$

which shows that this Koszul complex is a virtual resolution of $I(Z)$.

For sets of $3m + 3$ points, we choose two independent global sections of $\mathcal{O}_X(0, m, 1)$ and one independent global section of $\mathcal{O}_X(0, m+1, 1)$. For sets of $3m+4$ points, we choose one independent global section of $\mathcal{O}_X(0, m, 1)$ and two independent global sections of $\mathcal{O}_X(0, m + 1, 1)$. By the same dimension-counting argument, the respective Koszul complexes of these two sets of three forms is a virtual resolution of $I(Z)$.

3.4. **SFTV (3, 6).** SFTV(3, 6) is $\mathbb{Z}^3$-graded, with variable degrees given by

$$\begin{bmatrix} 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and the Hilbert function is given by

$$\dim H^0(X, \mathcal{O}_X(i,j,k)) = \sum_{p=0}^{k} \sum_{q=0}^{p+j} (q+1+i)$$

so that

$$\dim H^0(X, \mathcal{O}_X(m,0,1)) = 3m+4.$$

Let Z be a finite subscheme of general points. When Z contains $3m+1$ points, generality implies there exist three independent global sections f_1, f_2, f_3 of $\mathcal{O}_X(m,0,1)$ that vanish on Z. The Koszul complex K of (f_1, f_2, f_3) is

$$S \longleftarrow S(-m,0,-1)^3 \longleftarrow S(-2m,0,-2)^3 \longleftarrow S(-3m,0,-3) \longleftarrow 0.$$

Using the same conventions as before, we compute via Mathematica

$$\begin{aligned} \dim_{\mathbb{C}}[S/I]_d &= \dim_{\mathbb{C}} S_d - 3\dim_{\mathbb{C}}[S(-m,0,-1)]_d + 3 \cdot \dim_{\mathbb{C}}[S(-2m,0,-2)]_d - \dim_{\mathbb{C}}[S(-3m,0,-3)]_d \\ &= 3m+1, \end{aligned}$$

for d sufficiently positive, demonstrating that K is a virtual resolution of Z. If Z contains $3m+2$ points, then we choose two independent forms of multidegree $(m,0,1)$ and one form of degree $(m+1,0,1)$; taking the Koszul complex of the resulting ideal I and computing $\dim_{\mathbb{C}}[S/I]_d = 3m+2$ shows that we have found a virtual resolution. Similarly, if Z has $3m+3$ points, choose one independent form of multidegree $(m,0,1)$ and two of degree $(m+1,0,1)$, and the computation still goes through.

3.5. **SFTV(3, 8).** Similar procedures also suffice to obtain virtual resolutions for sets of general points in SFTV(3, 8), SFTV(3, 9), and SFTV(4,5). The process for generating the virtual resolutions is exactly like the previous cases already describe, so in the remaining sections we provide only the information specific to each variety.

The degree matrix of SFTV(3, 8) is

$$\begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and its Hilbert function is

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(i,j,k) = (k+1) \sum_{n=0}^{j} (i+1+n)$$

so that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(0,-1,-m) = 3m+4.$$

When $|Z| = 3m+1$, the virtual resolution is obtained by the Koszul complex of three independent forms of multidegree $(0,1,m)$ vanishing on Z. When $|Z| = 3m+2$, we choose two independent forms of degree $(0,1,m)$ and one of $(0,1,m+1)$, and when $|Z| = 3m+3$ we choose one independent form of degree $(0,1,m)$ and two of degree $(0,1,m+1)$. In all three of these cases, the alternating sum from the complex yields is constant and equal to $|Z|$.

3.6. **SFTV(3, 9).** The degree matrix of SFTV(3, 9) is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

and its Hilbert function is

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(i,j,k)) = \sum_{n=0}^{j} (i+1+n)(k+1+j-n),$$

so that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(0,-1,-m)) = 4m+3$$

and the same choices of forms corresponding to the number of points in Z works as in SFTV(3, 8).

3.7. **SFTV(4, 5).** The strategy we have employed is not limited to 3-folds. For example, If Z is a set of points in SFTV(4, 5), we can always find a Koszul complex of four forms to give a virtual resolution of $I(Z)$ using a similar algorithm as the previous five varieties.

The degrees of the six variables in the Cox ring of SFTV(4, 5) are

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Note the similarity to SFTV(3, 3). The Hilbert function is given by

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(i,j)) = \sum_{n=0}^{i} (j+k+1)\binom{i-n+2}{2}$$

so that

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(1,m)) = 4m+5$$

When $|Z| = 4m+1$, genericity of points implies that there exist four independent global sections of $\mathcal{O}_X(1,m)$ vanishing on Z. The Koszul complex of these four sections has multigraded Betti numbers

$$\beta_{0,(0,0)} = 1, \beta_{1,(1,m)} = 4, \beta_{2,(2,2m)} = 6, \beta_{3,(3,3m)} = 4, \beta_{4,(4,4m)} = 1$$

so that the alternating sum of the complex is

$$\dim_{\mathbb{C}}[S]_d - 4\dim_{\mathbb{C}}[S(-1,-m)]_d + 6\dim_{\mathbb{C}}[S(-2,-2m)]_d - 4\dim_{\mathbb{C}}[S(-3,-3m)]_d + \dim_{\mathbb{C}}[S(-4,-4)]_d$$

which simplifies to $4m+1$, showing that the Koszul complex is a virtual resolution of $I(Z)$.

For $|Z| = 4m+2$, there are three independent forms f_1, f_2, f_3 of degree $(1,m)$ vanishing on Z. There are 7 independent forms of degree $(1,m+1)$ vanishing on Z: they are $x_4f_1, x_5f_1, x_4f_2, x_5f_2, x_4f_3, x_5f_3,$

and f_4. Then the Koszul complex of (f_1, f_2, f_3, f_4) yields a virtual resolution of $I(Z)$ via the same computation as before.

For $|Z| = 4m+3$, we take two independent forms of degree $(1, m)$ and two of degree $(1, m+1)$; for $|Z| = 4m+4$ we take one form of degree $(1, m)$ and three of degree $(1, m+1)$. The same computation goes through to show that the respective Koszul complexes are virtual resolutions of $I(Z)$.

4. Comments

An astute reader may ask why this construction isn't performed for SFTV(3, 1) or SFTV(3, 2). (Note that SFTV(3, 0) is simply $\mathbb{P}^3$ and SFTV(3, 4) is $\mathbb{P}^1 \times \mathbb{P}^2$ and therefore already addressed in [BES17].) The two matrices below, respectively, are the multidegrees:

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -2 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 \end{bmatrix}.$$

These yield the following respective formulas for $\dim H^0(X, \mathcal{O}_X(i,j))$ for the two spaces:

$$\sum_{k=0}^{i} \binom{2k+j+2}{2}, \quad \sum_{k=0}^{i} \binom{k+j+2}{2}.$$

Note that $\dim H^0(X, \mathcal{O}_X(i,j))$ grows nonlinearly with respect to any choice of (i,j). Not only will this method of proof not work for these spaces, but we strongly believe based on computations in Macaulay2 that we need more than 3 forms to generate a virtual resolution of S/I in these spaces.

Call smooth Fano toric varieties *linear* if $\dim H^0(X, \mathcal{O}_X)$ grows linearly with respect to some choice of the grading. We hope that a general choice for all such linear SFTV's could be found. In particular, whenever X is a product of any toric variety with $\mathbb{P}^1$ is linear. However, the "natural" choice for SFTV(3, 10) (meaning 0 in one variable, 1 in another, and m or $m+1$ in the third) fails. We were unable to find a choice of 3 generators which does work for SFTV(3, 10). Therefore, linearity is not a sufficient condition for the heuristic construction to hold.

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