SMITH INVARIANTS OF UD AND DU LINEAR OPERATORS IN DIFFERENTIAL POSETS

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Abstract. In this report we explore the Smith invariants of certain maps in differential posets. These maps correspond to induction-restriction and restriction-induction maps when the poset is viewed in a representation theoretic way.

1. Introduction

Definition 1. [S1] For r a positive integer, a poset P is called *r-differential* if it satisfies the following:

(1) P is locally finite, graded, and has a $\hat{0}$ element.
(2) If $x \neq y$ are in P and exactly k elements of P are covered by both x and y, then exactly k elements of P cover both x and y.
(3) If $x \in P$ covers exactly k elements, then it is covered by exactly $k + r$ elements.

Given a locally finite poset P, we define two linear operators, called *up* and *down* maps; these map to and from KP, the K-vector space with basis P.

Definition 2. Let P be a locally finite poset and $x \in P$. Then the *up* and *down* maps U and D are defined by

$$Ux = \sum_{y \succ x} y \qquad \text{and} \qquad Dx = \sum_{z \prec x} z,$$

where $t \succ x$ denotes x is covered by t in P.

The following well-known theorem relates these two definitions, and is straightforward to prove.

Theorem 3. [S1] *Let P be a locally finite graded poset with a $\hat{0}$. If r is a positive integer, and P has only finitely many elements of each rank, then*

$$P \text{ is } r\text{-differential} \quad \text{if and only if} \quad DU - UD = rI.$$

A prototypical example of a 1-differential poset is *Young's lattice*, denoted by Y. This is the set of all partitions $\mathcal{P}$, ordered by inclusion of Young diagrams. We illustrate this poset in Figure 1.

The main focus of this paper is on the *Smith normal form*, abbreviated Snf, of DU and UD maps in r-differential posets. Because of this, we now go over some basic theory and language of the subject.

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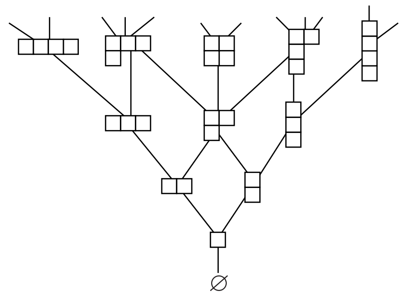

FIGURE 1. Young's lattice.

Definition 4. A unimodular matrix is a square integral matrix that has an integral inverse.

It is a standard exercise to show that an integral matrix is unimodular if and only if it has determinant ± 1, the units of $\mathbb{Z}$.

Definition 5. A (possibly rectangular) diagonal matrix D is a diagonal form for a matrix A if there exist unimodular matrices R and C such that $D = RAC$. It is called the *Smith normal form* of A if the diagonal entries $d_{11}, d_{22}, \ldots$ of D are non-negative and $d_{ii} \mid d_{jj}$ for all $i \leq j$; in this case, we say the *Smith entries* of A are $s_i = d_{ii}$.

For an integral matrix A, let $d_i(A)$ be the greatest common divisor of the determinants of all the $i \times i$ minors of A, where $d_i(A) = 0$ if all such $i \times i$ determinants are zero. The number $d_k(A)$ is called the kth *determinantal divisor* of A. The following is quite useful when studying the Snf.

Theorem 6. *The Smith normal form entries* $(s_1, s_2, \ldots)$ *of a matrix* A *are given by the equation*

$$s_j(A) = \frac{d_j(A)}{d_{j-1}(A)},$$

where $d_0(A)$ *is taken to be 1.*

The following new definition is central to our conjectures.

Definition 7. Let M be a nonsingular integral $n \times n$ matrix, and E be the multiset of eigenvalues for M. Define E_0 to be the largest subset of E that is not a multiset, and E_i to be the largest subset of $E - E_0 - \ldots - E_{i-1}$ that is not a multiset, for $i \geq 1$.

We say M possesses the ι-λ *relation*, or is of ι-λ-type, if its Smith normal form entries $\{s_i\}$ are given by

$$s_{n-k} = \prod_{e \in E_k} e,$$

where $0 \leq k \leq n-1$, and we take s_{n-j} to be 1 if $E_j = \emptyset$.

Example 8. A matrix with eigenvalues (superscripts denoting multiplicity)

$$\{1^8, 2^7, 3^4, 4^4, 5^2, 6^2, 7^1, 8^1, 10^1\}$$

and invariant factors

$$\{1^{23}, 2^3, 24^2, 720^1, 403200^1\}$$

is of ι-λ-type. Using the notation of Definition 7,

$E_0 = \{1, 2, \ldots, 8, 10\}$, $E_1 = \{1, \ldots, 6\}$, $E_2 = E_3 = \{1, \ldots, 4\}, E_4 = E_5 = E_6 = \{1, 2\}$, and $E_i = \{1\}$ for remaining i.

We are now in a position to state the main conjectures of our paper. For a graded poset P and nonnegative integer n, we denote the set of all elements of rank n by P_n, and the cardinality of this set by p_n.

Conjecture 9 (D Conjecture)**.** *If P is an r-differential poset, the linear map $D : \mathbb{C}P_n \to \mathbb{C}P_{n-1}$ has rank p_{n-1}, and its Snf contains only 1's and 0's.*

Conjecture 10 (UD Conjecture)**.** *If P is an r-differential poset, the linear map $UD : \mathbb{C}P_n \to \mathbb{C}P_n$ has rank p_{n-1}, and its Snf contains only 1's and 0's.*

Conjecture 11 (DU Conjecture)**.** *If P is an r-differential poset, the linear map $DU : \mathbb{C}P_n \to \mathbb{C}P_n$ possesses the ι-λ relation for each $n \geq 0$.*

One should note that the rank assertions in Conjectures 9 and 10 follow from the injectivity of U [S1].

In general, the eigenvalues of the DU map in an r-differential poset are very nice, which conjecturally make the Smith invariants nice and easy to compute. Let $\mathrm{Ch}(A)$ denote the characteristic polynomial of A in variable λ.

Theorem 12. [S1] *Let P be an r-differential poset and let $n \in \mathbb{N}$. Then*

$$\mathrm{Ch}(DU_n) = \prod_{i=0}^{n} (\lambda - r(i+1))^{\Delta p_{n-i}},$$

where $\Delta p_n = p_n - p_{n-1}$.

2. Results in Young's lattice

With Young's lattice, we can actually obtain a tight lower bound on the number of Smith invariants of DU_n equal to 1, and say what the last Smith invariant is. Moreover, in this setting, Conjecture 10 is a theorem. For future reference, we now rewrite Theorem 12 and Conjecture 11 for the special case of Y being our poset. We also define an ordering that we will often use.

Theorem 13. *For $n \geq 0$, the eigenvalues of $DU_n : \mathbb{C}Y_n \to \mathbb{C}Y_n$ are*

eigenvalue	*multiplicity*
$n+1$	1
$n-k$	$p(k+1) - p(k)$,

where $1 \leq k \leq n-1$, and $p(k)$ is the number of partitions of k.

Conjecture 14. *For an integer $n \geq 1$, we have that the Smith normal form entries of DU_n are*

entry	*multiplicity*
$(n+1)[(n-1)!]$	1
$(n-k)!$	$p(k+1) - 2p(k) + p(k-1)$
1	$p(n) - p(n-1) + p(n-2)$,

where $3 \leq k \leq n-2$.

Definition 15. For partitions λ and $\tilde{\lambda}$, we say $\lambda \leq \tilde{\lambda}$ *lexicographically* if $\lambda = \tilde{\lambda}$ or the first i for which $\lambda_i \neq \tilde{\lambda}_i$ has $\lambda_i < \tilde{\lambda}_i$.

In this section, we will assume basic knowledge of the representation theory of $\mathfrak{S}_n$, and its language in terms of symmetric functions. To brush up on such material, see [S3], [F] or [M]. Here we will also warn the reader of some notational abuses. Unfortunately, we use s's for Smith entries and Schur polynomials, and p's for ranks, partitions, and power sums. However, in most cases the meaning should be clear from the context. One additional notation we will mention is that we bracket linear maps when emphasizing we are thinking in matrix form. We also place a subscript j on DU and UD maps to make clear P_j is its domain, if it is not already clear.

It turns out that differential posets in general are associated with towers of algebras [G-H-J]; in this setting, induction and restriction play the roles of U and D, respectively. In Young's lattice, our towers are very nice:

$$\mathbb{C}\mathfrak{S}_0 \subset \mathbb{C}\mathfrak{S}_1 \subset \mathbb{C}\mathfrak{S}_2 \subset \cdots .$$

In this situation, our poset Y consists of the irreducible characters of symmetric groups. Instead of $\lambda \in Y$, as in the previous section, we have the irreducible character χ^λ of the Specht module S^λ indexed by λ. Moreover, our covering relation is defined by

$$\chi^\lambda \succ \chi^{\tilde{\lambda}}$$

if S^λ is a summand of $\mathrm{ind}_{\mathfrak{S}_n}^{\mathfrak{S}_{n+1}} S^{\tilde{\lambda}}$. That is, our *up* and *down* operators are induction and restriction. By *the branching rule*, it is obvious that our two descriptions of Y agree.

To help us realize the power of looking at Y through this representation theoretic lens, we use symmetric functions.

Let $R^n_{\mathbb{C}}$ be the space of complex class functions on $\mathfrak{S}_n$, let

$$\Lambda^n_{\mathbb{C}} = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$$

be the space of symmetric polynomials of degree n, and

$$R_{\mathbb{C}} = \bigoplus_{n\geq 0} R^n_{\mathbb{C}} \qquad \text{and} \qquad \Lambda_{\mathbb{C}} = \bigoplus_{n\geq 0} \Lambda^n_{\mathbb{C}}$$

be corresponding graded rings. Likewise, we let R^n be the $\mathbb{Z}$-module generated by the irreducible characters of $\mathfrak{S}_n$ and

$$\Lambda^n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$$

be *the* symmetric polynomials of degree n, with corresponding graded rings defined analogously:

$$R = \bigoplus_{n\geq 0} R^n \qquad \text{and} \qquad \Lambda = \bigoplus_{n\geq 0} \Lambda^n.$$

One can then endow R with a multiplication, which we omit, to obtain the following result.

Lemma 16. *The* characteristic *map* $\mathrm{ch} : R_{\mathbb{C}} \to \Lambda_{\mathbb{C}}$*, defined by*

$$\mathrm{ch}(f) = \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} f(\pi) p_\pi,$$

where $p_\pi = p_\mu$ *and* $\mu \vdash n$ *is the cycle type of* π*, is an isometric isomorphism of graded algebras. Moreover, it restricts to an isometric isomorphism of* R *onto* Λ*.*

It is then known [S1] that the up and down maps in R, given by induction and restriction, correspond to multiplication by the 1*st power sum symmetric function* p_1 and applying the linear operator $\partial/\partial p_1$, respectively, in Λ. With this, along with Lemma 16 in mind, it is not too hard to see the following important result:

Theorem 17. [S1] *The eigenvectors for $DU : \mathbb{C}Y_n \to \mathbb{C}Y_n$ are given by*

$$X_\mu = \sum_{\lambda \vdash n} \chi^\lambda(\mu)\lambda,$$

each belonging to the eigenvalue $\#\{$parts of μ equal to $1\} + 1$, where μ is any partition of n.

We are now able to prove a tight lower bound for the number of 1's in the Snf of $DU_n : \mathbb{C}Y_n \to \mathbb{C}Y_n$, which is conjectured to be $p(n) - p(n-1) + p(n-2)$.

Theorem 18. *There are at least $p(n) - p(n-1) + p(n-2)$ invariant factors of $DU_n : \mathbb{C}Y_n \to \mathbb{C}Y_n$ equal to 1.*

Proof. Recall that the *complete symmetric polynomials* $\{h_\lambda\}_{\lambda \vdash n}$ form a $\mathbb{Z}$-basis of Λ^n. Letting $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash n$, we have

$$\begin{aligned} DU(h_\lambda) &= \frac{\partial}{\partial p_1} p_1 h_\lambda \\ &= \frac{\partial}{\partial p_1} h_{(\lambda,1)} \\ &= h_\lambda + h_1 \sum_{i=1}^{\ell} h_{(\lambda_1,\ldots,\lambda_i - 1,\ldots,\lambda_\ell)} \\ &= h_\lambda + \sum_{i=1}^{\ell} h_{(\lambda_1,\ldots,\lambda_i-1,\ldots,\lambda_\ell,1)}, \end{aligned} \tag{1}$$

where in the last step we are using $p_1 = h_1$ and $\partial/\partial p_1 h_n = h_{n-1}$.

Let $[DU_n]_h$ denote the matrix of DU_n in the h basis, with rows and columns indexed by the partitions of n.

For $\lambda_1, \lambda_2 \vdash n$, we define $\lambda_1 <_h \lambda_2$ if

$$\#\{\text{1-parts in } \lambda_1\} < \#\{\text{1-parts in } \lambda_2\}$$

or

$$\#\{\text{1-parts in } \lambda_1\} = \#\{\text{1-parts in } \lambda_2\} \quad \text{and} \quad \lambda_1 > \lambda_2 \text{ in lexicographic order.}$$

Order the indexing λ's of $[DU_n]_h$ from left to right, top to bottom, in increasing order with $<_h$. One can then see $[DU_n]_h$ is given by

$$\begin{bmatrix} I_{p(n)-p(n-1)} & 0 \\ & \\ * & [DU_{n-1}] + I \end{bmatrix}. \tag{2}$$

Thus, it suffices to show there are at least $p(n-2)$ invariant factors of $DU_{n-1} + I$ equal to 1.

Let $[DU_{n-1}+I]_s$ be in the usual basis, indexed by the partitions from left to right, top to bottom, in decreasing lexicographical order. Consider now the submatrix of $[DU_{n-1} + I]_s$ whose columns are indexed by

$$\{\lambda \vdash n-1 \mid \lambda \text{ has a 1-part}\},$$

and whose rows are indexed by the conjugate set

$$\{\lambda \vdash n-1 \mid \lambda \text{ has exactly 1 largest part}\}.$$

One can then see this submatrix is lower triangular, with 1's down its diagonal. $\square$

Example 19. Consider the map $DU_5 : \mathbb{C}Y_5 \to \mathbb{C}Y_5$. We have $[DU_5]_s$ and $[DU_5]_h$ are given by

$$\begin{array}{c} \\ 5 \\ 41 \\ 32 \\ 31^2 \\ 2^21 \\ 21^3 \\ 1^5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} 5 & 41 & 32 & 31^2 & 2^21 & 21^3 & 1^5 \end{array} \\ \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c} \\ 5 \\ 32 \\ 41 \\ 2^21 \\ 31^2 \\ 21^3 \\ 1^5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} 5 & 32 & 41 & 2^21 & 31^2 & 21^3 & 1^5 \end{array} \\ \begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 2 & & & & \\ 0 & 1 & 0 & 2 & & & \\ 0 & 1 & 1 & 0 & 3 & & \\ 0 & 0 & 0 & 2 & 1 & 4 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{pmatrix} \end{array},$$

respectively.

We can also determine the last Smith entry $s_{p(n)}$ of DU_n, which matches the conjecture. In the proof, we will use the fact that if A is a nonsingular matrix, its largest Smith entry is the smallest integer m such that

$$mA^{-1}$$

is an integral matrix; this simply follows from Theorem 6 together with Cramer's rule. For a partition λ, we let

$$z(\lambda) = \prod_{r \geq 1} r^{m_r} m_r!,$$

where m_r is the number of times r occurs in λ. We will also use χ^λ_μ and $\chi^\lambda(\mu)$ interchangeably.

Theorem 20. *The largest Smith entry of $DU_n : \mathbb{C}Y_n \to \mathbb{C}Y_n$ is $(n-1)!(n+1)$.*

Proof. Let $\lambda_1, \ldots, \lambda_{p(n)}$ be the partitions of n, and define

$$d(\mu) = 1 + \#\{\text{1-parts of } \mu\}.$$

Letting M be the matrix $(\chi^{\lambda_i}_{\lambda_j})$, Theorem 17 says

$$(M^{-1}[DU_n]_s M)_{ij} = \begin{cases} 0 & \text{if } i \neq j; \\ d(\lambda_i) & \text{if } i = j. \end{cases}$$

Since

$$s_\lambda = \sum_\mu \frac{1}{z(\mu)} \chi^\lambda_\mu p_\mu,$$

we know

$$M^{-1} = \left(\frac{\chi^{\lambda_j}_{\lambda_i}}{z(\lambda_i)} \right).$$

Abusing notation, we let D denote the diagonal matrix of $[DU_n]$. Multiplying the above out, we have

$$\begin{aligned}
([DU_n]_s^{-1})_{ij} &= MD^{-1}M^{-1} \\
&= \sum_{i_1} \chi^{\lambda_i}_{\lambda_{i_1}} \frac{1}{z(\lambda_{i_1})} \chi^{\lambda_j}_{\lambda_{i_1}} d^{-1}_{i_1 i_1} \\
&= \sum_{\mu} \frac{1}{z(\mu)d(\mu)} \chi^{\lambda_i}_{\mu} \chi^{\lambda_j}_{\mu} \\
&= \frac{1}{|\mathfrak{S}_n|} \sum_{\pi \in \mathfrak{S}_n} \chi^{\lambda_i}_{\pi} \chi^{\lambda_j}_{\pi} \frac{1}{d(\pi)} \\
&= \left\langle \chi^{\lambda_i} \chi^{\lambda_j}, \frac{1}{d(\cdot)} \right\rangle_{\mathfrak{S}_n},
\end{aligned}$$

where $d(\pi) = d(\text{cycle type of } \pi)$.

Let f be the class function on $\mathfrak{S}_n$ defined by

$$f(\pi) = \frac{(n-1)!(n+1)}{d(\pi)}.$$

For an upper bound on Smith entry $s_{p(n)}$, we want to show

$$\langle \chi^{\lambda} \chi^{\tilde{\lambda}}, f \rangle_{\mathfrak{S}_n}$$

is an integer for all $\lambda, \tilde{\lambda} \vdash n$. Thus, we want to show that f is a virtual character of $\mathfrak{S}_n$, i.e. $\mathrm{ch}(f)$ is an element of Λ.

We have

$$\begin{aligned}
\mathrm{ch}(f) &= \frac{(n-1)!(n+1)}{n!} \sum_{\pi \in \mathfrak{S}_n} \frac{p_\pi}{d(\pi)} \\
&= \frac{n+1}{n} \sum_{\pi \in \mathfrak{S}_n} \left(\frac{1}{p_1} \int p_\pi \, dp_1 \right) \\
&= \frac{(n-1)!(n+1)}{p_1} \int h_n \, dp_1.
\end{aligned}$$

Letting $H(t) = \sum_{r \geq 0} h_r t^r$, it is straightforward to see

$$\sum_{r \geq 1} p_r \frac{t^r}{r} = \log H(t).$$

Thus,

$$\begin{aligned}
\int H(t) \, dp_q &= \int e^{\sum_{r \geq 1} p_r \frac{t^r}{r}} \, dp_1 \\
&= \frac{H(t)}{t} + C.
\end{aligned}$$

Plugging in 0 for p_1 to determine the constant of integration C, we find

$$C = -\frac{H(t)}{te^{p_1 t}}.$$

This gives, since $h_1 = p_1$,

$$\begin{aligned}\sum_{n\geq 0}\left(\int h_n\,dp_1\right)t^n &= \frac{H(t)}{t}(1-e^{-h_1 t})\\ &= (1+h_1t+h_2t^2+\ldots)(h_1-\frac{h_1^2}{2!}t+\frac{h_1^3}{3!}t^2-\ldots),\end{aligned}$$

so we have

$$\begin{aligned}\int h_n\,dp_1 &= \frac{h_nh_1}{1}-\frac{h_{n-1}h_1^2}{2!}+\frac{h_{n-2}h_1^3}{3!}-\ldots\pm\frac{h_2h_1^{n-1}}{(n-1)!}\mp\frac{h_1h_1^n}{n!}\pm\frac{h_1^{n+1}}{(n+1)!}\\ &= \frac{h_nh_1}{1}-\frac{h_{n-1}h_1^2}{2!}+\frac{h_{n-2}h_1^3}{3!}-\ldots\pm\frac{h_2h_1^{n-1}}{(n-1)!}\mp h_1^{n+1}\frac{n}{(n+1)!}.\end{aligned}$$

Thus, we see

$$\begin{aligned}\frac{(n-1)!(n+1)}{p_1}\int h_n\,dp_1 =&\frac{(n-1)!(n+1)}{1!}h_n-\frac{(n-1)!(n+1)}{2!}h_{n-1}h_1\\ &+\ldots\pm(n+1)h_2h_1^{n-2}\mp h_1^n,\end{aligned}\tag{3}$$

which is surely in Λ.

Having proved $(n-1)!(n+1)[DU_n]^{-1}$ is an integral matrix, if we can point out an entry that is ± 1, then $(n-1)!(n+1)$ is indeed the largest Smith invariant, as claimed.

We will use (3) for the image of our virtual character f under the characteristic map. Since

$$s_\lambda\cdot s_\mu=\sum_\nu c^\nu_{\lambda\mu}s_\nu,$$

it is clear that when $h_k\cdot h_1^\ell$ is written as a sum of s_λ's, for $k>1$, that $s_{(1^{k+\ell})}$ does not appear. Moreover, $s_{(1^\ell)}$ occurs only once in such an expansion of h_1^ℓ. Thus, by (3), we see $s_{(1^n)}$ occurs in $\mathrm{ch}(f)$ with coefficient ± 1. That is, the irreducible character of the alternating representation

$$\chi^{(1^n)}$$

appears in our virtual character f with coefficient ± 1. Therefore, the entry of

$$(n-1)!(n+1)([DU]_s^{-1})$$

indexed by (1^n) and (n) is

$$\langle\chi^{(1^n)}\chi^{(n)},f\rangle_{\mathfrak{S}_n}=\pm 1,$$

since $\chi^{(n)}$ is the trivial character, finishing the proof. □

We also have Conjectures 10 and 9 hold in Young's lattice. This follows from the fact that they hold for Cartesian products of Y, a fact we prove in the next section.

3. Cartesian products of Young's lattice

In this section, we obtain some results on the Smith invariants of DU_n in

$$Y^r = Y\times\cdots\times Y.$$

Definition 21. If P and Q are posets, we define their *Cartesian product* to be the poset $P \times Q$ on the set

$$\{(p,q) : p \in P \text{ and } q \in Q\}$$

such that $(p,q) \leq (p',q')$ if $p \leq p'$ in P and $q \leq q'$ in Q.

It is a standard exercise to prove the following.

Proposition 22. [S1] *If P is an r-differential poset and Q is an s-differential poset, then $P \times Q$ is an $(r+s)$-differential poset.*

In particular, Y^r is an r-differential poset. Conjecture 11 then claims

$$DU_n : \mathbb{C}Y_n^r \to \mathbb{C}Y_n^r$$

has exactly p_{n-1} Smith invariants equal to 1 when $r \geq 2$. We now prove this is indeed a lower bound on the number of 1's.

Theorem 23. *Let $r \geq 1$. Then $DU_n : \mathbb{C}Y_n^r \to \mathbb{C}Y_n^r$ has at least p_{n-1} invariant factors equal to* 1.

Proof. The elements of Y_n^r are the r-tuples whose parts partition integers summing to n. Define a natural double lexicographic order on Y^r as follows:
$(\lambda^1, \dots, \lambda^r) \leq (\tilde{\lambda}^1, \dots, \tilde{\lambda}^r)$ if they are equal or the first coordinate i they differ in has $\lambda^i < \tilde{\lambda}^i$ in the usual lexicographic ordering of Y.
Now order the rows and columns of $[DU_n]$, decreasing from left to right and top to bottom. Consider the set

$$P = \{(\lambda^1, \dots, \lambda^r) \in Y_n^r : \lambda^r \text{ has a 1-part}\},$$

and its conjugate set

$$P^{\text{conj}} = \{(\lambda^1, \dots, \lambda^r) \in Y_n^r : \lambda^1 \text{ has exactly one largest part}\}.$$

Now one can easily check the submatrix of $[DU_n]$ whose columns and rows are indexed by P and P^{conj}, respectively, is lower triangular with 1's down its diagonal. ☐

One can then follow this proof, almost verbatim, to prove Conjecture 10 in the case of Y^r.

Theorem 24. *For $r \geq 1$, the map UD_n in Y^r has rank p_{n-1} and p_{n-1} 1's in its Smith normal form.*

Proof. See the proof of Theorem 23. ☐

Moreover, we also see Conjecture 9 holds in Y^r.

Theorem 25. *For $r \geq 1$, the map D_n in Y^r has rank p_{n-1} and p_{n-1} 1's in its Smith normal form.*

Proof. Using the set P defined in the proof of Theorem 23, one observes that the $p_{n-1} \times p_{n-1}$ submatrix of $[D_n]$ whose columns are indexed by P is lower triangular with 1's down its diagonal. ☐

It should be mentioned that we expect our results in Young's lattice Y to naturally extend to Y^r through methods similar to those used in the previous section. However, we have not yet explored this thoroughly.

4. The Fibonacci poset

As a set, the *Fibonacci poset* Z consists of all finite words using alphabet $\{1, 2\}$. For two words $w, w' \in Z$, we define w to cover w' if either

(1) w' is obtained from w by changing a 2 to a 1, as long as only 2's occur to its left, or
(2) w' is obtained from w by deleting its first 1.

It is an easy exercise to show Z is a 1-differential poset [S1]. It is also easy to see why it has such a name: its jth rank has size f_j, the jth Fibonacci number.

In this section, we will prove Conjectures 9, 10, and 11 for the Fibonacci poset Z. Because our work is largely computational, for this section we will do away with our previous convention of bracketing linear maps to represent a matrix representation.

Whenever dealing with the matrix forms of D, U, and compositions thereof, acting on a rank of Z, we will index the rows and columns in decreasing lexicographic order, from left to right, top to bottom. One can then observe how DU_n is recursively defined. First, one observes

$$D_n = \begin{pmatrix} U_{n-2} & I_{f_{n-1}} \end{pmatrix}, \tag{4}$$

which is not too hard to verify; the $I_{f_{n-1}}$ is clear, and one simply checks $D(2v) = U(v)$ for $v \in Z$. From this, one can then obtain

$$DU_n = \begin{pmatrix} DU_{n-2} + I_{f_{n-2}} & U_{n-3} & I_{f_{n-2}} \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ I_{f_{n-2}} & 0 & 2I_{f_{n-2}} \end{pmatrix}. \tag{5}$$

Now (4) and (5) together with the fact $DU - UD = I$ immediately prove Conjectures 9 and 10, respectively, in the case of Z.

Theorem 26. *The rank of $D_n : \mathbb{C}Z_n \to \mathbb{C}Z_n$ is f_{n-1}, with exactly this many 1's in its Smith normal form.*

Theorem 27. *The rank of $UD_n : \mathbb{C}Z_n \to \mathbb{C}Z_n$ is f_{n-1}, with exactly this many 1's in its Smith normal form.*

We now look to our second objective: prove Conjecture 11 in the case of Z. To do this, we start with a lemma that is an instrumental tool in the proof.

Lemma 28. In the Fibonacci lattice Z, for $n \geq 1$ we have

$$\begin{pmatrix} aI_{f_n} & -(a-1)D_{n+1} \\ -aU_n & -bI_{f_{n+1}} \end{pmatrix} \sim \begin{pmatrix} I_{f_n} & 0 & 0 \\ 0 & bI_{f_{n-1}} & -a(a-1)D_n \\ 0 & -bU_{n-1} & -a(a+b-1)I_{f_n} \end{pmatrix}.$$

Proof. We simply use row and column operations together with (4). We have

$$\begin{aligned}
\begin{pmatrix} aI_{f_n} & -(a-1)D_{n+1} \\ -aU_n & -bI_{f_{n+1}} \end{pmatrix} &\sim \begin{pmatrix} aI_{f_n} & -(a-1)U_{n-1} & -(a-1)I_{f_n} \\ -aD_n & -bI_{f_{n-1}} & 0 \\ -aI_{f_n} & 0 & -bI_{f_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{f_n} & -(a-1)U_{n-1} & -(a-1)I_{f_n} \\ -aD_n & -bI_{f_{n-1}} & 0 \\ -(a+b)I_{f_n} & 0 & -bI_{f_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{f_n} & 0 & 0 \\ 0 & -a(a-1)DU_{n-1} - bI_{f_{n-1}} & -a(a-1)D_n \\ 0 & -(a+b)(a-1)U_{n-1} & -(a+b)(a-1)I_{f_n} - bI_{f_n} \end{pmatrix} \\
&\sim \begin{pmatrix} I_{f_n} & 0 & 0 \\ 0 & bI_{f_{n-1}} & -a(a-1)D_n \\ 0 & -bU_{n-1} & -a(a+b-1)I_{f_n} \end{pmatrix}.
\end{aligned}$$

□

Lemma 29.

$$\begin{aligned}
DU_n \sim{}& I_{f_{n-2}} \oplus I_{f_{n-3}} \oplus I_{f_{n-4}} \oplus 2!I_{f_{n-5}} \oplus 3!I_{f_{n-6}} \oplus \cdots \oplus (k-4)!I_{f_{n-k+1}} \\
&\oplus (k-3)! \begin{pmatrix} I_{f_{n-k}} & 0 & 0 \\ 0 & k(k-2)I_{f_{n-k-1}} & -(k-1)(k-2)D_{n-k} \\ 0 & -k(k-2)U_{n-k-1} & -(k+1)(k-1)(k-2)I_{f_{n-k}} \end{pmatrix},
\end{aligned}$$

for $3 \le k \le n-1$, *taking* $(-1)!$ *and* $0!$ *to be* 1.

Proof. We simply induct on k. Working to show the base case holds, we have

$$\begin{aligned}
DU_n &= \begin{pmatrix} DU_{n-2} + I_{f_{n-2}} & U_{n-3} & I_{f_{n-2}} \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ I_{f_{n-2}} & 0 & 2I_{f_{n-2}} \end{pmatrix} \sim \begin{pmatrix} DU_{n-2} + I_{f_{n-2}} & U_{n-3} & I_{f_{n-2}} \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ -DU_{n-2} & -U_{n-3} & I_{f_{n-2}} \end{pmatrix} \\
&\sim \begin{pmatrix} 2DU_{n-2} + I_{f_{n-2}} & 2U_{n-3} & 0 \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ 0 & 0 & I_{f_{n-2}} \end{pmatrix} \sim \begin{pmatrix} 2DU_{n-2} - 2UD_{n-2} + I_{f_{n-2}} & -2U_{n-3} & 0 \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ 0 & 0 & I_{f_{n-2}} \end{pmatrix} \\
&\sim \begin{pmatrix} 3I_{f_{n-2}} & -2U_{n-3} & 0 \\ D_{n-2} & 2I_{f_{n-3}} & 0 \\ 0 & 0 & I_{f_{n-2}} \end{pmatrix} \sim \begin{pmatrix} I_{f_{n-2}} & 0 & 0 \\ 0 & 2I_{f_{n-3}} & -D_{n-2} \\ 0 & -2U_{n-3} & -3I_{f_{n-2}} \end{pmatrix}.
\end{aligned}$$

Now a single application of Lemma 28 yields the base case:

$$DU_n \sim \begin{pmatrix} I_{f_{n-2}} & 0 & 0 \\ 0 & 2I_{f_{n-3}} & -D_{n-2} \\ 0 & -2U_{n-3} & -3I_{f_{n-2}} \end{pmatrix} \sim I_{f_{n-2}} \oplus \begin{pmatrix} I_{f_{n-3}} & 0 & 0 \\ 0 & 3I_{f_{n-4}} & -2D_{n-3} \\ 0 & -3U_{n-4} & -4 \cdot 2I_{f_{n-3}} \end{pmatrix}.$$

For the induction step, assume $3 \le k \le n-2$. Then Lemma 28 says

$$\begin{aligned}
&\begin{pmatrix} k(k-2)I_{f_{n-k-1}} & -(k-1)(k-2)D_{n-k} \\ -k(k-2)U_{n-k-1} & -(k+1)(k-1)(k-2)I_{f_{n-k}} \end{pmatrix} \\
&\quad \sim (k-2) \begin{pmatrix} I_{f_{n-k-1}} & 0 & 0 \\ 0 & (k+1)(k-1)I_{f_{n-k-2}} & -k(k-1)D_{n-k-1} \\ 0 & -(k+1)(k-1)U_{n-k-2} & -(k+2)(k)(k-1)I_{f_{n-k-1}} \end{pmatrix},
\end{aligned}$$

completing the induction. □

Theorem 30. *The map* DU_n *in the Fibonacci poset* Z *possesses the* ι-λ *relation.*

Proof. In particular, Lemma 29 says

$$DU_n \sim I_{2f_{n-2}} \oplus 2!I_{f_{n-5}} \oplus 3!I_{f_{n-6}} \oplus \cdots \oplus (n-4)!I_{f_1} \oplus (n-3)! \begin{pmatrix} (n-1) & (n-2) \\ -(n-1) & n(n-2) \end{pmatrix}.$$

Since

$$\begin{pmatrix} (n-1) & (n-2) \\ -(n-1) & n(n-2) \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & (n+1)(n-1)(n-2) \end{pmatrix},$$

we have

$$DU_n \sim I_{2f_{n-2}} \oplus 2!I_{f_{n-5}} \oplus \cdots \oplus (n-4)!I_1 \oplus (n-3)! \oplus (n+1)(n-1)!,$$

which one can check matches the conjectured Snf of DU_n in Z. □

One can also look at the Fibonacci poset in terms of towers of algebras, with induction and restriction operators. In fact, [O] developed a Z-analogue of the ring Λ of symmetric functions we used for our study of Young's lattice. Moreover, his analogue of the h's makes our $[DU]$ matrix lower triangular, just as we had seen in the case of Young's lattice.

To be more precise about the work done in [O], the main goal was to answer the following question posed by Stanley (which we quote verbatim):

> [S1, Problem 8] Fix $r \in \mathbb{P}$. Is there a "natural" sequence
>
> $$\mathscr{A}_0 \subset \mathscr{A}_1 \subset \cdots$$
>
> of semisimple algebras $\mathscr{A}_n$ (over $\mathbb{C}$, say) whose relationship to $Z(r)$[1] is analogous to the relationship between the group algebras
>
> $$\mathbb{C}S_0 \subset \mathbb{C}S_1 \subset \cdots$$
>
> of the symmetric groups S_n and Young's lattice?... The existence of abstract algebras $\mathscr{A}_n$ with the desired properties is easy to see; in fact, it follows from the "fundamental construction" of [G-H-J] that
>
> $$\mathscr{A}_{n+1} \cong (\mathrm{End}_{\mathscr{A}_{n-1}} \mathscr{A}_n) \oplus \mathscr{A}_n^r,$$
>
> where $\mathrm{End}_{\mathscr{A}_{n-1}} \mathscr{A}_n$ is defined by a certain embedding $\mathscr{A}_{n-1} \subset \mathscr{A}_n$ corresponding to the Brattelli diagram $Z(r)_{n-1,n}$. What is wanted is a "natural" combinatorial definition of $\mathscr{A}_n$.

However, we merely mention these facts for the interested reader.

It should also be mentioned that in the case of the Fibonacci poset, the nice recursive definition of $[DU_n]$ makes it plausible that one can prove Conjecture 11 with a more straightforward and generalizable induction. When looking at possible induction techniques, a proof of the following would be incredibly useful.

Conjecture 31. *If M is of ι-λ-type, then $M+1$ is of ι-λ-type.*

If true, combined with (2), it would immediately prove Conjecture 11 in the case of Young's lattice, a particularly interesting case.

[1] $Z(r)$ simply generalizes Z by having r "colors" of 1, instead of just one. See [S1] for more information.

5. Computation & natural questions

In this section, we begin by telling the reader what computations have been done to justify making Conjectures 11 and 10. We will also give the answers to some questions about generalizing this phenomenon.

In the case of Young's lattice, we have tested the DU_n map for the ι-λ relation up to $n = 20$. We have also tested various reflection extensions [S1] starting at ranks of Y, $Z(1)$, $Z(2)$, $Z(3)$, and some Cartesian products, all going up to ranks of at least 10.

A natural question to ask is does, say, $D_{n-1} \circ D_n \circ U_n \circ U_{n-1}$ always possess the ι-λ relation? The answer to this is no. For example, one can easily check that

$$D_4 D_5 U_4 U_3 : \mathbb{C}Y_3 \to \mathbb{C}Y_3$$

is given by the matrix

$$\begin{pmatrix} 7 & 6 & 1 \\ 6 & 14 & 6 \\ 1 & 6 & 7 \end{pmatrix},$$

which is not of ι-λ-type.

Another natural question is whether or not we have this phenomenon in *sequentially differential posets* [S2], a generalization of r-differential posets. Again, the answer is no. The Boolean algebra is known to be such a poset, and one can easily show the DU maps are not in general of ι-λ-type (where nonsingular, of course). Morover, we do not know of an analogues phenomenon for these generalized differential posets.

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